# Rotating-wave approximation in high-gain lasers

Lee W. Casperson*
*Department of Physics, University of Otago, P.O. Box 56, Dunedin, New Zealand*
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Semiclassical models for lasers typically incorporate a fundamental simplification called the rotating-wave approximation. In this study the rotating-wave approximation is reexamined, and its implications for the simplest problem of a steady-state laser oscillator are considered in detail. It is found that for practical laser operating conditions the errors resulting from this approximation may not always be negligible.

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## I. INTRODUCTION

Every analysis of laser behavior typically involves a series of approximations. Some of these approximations are noted explicitly as a calculation proceeds, while others are already implicit in the starting formalism. Among the first approximations that one encounters in a semiclassical model are certain assumptions about the field derivatives and frequency differences appearing in Maxwell's equations. These assumptions were clearly noted in the Maxwell-Schrödinger laser models developed by Lamb and others [1], and they are sometimes identified in part as slowly-varying-amplitude approximations. In recent studies the author has considered in detail the consequences and possible limitations of these assumptions for steady-state and unstable laser amplifiers and oscillators [2]. A different but equally fundamental approximation involves the neglect of harmonics of the field and polarization oscillation frequencies that always arise in the intrinsically nonlinear Maxwell-Schrödinger models. For historical reasons this neglect is often referred to as the rotating-wave approximation, and it too has long been recognized [1]. The purpose of this study is to reexamine the rotating-wave approximation and obtain an estimate of the errors that it might introduce into a laser calculation. The emphasis here is on the simplest problem of steady-state laser oscillation, and it is shown that the errors resulting from this approximation may not always be negligible.

A few investigations of the effects of the rotating-wave approximation on laser behavior have been carried out previously, but for the most part those studies use highly simplified models and are not directly applicable to practical laser systems. Typical simplifications in the semiclassical field-atom models include the neglect of pumping, spontaneous decay, coherence decay, cavity losses, and propagation effects. These models are found to sometimes exhibit chaotic oscillatory behavior [3,4], and exact periodic solutions have also been found [5,6]. For practical laser operating conditions, the rotating-wave approximation is generally understood to lead to only small quantitative differences from the predictions of more exact treatments, and the Bloch-Siegert shift is an example [7]. Corrections are also believed to only be important at very high intensities [8]. This study focuses on the simplest possible implications of the rotating-wave approximation in more general models that include the various relaxation processes that have just been mentioned. In particular, the results obtained here permit one to estimate the effects of the rotating-wave approximation on the output intensity and oscillation frequency of laser oscillators.

The basic semiclassical formalism for a cw one-directional ring laser oscillator is developed in Sec. II, and the rotating-wave and frequency approximations are avoided. In Sec. III this model is reduced to the case of a periodic electromagnetic field and in Sec. IV to a single-frequency sinusoidal field. Detailed numerical solutions for a homogeneously broadened laser are presented in Sec. V, and it is found that the rotating-wave approximation can have a significant effect on the laser oscillation frequency and intensity.

## II. GENERAL MODEL

The emphasis of this study is on the development of a model for laser oscillation that avoids the rotating-wave approximation. For this purpose many other complicating features that have been included in more specialized treatments may be largely neglected. The starting point for this study is the same basic one-dimensional semiclassical model that has been employed in Ref. [2] and elsewhere. The density-matrix equations in this model take the form

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right]\rho_{ab}(v,\omega_\alpha,z,t) = -(i\omega_\alpha+\gamma)\rho_{ab}(v,\omega_\alpha,z,t) - \frac{i\mu}{\hbar}E(z,t)[\rho_{aa}(v,\omega_\alpha,z,t)-\rho_{bb}(v,\omega_\alpha,z,t)] , \quad (1)$$

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right]\rho_{aa}(v,\omega_\alpha,z,t) = \lambda_a(v,\omega_\alpha,z,t)-\gamma_a\rho_{aa}(v,\omega_\alpha,z,t) + \left[\frac{i\mu}{\hbar}E(z,t)\rho_{ba}(v,\omega_\alpha,z,t)+\text{c.c.}\right] , \quad (2)$$

 

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right]\rho_{bb}(v,\omega_\alpha,z,t)$$
$$=\lambda_b(v,\omega_\alpha,z,t)-\gamma_b\rho_{bb}(v,\omega_\alpha,z,t)+\gamma_{ab}\rho_{aa}(v,\omega_\alpha,z,t)-\left[\frac{i\mu}{\hbar}E(z,t)\rho_{ba}(v,\omega_\alpha,z,t)+\text{c.c.}\right], \quad (3)$$

$$\rho_{ba}(v,\omega_\alpha,z,t)=\rho^*{}_{ab}(v,\omega_\alpha,z,t), \quad (4)$$

where the subscripts $a$ and $b$ denote the upper and lower laser levels, respectively, $\gamma_a$ and $\gamma_b$ are the total decay rates for these levels, $\gamma_{ab}$ is the rate of direct decays from level $a$ to level $b,\gamma$ is the decay rate for the off-diagonal elements, $\lambda_a$ and $\lambda_b$ are the pumping rates, $\mu$ is the dipole moment for the laser transition, and the notation c.c. means the complex conjugate of the preceding terms. The laser medium is assumed to have both Doppler- and non-Doppler-broadening mechanisms, with $v$ being the $z$ component of the velocity and $\omega_\alpha$ the center frequency of the laser transition for members of an atomic or molecular class $\alpha$. The decay process represented by $\gamma_{ab}$ is often missing from theoretical studies, but it can have an important effect on the population of the lower level of the laser transition.

