Dark solitary waves in a generalized version of the nonlinear Schrödinger equation

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We investigate dark-wave solutions of the nonlinear parabolic equation with a nonlinearity of rather general type and nonzero boundary conditions at infinity. Traveling-wave solutions of some polynomial models are presented in the evident forms. The bistability of dark pulses as the possibility for two different waves to exist under the same boundary conditions is discussed. The consideration is supported by the numerical treatment of the polynomial nonlinearity allowing the bistability regime. It is found that some different boundary conditions can be coordinated with the nonlinear equation in the case of an N-shaped nonlinearity. In the small-amplitude limit the nonlinear Schrödinger equation is reduced to the Korteveg–de Vries equation. The stability of small-amplitude solutions is determined by both the kind of the nonlinearity and the intensity level at infinity. The occurrence of antidark pulses is pointed out.

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I. INTRODUCTION

In the present paper we study traveling-wave solutions of the highly nonlinear Schrödinger equation (HNSE)

$$i\psi_t+\psi_{xx}-2\psi f(I)=0 \tag{1}$$

(where $I=|\psi|^2$ and, for the meantime, $f(I)$ is an arbitrary nonlinear function) satisfying nonzero boundary conditions at infinity. As is evident, the simplest case $f(I)=kI$ of Eq. (1) is the conventional nonlinear Schrödinger equation (NSE), which has many applications in various branches of modern physics (see Ref. [1] for a review). In particular, the NSE is of great importance for nonlinear optics [2], where it is successfully used for a description of picosecond pulse propagation in monomode fibers. In that case, the nonlinearity if stipulated by the dependence of a fiber refractive index on the field intensity. The NSE can also be considered as a leading order of the expansion of Eq. (1) with respect to small nonlinearity. Hence, physical factors contributing to the increase of nonlinearity may require the employment of the HNSE with a more general function $f(I)$. For example, multiphoton resonances and light-induced phase transitions are such mechanisms in optics [3].

It is well known [4] that the NSE, which is exactly integrable by means of the inverse scattering technique, possesses two kinds of fundamental traveling-wave solutions depending on the sign of k. The first one is often called a bright soliton. It may exist under zero boundary conditions: $\psi\to 0$ as $|x|\to\infty$ if $k<0$. Another solution, called a dark soliton, is possible when $k>0$. It has the form

$$\psi(x,t)=q(x-vt)\exp(-2i\rho^2 t) \tag{2}$$

(where v and ρ are a real and a positive constant, respectively), with $q(x)$ satisfying boundary conditions

$$q\to\begin{cases}\rho & \text{as } x\to-\infty\\ \rho\exp(i\theta) & \text{as } x\to\infty\ ,\end{cases} \tag{3}$$

where θ is a constant belonging to the interval $[0,2\pi]$. A dark soliton is an intensity hole on the constant cw background of the amplitude equal to ρ.

In the case when $f(I)$ is a nonlinear function, stationary bright-soliton-like solutions also exist. Conditions for their existence as well as stability aspects of the problem have been thoroughly investigated by Kaplan [3], Enns, Ragnekar, and Kaplan [5], and Enns and Ragnekar [6]. In this way Kaplan discovered the bistability of bright solutions [3]. The associated stable solutions according to him are those that have the same energy but different parameters.

Dark pulses of the HNSE with $f(I)=-I+\alpha I^2$ are known as well. Their evident form has been obtained by Barashenkov and Makhankov [7], and stability has been studied numerically by Barashenkov and Kholmurodov [8]. Such systems turned out to have no bistable regimes.

The essential feature of the dark NSE soliton compared with the bright one is that it has only one parameter—a pulse velocity. It is determined by the boundary conditions (3). Other characteristics of the dark soliton (amplitude and width) are expressed through the velocity. Such a situation is also held in the case of generic nonlinearity $f(I)$ (see below). Therefore, one can foresee that the straightforward treatment of the bistability in the manner mentioned above is no longer available. Even application of the "bistability" term in the sense generally accepted [9] is questionable for the dark-pulse dynamics. Nevertheless, in the present paper we make use of this conception (being conscious of the conditional character of this term) in order to designate two kinds of solitary dark solutions of nonlinear equation (1) with different initial but the same boundary conditions.

It should be noted here that the bistability of dark optical pulses has been reported by Mulder and Enns [10] and Enns and Mulder [11]. In contrast to our case, in these papers, moving solutions of Eq. (1) were considered under boundary conditions with phases varying in time at infinity. In that case, the bistability of dark holes was understood in the same sense as that of bright solitons.

Only stationary solutions were treated under fixed phases.