To the density-matrix equations for the atomic or molecular populations and polarizations must be added an equation for the electric field. The wave equation for the electric field of a linearly polarized wave in a laser medium can be written

$$\frac{\partial^2 E(z,t)}{\partial z^2}-\mu_1\sigma\frac{\partial E(z,t)}{\partial t}-\mu_1\epsilon_1\frac{\partial^2 E(z,t)}{\partial t^2}=\mu_1\frac{\partial^2 P(z,t)}{\partial t^2}. \quad (5)$$

The permeability $\mu_1$ and permittivity $\epsilon_1$ should be understood to include all of the magnetic and dielectric properties of the laser medium except for the polarization $P(z,t)$, which is due to the lasing atoms or molecules. The polarization driving this equation can be related back to the off-diagonal density-matrix elements by

$$P(z,t)=\int_0^\infty\int_{-\infty}^\infty \mu\rho_{ab}(v,\omega_\alpha,z,t)dv\,d\omega_\alpha+\text{c.c.} \quad (6)$$

Equations (1)–(6) are a complete set from which the time and space dependences of the electric field and of the atomic or molecular parameters can be determined, subject to the boundary conditions at the resonator mirrors.

It is convenient to explicitly separate the off-diagonal matrix element into its real and imaginary parts using the definition

$$\rho_{ab}(v,\omega_\alpha,z,t)=[P_r(v,\omega_\alpha,z,t)+iP_i(v,\omega_\alpha,z,t)]/2\mu, \quad (7)$$

where the subscripts $r$ and $i$ denote the real and imaginary parts, respectively. It is also helpful to introduce the population difference and sum according to

$$D(v,\omega_\alpha,z,t)=\rho_{aa}(v,\omega_\alpha,z,t)-\rho_{bb}(v,\omega_\alpha,z,t), \quad (8)$$

$$M(v,\omega_\alpha,z,t)=\rho_{aa}(v,\omega_\alpha,z,t)+\rho_{bb}(v,\omega_\alpha,z,t). \quad (9)$$

With these substitutions Eqs. (1)–(4) become

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right]P_i(v,\omega_\alpha,z,t)=-\omega_\alpha P_r(v,\omega_\alpha,z,t)-\gamma P_i(v,\omega_\alpha,z,t)-\frac{2\mu^2}{\hbar}E(z,t)D(v,\omega_\alpha,z,t), \quad (10)$$

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right]P_r(v,\omega_\alpha,z,t)=\omega_\alpha P_i(v,\omega_\alpha,z,t)-\gamma P_r(v,\omega_\alpha,z,t), \quad (11)$$

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right]D(v,\omega_\alpha,z,t)=\lambda_a(v,\omega_\alpha,z,t)-\lambda_b(v,\omega_\alpha,z,t)-\frac{\gamma_a+\gamma_{ab}+\gamma_b}{2}D(v,\omega_\alpha,z,t)$$
$$-\frac{\gamma_a+\gamma_{ab}-\gamma_b}{2}M(v,\omega_\alpha,z,t)+\frac{2}{\hbar}E(z,t)P_i(v,\omega_\alpha,z,t), \quad (12)$$

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right]M(v,\omega_\alpha,z,t)=\lambda_a(v,\omega_\alpha,z,t)+\lambda_b(v,\omega_\alpha,z,t)-\frac{\gamma_a-\gamma_{ab}-\gamma_b}{2}D(v,\omega_\alpha,z,t)-\frac{\gamma_a-\gamma_{ab}+\gamma_b}{2}M(v,\omega_\alpha,z,t). \quad (13)$$

Also, from Eq. (6) the polarization can be written

$$P(z,t)=\int_0^\infty\int_{-\infty}^\infty P_r(v,\omega_\alpha,z,t)dv\,d\omega_\alpha. \quad (14)$$

## III. PERIODIC FIELD

It usually is not useful to look directly for general solutions of Eqs. (5) and (10)–(14), and some constraints will be imposed here at the outset. To be specific, we will only consider solutions in which each of the dependent variables can be represented as a sum of harmonics of a fundamental traveling-wave field. While this constraint limits somewhat the kinds of spatial and temporal dynamics that can be investigated, it provides a first check on the implications of the rotating-wave approximation for most ordinary cw laser oscillators. These lasers are generally understood to produce a steady-state output intensity when one looks on a time scale long compared to an optical cycle or a distance scale long compared to a wavelength. If the losses in a unidirectional ring laser oscillator can be considered to be uniformly distributed and the pump rates constant, then the time and space variations of the dependent variables in Eqs. (5) and (10)–(14) can be represented by the summations

$$P_r(v,\omega_\alpha,z,t)=\sum_{j=-\infty}^{\infty} P_{r,2j+1}(v,\omega_\alpha)\times\exp[i(2j+1)(kz-\omega t)]\ , \quad (15)$$

$$P_i(v,\omega_\alpha,z,t)=\sum_{j=-\infty}^{\infty} P_{i,2j+1}(v,\omega_\alpha)\times\exp[i(2j+1)(kz-\omega t)]\ , \quad (16)$$

$$D(v,\omega_\alpha,z,t)=\sum_{j=-\infty}^{\infty} D_{2j}(v,\omega_\alpha)\exp[i(2j)(kz-\omega t)]\ , \quad (17)$$

$$M(v,\omega_\alpha,z,t)=\sum_{j=-\infty}^{\infty} M_{2j}(v,\omega_\alpha)\exp[i(2j)(kz-\omega t)]\ , \quad (18)$$

$$E(z,t)=\tfrac{1}{2}\sum_{j=-\infty}^{\infty} E_{2j+1}\exp[i(2j+1)(kz-\omega t)]\ . \quad (19)$$

Since the basic laser variables appearing on the left-hand sides of Eqs. (15)–(19) are real, the expansion coefficients must satisfy the relationships

$$P_{r,j}(v,\omega_\alpha)=P^*_{r,-j}(v,\omega_\alpha)\ , \quad (20)$$

$$P_{i,j}(v,\omega_\alpha)=P^*_{i,-j}(v,\omega_\alpha)\ , \quad (21)$$

$$D_j(v,\omega_\alpha)=D^*_{-j}(v,\omega_\alpha)\ , \quad (22)$$

$$M_j(v,\omega_\alpha)=M^*_{-j}(v,\omega_\alpha)\ , \quad (23)$$

$$E_j=E^*_{-j}\ . \quad (24)$$

When Eqs. (15)–(19) are substituted into Eqs. (10)–(13) one obtains the equations

$$i(2j+1)(kv-\omega)P_{i,2j+1}(v,\omega_\alpha)=-\omega_\alpha P_{r,2j+1}(v,\omega_\alpha)-\gamma P_{i,2j+1}(v,\omega_\alpha)-\frac{\mu^2}{\hbar}\sum_l E_{2l+1}D_{2j-2l}(v,\omega_\alpha)\ , \quad (25)$$

$$i(2j+1)(kv-\omega)P_{r,2j+1}(v,\omega_\alpha)=\omega_\alpha P_{i,2j+1}(v,\omega_\alpha)-\gamma P_{r,2j+1}(v,\omega_\alpha)\ , \quad (26)$$

$$i(2j)(kv-\omega)D_{2j}(v,\omega_\alpha)=[\lambda_a(v,\omega_\alpha)-\lambda_b(v,\omega_\alpha)]\delta_{j0}-\frac{\gamma_a+\gamma_{ab}+\gamma_b}{2}D_{2j}(v,\omega_\alpha)-\frac{\gamma_a+\gamma_{ab}-\gamma_b}{2}M_{2j}(v,\omega_\alpha)+\frac{1}{\hbar}\sum_l E_{2l+1}P_{i,2j-2l-1}(v,\omega_\alpha)\ , \quad (27)$$

$$i(2j)(kv-\omega)M_{2j}(v,\omega_\alpha)=[\lambda_a(v,\omega_\alpha)+\lambda_b(v,\omega_\alpha)]\delta_{j0}-\frac{\gamma_a-\gamma_{ab}-\gamma_b}{2}D_{2j}(v,\omega_\alpha)-\frac{\gamma_a-\gamma_{ab}+\gamma_b}{2}M_{2j}(v,\omega_\alpha)\ . \quad (28)$$

The same substitutions reduce Eqs. (5) and (14) to the new field equation

$$[-(2j+1)^2k^2+i(2j+1)\omega\mu_1\sigma+(2j+1)^2\omega^2\mu_1\epsilon_1]E_{2j+1}=-2(2j+1)^2\omega^2\mu_1\int_0^\infty\int_{-\infty}^{\infty}P_{r,2j+1}(v,\omega_\alpha)dv\,d\omega_\alpha\ . \quad (29)$$

It is sometimes convenient to introduce the new frequency parameter $\Omega=k(\mu_1\epsilon_1)^{-1/2}$ and the cavity lifetime $t_c=\epsilon_1/\sigma$, and then Eq. (29) can be written

$$-\frac{E_{2j+1}}{2t_c(2j+1)}+i\frac{\omega^2-\Omega^2}{2\omega}E_{2j+1}=-\frac{i\omega}{\epsilon_1}\int_0^\infty\int_{-\infty}^{\infty}P_{r,2j+1}(v,\omega_\alpha)dv\,d\omega_\alpha\ . \quad (30)$$