Another peculiarity of the problem under consideration consists of coordination of the nonzero boundary conditions (3) with the nonlinear evolution equation (1). (It is the requirement that gives ρ^2 in the representation (2) for the dark NSE soliton [4].) In the event $f(I)$ is a nonlinear function, the corresponding requirement may give rise to additional complexity.

The organization of the paper is as follows. Section II is devoted to both the derivation of general equations and a discussion of the main properties of traveling solutions. Examples of polynomial models—the simplest one, $f(I)=I+\alpha I^2$, and those for which the bistability of bright pulses have been observed [5], $f(I)=I-bI^3+cI^5$ (a, b, and c are constants)—are treated in Sec. III. The main results are briefly summarized in the Conclusion.

II. GENERAL APPROACH

Let us search for a solution of Eq. (1) in the form $\psi(x,t)=q(x,t)\exp(-2i\omega t)$, where ω is a real constant (for the sake of definitenesses we will call it a cw frequency). Then $q(x,t)$ is governed by the equation

$$iq_t+q_{xx}-2q[f(I)-\omega]=0 \ . \tag{4}$$

Apparently, Eq. (4) is compatible with the boundary conditions (3) if

$$\omega=f(\rho^2) \ . \tag{5}$$

This relation between ω and ρ is not, generally speaking, one to one (cf., the case of the conventional NSE [4]) if $f(I)$ is a nonlinear function. Depending on the type of $f(I)$ there can be a set of values $\{\rho_i\}$ $(i=1,2,\dots,N)$ under which the formulation of the boundary problem is possible for the same ω. In other words, there is a set of the cw backgrounds with amplitudes ρ_i against which different dark pulses with the same cw frequency can propagate. Illustration of this situation for an N-shaped function $f(I)$ is represented in Fig. 1.

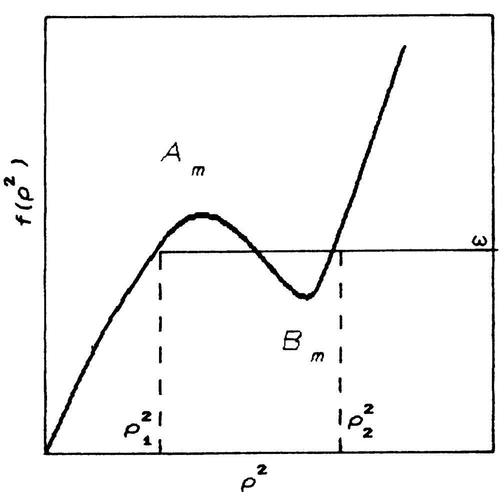

FIG. 1. The model $f(I)=I-bI^3+cI^5$. A_m and B_m are extrema of $f(I)$. Values ρ_1^2 and ρ_2^2, which are the crossing points of the graph with the level ω, are evidence of the two appropriate backgrounds.

It is not difficult to state that Eq. (4) possesses at least three integrals of motion for each given ρ^2 (cf. Refs. [4] and [5]):

$$I_1=\int_{-\infty}^{\infty}(|q|^2-\rho^2)dx \ , \tag{6a}$$

$$I_2=(1/2i)\int_{-\infty}^{\infty}(q_x q^*-q_x^* q)dx \ , \tag{6b}$$

$$I_3=\int_{-\infty}^{\infty}\left[|q_x|^2+2\int_{\rho^2}^{|q|^2}[f(s)-\omega]ds\right]dx \ . \tag{6c}$$

As usual, the third integral is the Hamiltonian of the system so that one can represent Eq. (4) in the form,

$$q_t=-i(\delta I_3/\delta q^*) \ .$$

A. Basic relations for a traveling-wave solution

Since we are interested in traveling-wave solutions, we represent function $q(x,t)$ in the form

$$q(x,t)=u(y)\exp[i\phi(y)] \ , \tag{7}$$

where a real amplitude $u(y)$, phase $\phi(y)$, and moving variable $y=x-vt$ with a real constant v being a pulse velocity, are introduced. Substituting the representation (7) into Eq. (4) and separating real and imaginary parts of the resulting equation, one derives the system

$$u\phi_{yy}+2u_y\phi_y-vu_y=0 \ , \tag{8a}$$

$$u_{yy}-u\{\phi_y^2-v\phi_y+2[f(I)-\omega]\}=0 \tag{8b}$$

(evidently now $I=u^2$). The boundary conditions for new unknown functions $u(y)$ and $\phi(y)$ follow directly from Eqs. (3) and (7):

$$u\to\rho \quad \text{as } |x|\to\infty \ , \tag{9a}$$

$$\phi\to\begin{cases}0 & \text{as } x\to-\infty \quad (9b)\\ \theta & \text{as } x\to\infty \ . \quad (9c)\end{cases}$$

To be rigorous in accordance with Eq. (5) one has to write ρ_i and, hence, u_i in Eq. (9a) and in all expressions in what follows. We do not do it to shorten formulas, however, bearing in mind that each relation obtained below corresponds to the definite boundary conditions determined by Eqs. (5) or (9).