## IV. SINUSOIDAL FIELD

Equations (25)–(28) and (30) represent a general model for the behavior of a ring laser oscillator so long as the fields, polarizations, and populations vary periodically in time and space. In practice, however, the fields in lasers are usually found to vary sinusoidally with little or no higher harmonic content. This empirical result may be attributable to the small amplitude or incorrect phase velocity of the higher polarization harmonics in Eq. (30) or to the shorter cavity lifetime of the higher harmonics. Equation (30) has been written as if all harmonics had the same lifetime, but in practice $t_c$ would usually have a strong frequency dependence due to the characteristics of the mirror coatings and amplifier losses. It is possible, of course, that the higher field harmonics might coincidentally or intentionally occur at frequencies of high mirror reflectivity and good transmission of the other laser elements. In that case significant field amplitudes might be possible at the odd harmonics of the fundamental lasing frequency. However, the practical absence of field harmonics suggests that at least initially this same simplification should be made in the theoretical model. If the field harmonics are set equal to zero, Eqs. (25)–(28) and (30) reduce to

$$0=-[i(2j+1)(kv-\omega)+\gamma]P_{i,2j+1}(v,\omega_\alpha)-\omega_\alpha P_{r,2j+1}(v,\omega_\alpha)-\frac{\mu^2}{\hbar}[E_1D_{2j}(v,\omega_\alpha)+E_{-1}D_{2j+2}(v,\omega_\alpha)]\ , \quad (31)$$

$$0=-[i(2j+1)(kv-\omega)+\gamma]P_{r,2j+1}(v,\omega_\alpha)+\omega_\alpha P_{i,2j+1}(v,\omega_\alpha)\ , \tag{32}$$

$$\begin{aligned}0=&[\lambda_a(v,\omega_\alpha)-\lambda_b(v,\omega_\alpha)]\delta_{j0}\\&-\left[i(2j)(kv-\omega)+\frac{\gamma_a+\gamma_{ab}+\gamma_b}{2}\right]D_{2j}(v,\omega_\alpha)\\&-\frac{\gamma_a+\gamma_{ab}-\gamma_b}{2}M_{2j}(v,\omega_\alpha)\\&+\frac{1}{\hbar}[E_1P_{i,2j-1}(v,\omega_\alpha)+E_{-1}P_{i,2j+1}(v,\omega_\alpha)]\ ,\end{aligned} \tag{33}$$

$$\begin{aligned}0=&[\lambda_a(v,\omega_\alpha)+\lambda_b(v,\omega_\alpha)]\delta_{j0}\\&-\left[i(2j)(kv-\omega)+\frac{\gamma_a-\gamma_{ab}+\gamma_b}{2}\right]M_{2j}(v,\omega_\alpha)\\&-\frac{\gamma_a-\gamma_{ab}-\gamma_b}{2}D_{2j}(v,\omega_\alpha)\ ,\end{aligned} \tag{34}$$

$$-\frac{E_1}{2t_c}+i\frac{\omega^2-\Omega^2}{2\omega}E_1=-\frac{i\omega}{\epsilon_1}\int_0^\infty\int_{-\infty}^\infty P_{r,1}(v,\omega_\alpha)dv\,d\omega_\alpha\ . \tag{35}$$

It is simplest now to require the electric field amplitude to have the real value $E_1=E_{-1}=E_r$. Equation (31) and (32) can be solved for the polarization components, and the results can be written

$$P_{r,2j+1}(v,\omega_\alpha)=\frac{\omega_\alpha}{i(2j+1)(kv-\omega)+\gamma}P_{i,2j+1}(v,\omega_\alpha)\ , \tag{36}$$

$$P_{i,2j+1}(v,\omega_\alpha)=-\frac{\mu^2}{\hbar\gamma}\alpha_j(v,\omega_\alpha)\times[D_{2j}(v,\omega_\alpha)+D_{2j+2}(v,\omega_\alpha)]E_r\ , \tag{37}$$

where the coefficient $\alpha_j(v,\omega_\alpha)$ is given by

$$\alpha_j(v,\omega_\alpha)=\frac{\gamma/2}{i(2j+1)(kv-\omega)+i\omega_\alpha+\gamma}+\frac{\gamma/2}{i(2j+1)(kv-\omega)-i\omega_\alpha+\gamma}\ . \tag{38}$$