Equations (8) can be integrated once. This gives

$$\phi(y)=(\tfrac{1}{2})v\int_{-\infty}^{y}[(I-\rho^2)/I]dy \ , \tag{10a}$$

$$I_y^2=Q(I)-v^2(I-\rho^2)^2 \ . \tag{10b}$$

Here the designation

$$Q(I)=8I\int_{\rho^2}^{I}[f(s)-f(\rho^2)]ds \tag{11}$$

is introduced. The straightforward algebra allows us to verify that now the Hamiltonian of the system is expressed directly through the function $Q(I)$:

$$I_3=(\tfrac{1}{2})\int_{-\infty}^{\infty}[Q(I)/I]dI \ . \tag{12}$$

While obtaining Eqs. (10) we have taken into account the boundary conditions [(9a) and (9b)] only. As it follows from Eq. (10a), in order to satisfy the limit (9c) one must require

$$(\tfrac{1}{2})v\int_{-\infty}^{\infty}[(I-\rho^2)/I]dy=\theta \ . \tag{13}$$

This formula allows us to determine the pulse velocity for given θ, ρ, and $f(I)$. It is important to stress that Eq. (13) is a transcendental equation with respect to v, since according to Eq. (10b) $I(y)$ depends on parameter v^2 in the evident form (rather than through the variable y only).

Equation (13) is the basic formula of our approach. It provides the opportunity to answer the questions of whether the traveling-wave solutions of the boundary problems (3) and (4) exist and how many there are.

Depending on the problem parameters, Eq. (13) may have no roots, (it corresponds to the lack of solitary waves), one root, and more than one root. The latter means in fact that there are various traveling dark pulses that can propagate against the same cw background ρ with the same constant phase shift at infinity θ and cw frequency ω.

Thus we can formulate the bistability (multistability) problem. The event when two (or more) dark pulses governed by Eq. (4) with the *same* boundary conditions (3) but having *different* parameters (amplitudes, velocities, etc.) exist will be called bistability (multistability).

As it follows from Eq. (10b), I_1 defined by Eq. (6a) has an evident dependence on v and, hence, the first motion integrals corresponding to different bistable pulses are different. In this sense, our definition of the bistability differs from that used by Kaplan [3].

It is useful to point out that by analogy with Ref. [3] one need not solve the Eq. (10b) in order to calculate v. Indeed, with the help of Eq. (10b) it is possible to pass from the integration over y in the left-hand side of Eq. (13) to the integration over I. In this way Eq. (13) can be rewritten as follows:

$$v\int_{\rho^2}^{I_m(v^2)}dI\,[IP(I)]^{-1}=\theta \ . \tag{14}$$

Here $P(I)=|I_y/(I-\rho^2)]|$ and $I_m(v^2)$ is a root of the equation

$$Q(I)/(I-\rho^2)^2=v^2 \ , \tag{15}$$

being the closest one to ρ^2. It is by definition the intensity extremum of the pulse, or, more precisely, the deepest point of the intensity hole in terms of dark-pulse dynamics. The particular case when $I_{yy}=0$ at $I=I_m$ is beyond our consideration.

We mainly restrict the present investigation to the case of dark-pulse dynamics that corresponds to $I<\rho^2$ and, hence, $I_m(v^2)<\rho^2$. Consequently, it is easy to see that this case implies that $I(y)$ belongs to the interval $[I_m(v^2),\rho^2]$. As a necessary condition for the problem to be solvable in accordance with Eq. (10b), one can obtain the inequality

$$v^2<Q(I)/(I-\rho^2)^2=D(I) \ , \tag{16}$$

which must be held for all I from the interval mentioned. Evidently, if it is not valid at least for $I=\rho^2$, no dark pulses may exist. Substituting this value in the inequality (16), we obtain the following restriction for the velocity:

$$v^2<4\rho^2 f'(\rho^2)=v_m^2 \tag{17}$$

[the prime means the derivation of $f(I)$ with respect to its argument]. The first consequence of this relation is that the traveling dark pulses may propagate only against the background with an amplitude satisfying the requirement

$$f'(\rho^2)>0. \tag{18}$$

Employing Fig. 1 as an example, one can determine that there are two possible cw backgrounds (they are designated by points ρ_1^2 and ρ_2^2).