Equations (33) and (34) can be combined to yield the population difference

$$\begin{aligned}D_{2j}(v,\omega_\alpha)=&-\frac{E_r}{2\hbar}\frac{\gamma_a-\gamma_{ab}+\gamma_b}{\gamma_a\gamma_b}\beta_j(v,\omega_\alpha)\\&\times[P_{i,2j-1}(v,\omega_\alpha)+P_{i,2j+1}(v,\omega_\alpha)]\\&+N(v,\omega_\alpha)\delta_{j0}\ ,\end{aligned} \tag{39}$$

where the coefficient $\beta_j(v,\omega_\alpha)$ is

$$\beta_j(v,\omega_\alpha)=\frac{\gamma_a\gamma_b}{\gamma_a-\gamma_{ab}+\gamma_b}\left\{\frac{i(4j)(kv-\omega)+\gamma_a-\gamma_{ab}+\gamma_b}{[i(2j)(kv-\omega)+\gamma_a][i(2j)(kv-\omega)+\gamma_b]}\right\}\ , \tag{40}$$

and the fundamental unsaturated population difference is

$$N(v,\omega_\alpha)=(1-\gamma_{ab}/\gamma_b)\lambda_a(v,\omega_\alpha)/\gamma_a-\lambda_b(v,\omega_\alpha)/\gamma_b\ . \tag{41}$$

Equations (37) and (39) may be combined, and one result is

$$P_{i,1}(v,\omega_\alpha)=-\frac{\mu^2}{\hbar\gamma}W(v,\omega_\alpha)D_0(v,\omega_\alpha)E_r\ , \tag{42}$$

where the function $W(v,\omega_\alpha)$ is the continued fraction

$$W(v,\omega_\alpha)=\cfrac{\alpha_0(v,\omega_\alpha)}{1+\cfrac{\alpha_0(v,\omega_\alpha)\beta_1(v,\omega_\alpha)sI}{1+\cfrac{\alpha_1(v,\omega_\alpha)\beta_1(v,\omega_\alpha)sI}{1+\cfrac{\alpha_1(v,\omega_\alpha)\beta_2(v,\omega_\alpha)sI}{1+\cdots}}}}\ , \tag{43}$$

and the normalized intensity is

$$sI=\frac{\mu^2E_r^2}{2\hbar^2}\frac{\gamma_a-\gamma_{ab}+\gamma_b}{\gamma\gamma_a\gamma_b}\ . \tag{44}$$

Similar solutions are obtained within the rotating-wave approximation for standing-wave lasers [9-13]. Using Eqs. (21) and (42), one finds from Eq. (39) that the fundamental population difference is

$$D_0(v,\omega_\alpha)=\frac{N(v,\omega_\alpha)}{1+2W_r(v,\omega_\alpha)sI}\ , \tag{45}$$

where again the subscript $r$ refers to the real part. Then, with Eqs. (36) and (42) the polarization component $P_{r,1}(v,\omega_\alpha)$ can be written

$$P_{r,1}(v,\omega_\alpha)=-\frac{\mu^2E_r}{\gamma\hbar}\frac{\omega_\alpha}{i(kv-\omega)+\gamma}\frac{W(v,\omega_\alpha)N(v,\omega_\alpha)}{1+2W_r(v,\omega_\alpha)sI}\ . \tag{46}$$

The polarization given by Eq. (46) may be substituted into Eq. (35), and the real and imaginary parts of the result are the two equations

$$\frac{1}{2t_c}=\frac{\omega\mu^2}{\gamma\hbar\epsilon_1}\int_0^\infty\int_{-\infty}^\infty\frac{(\omega_\alpha/\gamma)N(v,\omega_\alpha)}{1+2W_r(v,\omega_\alpha)sI}\times\mathrm{Im}\left[\frac{W(v,\omega_\alpha)}{1+i(kv-\omega)/\gamma}\right]dv\,d\omega_\alpha\ , \tag{47}$$

$$\frac{\omega^2-\Omega^2}{2\omega}=\frac{\omega\mu^2}{\gamma\hbar\epsilon_1}\int_0^\infty\int_{-\infty}^\infty\frac{(\omega_\alpha/\gamma)N(v,\omega_\alpha)}{1+2W_r(v,\omega_\alpha)sI}\mathrm{Re}\left[\frac{W(v,\omega_\alpha)}{1+i(kv-\omega)/\gamma}\right]dv\,d\omega_\alpha\,. \tag{48}$$

These results can also be written

$$\frac{1}{r}=\frac{\int_0^\infty\int_{-\infty}^\infty\frac{(\omega_\alpha/\gamma)N(v,\omega_\alpha)}{1+2W_r(v,\omega_\alpha)sI}\mathrm{Im}\left[\frac{W(v,\omega_\alpha)}{1+i(kv-\omega)/\gamma}\right]dv\,d\omega_\alpha}{\int_0^\infty\int_{-\infty}^\infty(\omega_\alpha/\gamma)N(v,\omega_\alpha)\mathrm{Im}\left[\frac{\alpha_0(v,\omega_\alpha)}{1+i(kv-\omega)/\gamma}\right]_{\omega_0}dv\,d\omega_\alpha}\,, \tag{49}$$

$$\frac{(\omega^2-\Omega^2)t_c}{\omega r}=\frac{\int_0^\infty\int_{-\infty}^\infty\frac{(\omega_\alpha/\gamma)N(v,\omega_\alpha)}{1+2W_r(v,\omega_\alpha)sI}\mathrm{Re}\left[\frac{W(v,\omega_\alpha)}{1+i(kv-\omega)/\gamma}\right]dv\,d\omega_\alpha}{\int_0^\infty\int_{-\infty}^\infty(\omega_\alpha/\gamma)N(v,\omega_\alpha)\mathrm{Im}\left[\frac{\alpha_0(v,\omega_\alpha)}{1+i(kv-\omega)/\gamma}\right]_{\omega_0}dv\,d\omega_\alpha}\,, \tag{50}$$

where $r$ is a threshold parameter which references the pump rate to the value it would have at threshold ($sI=0$) for a mode at an arbitrary frequency $\omega_0$ which is characteristic of the transition. Typically, $\omega_0$ would be chosen to be the center frequency, or the frequency of the gain maximum for asymmetric inhomogeneous gain profiles. It is clear from Eq. (43) that with this definition of threshold ($sI=0,\omega=\omega_0$) the continued fraction $W(v,\omega_\alpha)$ reduces to $\alpha_0(v,\omega_\alpha,\omega=\omega_0)$. Equations (49) and (50) are a coupled set which may, in principle, be solved for the mode frequency $\omega$ and intensity $sI$. These equations are the basis for the more specific calculations described in Sec. V, and they could also provide a basis for calculating the gain that would be seen by the higher field harmonics.

At this point it may be of interest to see how these new formulas for intensity and frequency reduce to known ring laser results when one makes the rotating-wave approximation. If one keeps only the leading term in the continued fraction, $W(v,\omega_\alpha)$ in Eq. (43) is replaced by $\alpha_0(v,\omega_\alpha)$, and Eqs. (49) and (50) reduce first to

$$\frac{1}{r}\approx\frac{\int_0^\infty\int_{-\infty}^\infty\frac{(\omega_\alpha/\gamma)N(v,\omega_\alpha)}{1+2\alpha_{0r}(v,\omega_\alpha)sI}\mathrm{Im}\left[\frac{\alpha_0(v,\omega_\alpha)}{1+i(kv-\omega)/\gamma}\right]dv\,d\omega_\alpha}{\int_0^\infty\int_{-\infty}^\infty(\omega_\alpha/\gamma)N(v,\omega_\alpha)\mathrm{Im}\left[\frac{\alpha_0(v,\omega_\alpha)}{1+i(kv-\omega)/\gamma}\right]_{\omega_0}dv\,d\omega_\alpha}\,, \tag{51}$$

$$\frac{(\omega^2-\Omega^2)t_c}{\omega r}\approx\frac{\int_0^\infty\int_{-\infty}^\infty\frac{(\omega_\alpha/\gamma)N(n,\omega_\alpha)}{1+2\alpha_{0r}(v,\omega_\alpha)sI}\mathrm{Re}\left[\frac{\alpha_0(v,\omega_\alpha)}{1+i(kv-\omega)/\gamma}\right]dv\,d\omega_\alpha}{\int_0^\infty\int_{-\infty}^\infty(\omega_\alpha/\gamma)N(v,\omega_\alpha)\mathrm{Im}\left[\frac{\alpha_0(v,\omega_\alpha)}{1+i(kv-\omega)/\gamma}\right]_{\omega_0}dv\,d\omega_\alpha}\,. \tag{52}$$

For optical frequencies that are large compared to the decay rate $\gamma$, the leading $\omega_\alpha$ is approximately equal to the constant $\omega_0$, and the parameter $\alpha_0(v,\omega_\alpha)$ from Eq. (38) reduces to

$$\alpha_0(v,\omega_\alpha)\approx\frac{1/2}{1-i(\omega-\omega_\alpha-kv)/\gamma}\,. \tag{53}$$

The other complex factors in the laser equations reduce according to

$$\frac{1}{1+i(kv-\omega)/\gamma}\approx i\frac{\gamma}{\omega_0}\,. \tag{54}$$

Thus, in the rotating-wave approximation, the laser model described by Eqs. (49) and (50) can be written

$$\frac{1}{r}\approx\frac{\int_0^\infty\int_{-\infty}^\infty\frac{N(v,\omega_\alpha)}{1+[(\omega-\omega_\alpha-kv)/\gamma]^2+sI}dv\,d\omega_\alpha}{\int_0^\infty\int_{-\infty}^\infty\frac{N(v,\omega_\alpha)}{1+[(\omega_0-\omega_\alpha-kv)/\gamma]^2}dv\,d\omega_\alpha}\,, \tag{55}$$

$$\frac{(\omega^2-\Omega^2)t_c}{\omega r} \approx -\frac{\int_0^\infty \int_{-\infty}^\infty \frac{[(\omega-\omega_\alpha-kv)/\gamma]N(v,\omega_\alpha)}{1+[(\omega-\omega_\alpha-kv)/\gamma]^2+sI} dv\, d\omega_\alpha}{\int_0^\infty \int_{-\infty}^\infty \frac{N(v,\omega_\alpha)}{1+[(\omega_0-\omega_\alpha-kv)/\gamma]^2} dv\, d\omega_\alpha}, \quad (56)$$

and these results are the same as the conventional ring laser formulas given previously as Eqs. (30) and (31) in Ref. [14].