The second consequence consists of the upper limit v_m for dark-pulse velocities. As is clear, it can be achieved at least by small-amplitude solutions. The latter allow consideration in more detail.

B. Small-amplitude pulses

As has been stated in Ref. [7], small-amplitude dark solutions of the higher nonlinear model $f(I)=-I+\alpha I^2$ having velocities close to critical v_m are described by the Korteveg–de Vries equation (KdV). A similar result for dark solitons of the NSE $[f(I)=I]$ was obtained by Kivshar and Afanasyev [12]. It is not difficult to state analogous behavior for the HNSE of a general type. To this end we represent the wave amplitude in the form

$$u(x,t)=\rho+\alpha(x,t) \ , \tag{19}$$

assuming that $\alpha(x,t)$ depends on provisional variable x and t instead of the traveling variable y and $|\alpha(x,t)|\ll\rho$. Then we follow to Ref. [12]. Namely $\alpha(x,t)$ and $\phi(x,t)$ are represented in the form of expansions:

$$\phi(x,t)=\epsilon\phi_0(\nu,\tau)+\epsilon^3\phi_1(\nu,\tau)+\cdots \ , \tag{20a}$$

$$\alpha(x,t)=\epsilon^2\alpha_0(\nu,\tau)+\epsilon^4\alpha_1(\nu,\tau)+\cdots \tag{20b}$$

(where ϵ is a small parameter introduced for convenience), and slow variables

$$\nu=\epsilon(x-v_m t) \ \text{ and } \ \tau=\epsilon^3 t \tag{21}$$

are used. Since these calculations closely follow those in Ref. [12], we omit them and write only the final answer. The function $\alpha_0(\nu,\tau)$ solves the KdV equation:

$$2v_m\alpha_{0\tau}+g\alpha_0\alpha_{0\nu}-\alpha_{0\nu\nu\nu}=0 \ , \tag{22}$$

where the constant g is determined by

$$g=8\rho[3f'(\rho^2)+\rho^2 f''(\rho^2)] \ . \tag{23}$$

Equation (22) is integrable and has one soliton solution [1] that in the case under consideration has the form

$$\alpha_0(\nu\tau)=-[(12\kappa^2)/g]\text{sech}^2\{\kappa[\tau-(2\kappa^2/v_m)\nu]\} \ . \tag{24}$$

Here the constant κ characterizes the soliton amplitude and is determined by the initial conditions for the HNSE.

Depending on the sign of g, the value $\alpha_0(x,t)$ is negative ($g>0$) or positive ($g<0$). The last case corresponds to I being greater than ρ^2 so that one has an anti-dark solution (as a similar bright splash against the background has been called in Ref. [12]). Its appearance is caused by the higher nonlinearity under the condition $\rho^2 f''(\rho^2) < -3f'(\rho^2)$.

Returning to the investigation of the dark-pulse dynamics, we can conclude that a stable soliton in the small-amplitude limit is available if

$$\rho^2 f''(\rho^2) > -3f'(\rho^2) \ . \tag{25}$$

The requirements for the small-amplitude approximation to be valid have been discussed in Ref. [12]. In our case one more restriction has to be satisfied: $|g| \ll \epsilon^{-2}$.

The relation (25) just obtained can be also derived from the general approach described in the previous section. Indeed, let us analyze Eq. (15) graphically as is shown in Fig. 2. It is easy to see that the value $I_m(v^2)$ at $v \to v_m$ crucially depends on the sign of $D'(I)$ at $I=\rho^2$. Taking into account the representation (11), one can verify that $\mathrm{sgn}[D'(\rho^2)] = \mathrm{sgn}[g(\rho^2)]$. For $v \to v_m$ and $g>0$ the corresponding dark pulses are of small amplitude [see Figs. 2(a) and 2(c)], entirely in agreement with the KdV approximation considered above. However, if $g<0$, the corresponding pulses have sufficiently large amplitudes [see Fig. 2(b)] so that the KdV approximation is impossible.

To conclude this section, in the case of traveling dark solutions we have reduced the initially formulated boundary problem [(3) and (4)] to the system of equations (10)–(12), which, together with the representation (7), give a full description of the solutions we are seeking.

III. EXAMPLES

Now we illustrate the employment of the general relations just obtained for investigation of polynomial models. This choice allows us to obtain exact solutions of HNSE having limiting transitions to the conventional NSE dark solitons and to observe bistability. Also, a polynomial model provides an N-shaped nonlinearity and, hence, gives an opportunity to investigate a case of two stable backgrounds.

A. Simple polynomial models

The first nontrivial case we consider is as follows:

$$f(I) = I \ . \tag{26}$$

Inserting Eq. (26) into Eqs. (10) and (14), one can easily derive the conventional NSE dark soliton [4] that is described by formulas

$$I(y) = \rho^2 - \eta^2 \mathrm{sech}^2(\eta y), \tag{27a}$$

$$\phi(y) = (\theta/2) + \arctan[\tanh(\eta y)\mathrm{tang}(\theta/2)] \ , \tag{27b}$$

where

$$2\eta = [v_m^2 - v^2]^{1/2}, \quad v = -2\rho\cos(\theta/2), \quad v_m^2 = 4\rho^2 \ . \tag{27c}$$

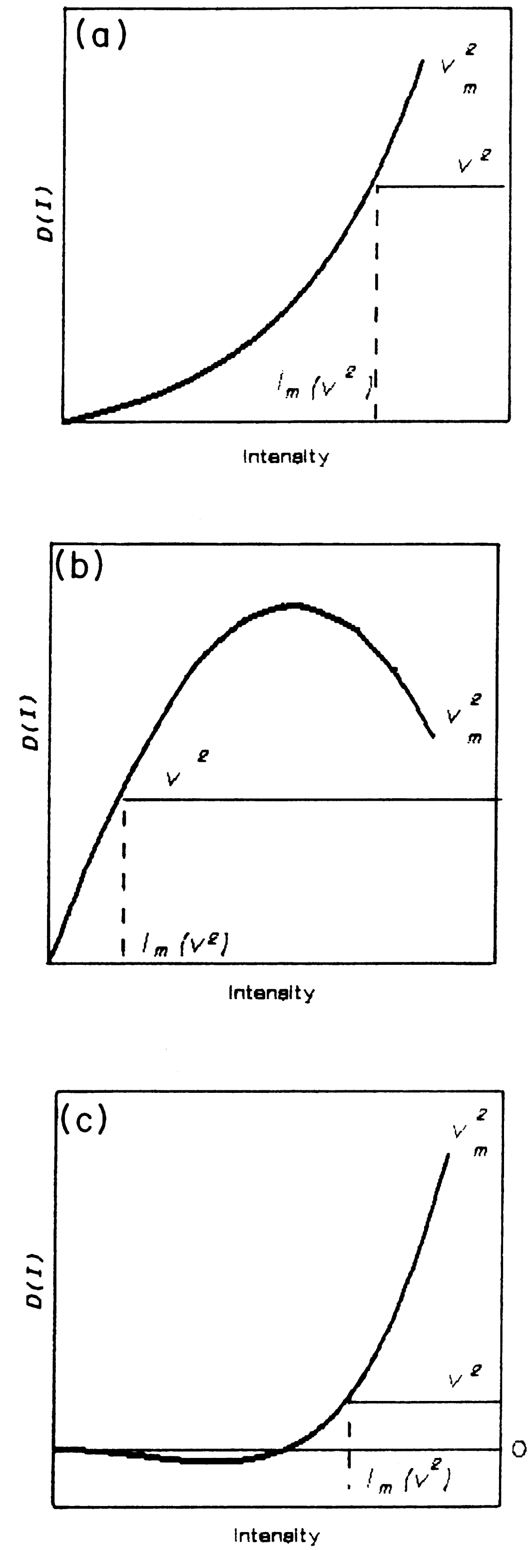

FIG. 2. Graphical solution of Eq. (15) for the highly nonlinear model. Within the chosen range of the problem parameters, we have three qualitatively different types of $D(I)$ dependence corresponding to three types of $\theta(v)$ behavior (a, b, and c correspond to the first, second, and third types, respectively). In the dark-pulse case, I can belong to the interval $[0,\rho^2]$. The point $I_m(v^2)$ is the root of Eq. (15) we are seeking.

Another simple polynomial model that can be treated analytically has a nonlinearity of the form

$$f(I)=I+(3a/2)I^2 , \tag{28}$$

where α is a real constant and the factor $(\frac{3}{2})$ is introduced for convenience. Note that the related HNSE differs from that considered in Ref. [7]. The solitary solution obtained there with the appropriate background tends to zero at $\alpha\to 0$. At the same time, the model described by Eq. (28) allows the limiting transition to the NSE dark solution (27). Substituting Eq. (28) into Eq. (5), one can conclude that for each ρ^2 satisfying the inequality (18) there is one-to-one correspondence between the background amplitude and frequency under which the formulation of the problem is possible. The intensity minimum of the dark solution is equal to