The emphasis here has been on calculating the intensity and frequency of the single dominant electromagnetic frequency component in a high-gain laser oscillator. It may be noted that there still may be some higher harmonic content in the material variables, and this content can also be calculated. For example, from Eq. (37) the fundamental polarization component is related to the higher population component by the equation

$$P_{i,1}(v,\omega_\alpha) = -\frac{\mu^2}{\hbar\gamma}\alpha_0(v,\omega_\alpha)[D_0(v,\omega_\alpha)+D_2(v,\omega_\alpha)]E_r \,. \quad (57)$$

But this polarization component has already been solved for in Eq. (42). Eliminating the polarization between Eqs. (42) and (57) leads to

$$-\frac{\mu^2}{\hbar\gamma}W(v,\omega_\alpha)D_0(v,\omega_\alpha)E_r = -\frac{\mu^2}{\hbar\gamma}\alpha_0(v,\omega_\alpha)[D_0(v,\omega_\alpha)+D_2(v,\omega_\alpha)]E_r \,. \quad (58)$$

This equation may readily be solved for $D_2(v,\omega_\alpha)$, and the result is

$$D_2(v,\omega_\alpha) = D_0(v,\omega_\alpha)[W(v,\omega_\alpha)/\alpha_0(v,\omega_\alpha)-1] \,. \quad (59)$$

Thus it is clear from Eq. (43) that this population harmonic is related in a simple way to the secondary terms in the continued fraction. Using this value of $D_2(v,\omega_\alpha)$, the next polarization component $P_{i,3}(v,\omega_\alpha)$ can be obtained from Eq. (39). In a similar way one can obtain all of the harmonic components of the material variables. These components would only be significant when the secondary terms in the continued fraction are substantial.

## V. RESULTS

In principle, Eqs. (49) and (50) may be solved numerically to obtain the intensity and frequency of the fields in a laser oscillator without incorporating the rotating-wave approximation, and the results could be compared with the more approximate formulas given in Eqs. (55) and (56). To be more specific, we will now restrict these formulas to the special case of homogeneous line broadening. In the homogeneous limit the atoms all have the same center frequency $\omega_\alpha=\omega_0$ and velocity $v=0$. Then all terms but the population density may be removed from the integrals in Eqs. (49) and (50), and the intensity and frequency are governed by the simpler formulas

$$\frac{1}{r} = \frac{1}{1+2W_r sI}\,\mathrm{Im}\left[\frac{W}{1-i\omega/\gamma}\right] \Big/ \mathrm{Im}\left[\frac{\alpha_0}{1-i\omega/\gamma}\right]_{\omega_0}, \quad (60)$$

$$\frac{(\omega^2-\Omega^2)t_c}{\omega r} = \frac{1}{1+2W_r sI} \times \mathrm{Re}\left[\frac{W}{1-i\omega/\gamma}\right] \Big/ \mathrm{Im}\left[\frac{\alpha_0}{1-i\omega/\gamma}\right]_{\omega_0}, \quad (61)$$

where from Eq. (43) $W$ is the continued fraction

$$W = \cfrac{\alpha_0}{1+\cfrac{\alpha_0\beta_1 sI}{1+\cfrac{\alpha_1\beta_1 sI}{1+\cfrac{\alpha_1\beta_2 sI}{1+\cdots}}}} \,. \quad (62)$$

From Eqs. (38) and (40) the functions $\alpha_j$ and $\beta_j$ reduce to

$$\alpha_j = \frac{\gamma/2}{-i(2j+1)\omega+i\omega_0+\gamma} + \frac{\gamma/2}{-i(2j+1)\omega-i\omega_0+\gamma}, \quad (63)$$

$$\beta_j = \frac{\gamma_a\gamma_b}{\gamma_a-\gamma_{ab}+\gamma_b} \times \left\{\frac{-i(4j)\omega+\gamma_a-\gamma_{ab}+\gamma_b}{[-i(2j)\omega+\gamma_a][-i(2j)\omega+\gamma_b]}\right\}. \quad (64)$$

For comparison, the homogeneous limit of the rotating-wave approximated model given in Eqs. (55) and (56) is

$$\frac{1}{r} \approx \frac{1}{1+[(\omega-\omega_0)/\gamma]^2+sI}, \quad (65)$$

$$\frac{(\omega^2-\Omega^2)t_c}{\omega r} \approx -\frac{(\omega-\omega_0)/\gamma}{1+[(\omega-\omega_0)/\gamma]^2+sI} \,. \quad (66)$$

The explicit appearance of the threshold parameter and the intensity in the dispersion formula given in Eq. (61) can be eliminated by substitution of Eq. (60), and the result is

$$\frac{(\omega^2-\Omega^2)t_c}{\omega} = \mathrm{Re}\left[\frac{W}{1-i\omega/\gamma}\right] \Big/ \mathrm{Im}\left[\frac{W}{1-i\omega/\gamma}\right]. \quad (67)$$

Similarly, for the rotating-wave approximation, Eqs. (65) and (66) may be combined to obtain

$$\frac{(\omega^2-\Omega^2)t_c}{\omega} \approx -(\omega-\omega_0)/\gamma \,. \quad (68)$$

According to Eq. (68) the oscillation frequency in a homogeneously broadened laser is independent of the threshold parameter, but the more accurate dispersion formula in Eq. (67) shows that the oscillation frequency actually depends on the intensity through the continued fraction $W$.