$$I_m(v^2)=[(\delta^2+\alpha v^2)^{1/2}-\delta]/(2a) , \tag{29}$$

where $\delta=1+2\alpha\rho^2$. Equation (14) can be also solved, providing the following expression for the pulse velocity:

$$v^2=2\rho^2[\beta-\alpha\rho^2\sin\theta+\cos\theta(\beta^2-a\rho^2\sin^2\theta)^{1/2}] , \tag{30}$$

where $\beta=1+3\alpha\rho^2$ and $\mathrm{sign}(v)=\mathrm{sign}(\pi-\theta)$. For the mathematical correctness of Eq. (30), one must require $\beta^2>(\alpha\rho^2\sin\theta)^2$. Therefore, dark solutions exist for those $\alpha\rho^2$ in the interval $[-(6+2\sin\theta)/(9-\sin^2\theta), -(6-2\sin\theta)/(9-\sin^2\theta)]$.

As it follows from Eq. (30), $v(\theta)$ is a single-valued function (remember, θ belongs to the interval $[0,2\pi]$) and, consequently, for given quantities of ρ and θ, there exists the only traveling-wave dark solution. In the bright case, for the analogous polynomial model, no bistability is observed, as well [3].

In the case under consideration, it is easy to obtain the evident expression for $I(y)$:

$$I(y)=\rho^2-(2\eta)^2/[\gamma+(\gamma^2-4av^2)^{1/2}\cosh^2(\eta y)] . \tag{31}$$

Here $\gamma=1+4\alpha\rho^2$, and η is expressed through parameters v and v_m, as previously in Eq. (27c), but now v^2 is defined by Eq. (30) and $v_m^2=4\rho^2\beta$. At a equal to zero, Eqs. (30) and (31) coincide with Eqs. (27).

Qualitatively, the main features of both solutions presented by Eqs. (27) and (31) are similar. The function $v(\theta)$ in both cases is monotonic, with $v(\pi)=0$ and $v(0,2\pi)=\mp v_m$. The intensity minima of both dark pulses reach zero at $\theta=\pi$. Such waves are called black pulses. The expressions for the inverse widths of the pulses η and their phases $\phi(y)$ are identical [the latter statement can be verified by substituting Eq. (31) into Eq. (10a)].

B. Dark-pulse bistability

As has been stated by Kaplan [3] it is the nonlinearity of the type

$$f(I)=I-bI^3+cI^5 , \tag{32}$$

where b and c are real constants, that allows a bistable regime of bright pulse propagation. We also employ this model in order to demonstrate a wide range of traveling dark solutions that may exist under the nonzero boundary conditions. Since it seems to be impossible to treat analytically Eqs. (10) and (14) with $f(I)$ defined by Eq. (32), we use numerical study.

Substituting Eq. (32) into Eq. (5), it is easy to verify that, depending on the values b and c, there can be one or two available [i.e., satisfying the inequality (18)] background amplitudes corresponding to the same frequency ω. The last event can occur if one chooses b and c so that $f(I)$ has two local extrema for the positive argument, say, A_m and B_m, as is shown in Fig. 1. It is possible only if $0<c<9/(20b^2)$. We let $b=1$ and vary c from zero to 0.44. For each value c, the quantity ρ varies in the interval $[A_m,B_m]$ [see the inequality (18)]. After the determination of v_m for a given ρ in accordance with Eq. (17), we find numerically $I_m(v^2)$ for each v^2 ($v^2\in[0,v_m^2]$) and then calculate the left-hand side of Eq. (14) as a function of v, which hereafter is designated as $\theta(v)$. $\theta(v)$ is an antisymmetric function of the argument that follows from Eq. (14). Since we assume that it belongs to the interval $[0,2\pi]$, we have $\theta=\pi$ as the center of $\theta(v)$ symmetry. It allows us to restrict our consideration to the positive v, i.e., $\theta\in[\pi,2\pi]$.

Qualitatively, one can distinguish three general types of the observed $\theta(v)$ behavior. They are reflected in Figs. 3–6. However, before discussion, we note that there is a close relation between the functions $I_m(v^2)$ and $\theta(v)$. So in order to explain the dependence $\theta(v)$, it is useful to start with Eq. (15), which determines $I_m(v^2)$.

The change of the pulse intensity extrema depends crucially on the function $D(I)$. Three observed kinds of $D(I)$ are plotted in Fig. 2. The case presented in Fig. 2(a) provides a monotonic $I_m(v^2)$ dependence with $I_m(0)=0$ and $I_m(v_m^2)=\rho^2$. The corresponding graph of $\theta(v)$ [the first type of the $\theta(v)$ behavior] is shown in Fig. 3. Qualitatively, it is a monotonic function as well, similar to those observed for the models considered in the previous section [see Eqs. (26) and (29)]. This type of graph is ob-

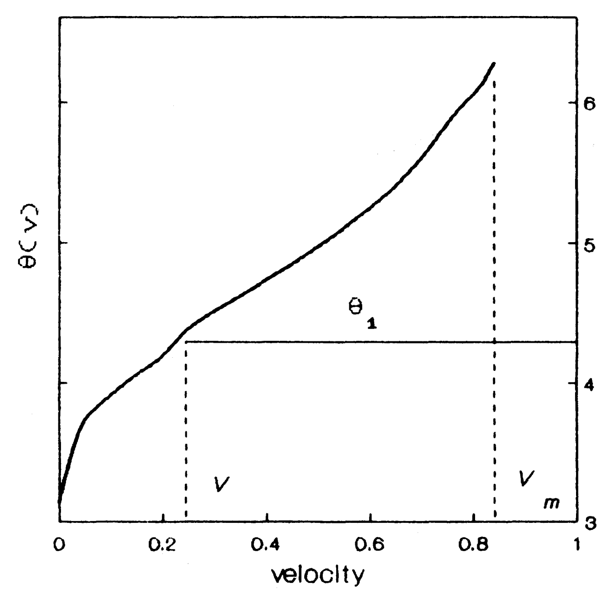

FIG. 3. Example of the first type of $\theta(v)$ behavior; $c=0.4$, $\rho^2=0.2$, $v_m=0.84$.

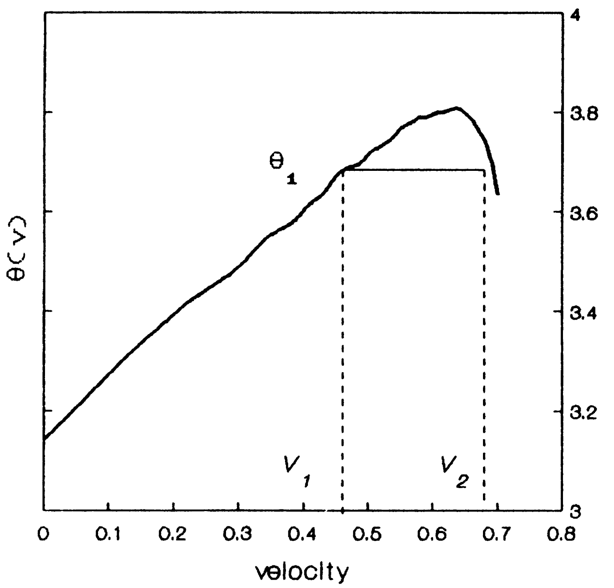

FIG. 4. Example of the second type of $\theta(v)$ behavior. It was observed for every c $(0<c<0.44)$ and ρ^2 being rather close to A_m (see Fig. 1), e.g., at $c=0.01$ for $0.45<\rho^2<0.58$ and at $c=0.4$ for $0.52<\rho^2<0.71$. The crossing of the level $\theta=\theta_1$ with the curve $\theta(v)$ produces two roots of Eq. (14) and hence provides evidence for the bistability regime; $c=0.3$, $\rho^2=0.55$, $v_m=0.71$.

served for every c when the value of ρ^2 is far enough from the external points A_m and B_m depicted in Fig. 1.

Function $D(I)$ for ρ^2 close to A_m (i.e., for the smaller background amplitudes) is as shown in Fig. 2(b). It is easy to verify that in this case there is a break in the $I_m(v^2)$ dependence as $v^2 \to v_m^2$. In other words, pulses

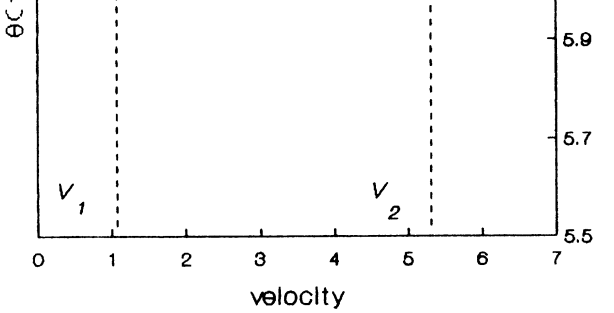

FIG. 5. Example of the third type of $\theta(v)$ behavior. It was observed at every c as well and for ρ^2 being close enough to B_m (see Fig. 1), e.g., at $c=0.2$ for $1.62<\rho^2<1.93$. As is seen in this case, the dark-pulse bistability can also exist; $c=0.1$, $\rho^2=2.6$, $v_m=6.09$.

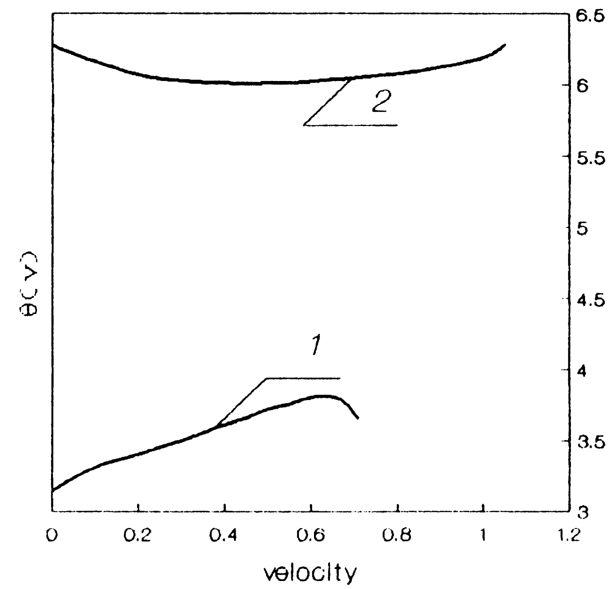

FIG. 6. Examples of the second and third classes corresponding to the same ω but different ρ. $c=0.4$, $\omega=0.41$. 1: $\rho^2=0.58$, $v_m=0.72$. 2: $\rho^2=1.09$, $v_m=1.05$.

having rather small amplitudes do not exist in the system. It is the effect that has been predicted earlier in Sec. II B. Corresponding examples of $\theta(v)$ are shown in Figs. 4 and 6 and make up the second type of $\theta(v)$ behavior.

As follows from Fig. 4 there can be values of θ for which two roots of Eq. (14) are possible. It is the situation that has been called here the dark-pulse bistability. Two localized dark traveling waves having different shapes and velocities but the same frequency and phase shift at infinity propagate against the same background.

The third type of the $\theta(v)$ behavior possesses the bistability regime, too. The main features of it are as follows: $I_m(0)\neq 0$ and $I_m(v_m^2)=\rho^2$ [see Fig. 2(c)]. It means that the existence of so-called black pulses is impossible in this case. It is worth noting that $I_m(0)=0$ is always the root of Eq. (15), but now there can be another root that is closer to ρ^2 than zero. Hence, as follows from Eq. (14), $\theta(0)=0$ and $\theta(v_m)=0$, so that the bistability regime is possible as well. This type of $\theta(v)$ dependence has been observed for ρ^2 close enough to B_m (i.e., for solutions against more intensive backgrounds).

It is interesting to point out that for values of c large enough $(0.38<c<0.44)$, both bistability regimes are observed for the same value ω (see Fig. 6). Thus in this case we have multistability in a definite sense. Indeed, there are two localized solutions against both backgrounds corresponding to the same frequency.

Our numerical results allow us to conclude that the occurrence of the dark-pulse bistability is a thresholdless process with respect to c (it has been observed even for $c=10^{-4}$). It is unlike the bright case when the bistability takes place for those values c that exceed some critical value [5]. We also note that formation of the conventional NSE solitons is a thresholdless process as well [13].

IV. CONCLUSION

To conclude, we have investigated localized traveling dark solutions of the HNSE satisfying nonzero boundary conditions (3). The problem considered is analogous with the treatment of the bright-wave dynamics. There is a bistability regime of the dark pulses, as well, which is observed, in particular, for the N-shaped polynomial model. In addition, the dark-pulse bistability has peculiarities in comparison with the bright one due to the nonzero boundary conditions, and, therefore, one parametric character of a solitary-wave solution.

A number of peculiarities we observed are as follows. The bistability, which enables the existence of solitary dark pulses with equal frequencies that can propagate against different backgrounds, has a thresholdless character. From the mathematical point of view, it means that there are two or more (for a general case) formulations of the boundary problem coordinated with the evolution equation (4) (cf. the conventional NSE [4]). From the optical point of view, it means that two levels of the cw background can be realized. As a consequence, one can expect the possibility for switching of the wave propagation in the system. Also, various multistable regimes in such a system for other kinds of nonlinearity can occur, say, for saturation models like those considered earlier for bright solutions [5].

Speaking about small-amplitude pulses, one has to note that for the rather general function $f(I)$ they are described by the KdV equation. It allows us to suppose that solutions allowing limiting transitions to the KdV solitons correspond to stable nonlinear dark pulses. However, a general discussion of the dark-wave stability was out of the scope of the present consideration.

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