Computer programs have been written to solve Eqs. (60) and (67). To reduce the number of parameters involved in the solutions, it is assumed that the spontaneous decay rates $\gamma_a$, $\gamma_{ab}$, and $\gamma_b$ are all equal. Figure 1 includes plots of the normalized laser intensity $sI$ as a function of the threshold parameter $r$ for various values of the decay rate ratio $\gamma_a/\gamma$ in a laser tuned to the line center frequency $\omega=\omega_0$ with $\omega_0/\gamma=10$. It is evident from the figure that when the population decay rate $\gamma_a$ is small compared to the coherence decay rate $\gamma$ the intensity becomes close to the rotating-wave approximated value $sI \approx r-1-(\omega-\omega_0)^2/\gamma$ implied by Eq. (65). The reason that the intensity becomes independent of $\gamma_a$ for small $\gamma_a$ is that in this limit $\beta_j$ in Eq. (64) vanishes for the frequencies of interest, and the continued fraction $W$ in Eq. (62) tends to reduce to a single term. The magnitude of $\alpha_j$ in Eq. (63) is at most of the order of unity, and the continued fraction also tends to reduce to the leading term for small values of the intensity. On the other hand, it is clear from the figure that for strong saturation the intensity behavior of a wideband laser may be much more complicated than expected from formulas based on the rotating-wave approximation.

The dispersion factor $(\omega^2-\Omega^2)t_c/\omega$ from Eq. (67) is plotted as a function of the threshold parameter in Fig.2 for the same conditions as Fig. 1. It is clear from these results that the oscillation frequency of a laser depends on the pumping level. With the rotating-wave approximation, on the other hand, the oscillation frequency of a homogeneously broadened laser is independent of the pump rate.

The tuning curves of the laser are also corrected significantly when the rotating-wave approximation is avoided. Figure 3 shows the normalized intensity as a function of the frequency detuning $(\omega-\omega_0)/\gamma$ for two values of the threshold parameter with the line center frequency $\omega_0/\gamma=10$ and the decay rate ratio $\gamma_a/\gamma=1$. The dashed lines in the figure are the corresponding approximate results from Eq. (65). It is evident from the figure that for operation well above threshold the actual tuning curve is displaced from the approximate result. Similarly, the dispersion as a function of frequency detuning is plotted in Fig. 4 and compared to the approximate result from Eq. (68). Again, there is displacement of the actual curves from those predicted by the rotating-wave approximation.

As a last example, we consider an important special case. In some of the widest bandwidth lasers, the spontaneous decay rates are very small compared to both the

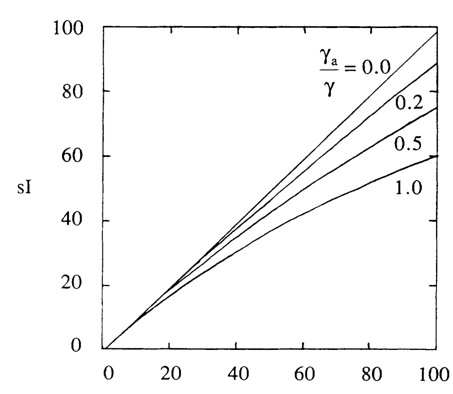


FIG. 1. Normalized laser intensity $sI$ as a function of the threshold parameter $r$ for various values of the decay rate ratio $\gamma_a/\gamma$ in a laser tuned to the line center frequency $\omega=\omega_0$ with $\omega_0/\gamma=10$. For small values of $\gamma_a/\gamma$ the intensity is close to the rotating-wave approximate result $sI \approx r-1-(\omega-\omega_0)^2/\gamma^2$.

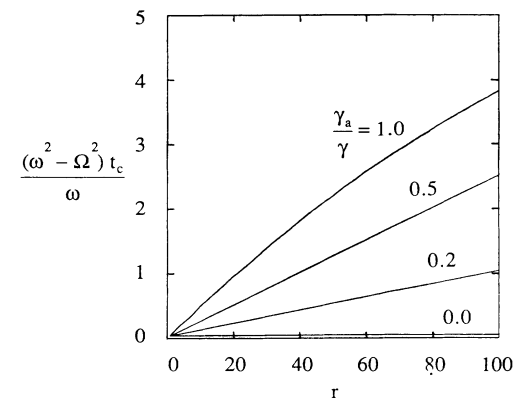


FIG. 2. Dispersion factor $(\omega^2-\Omega^2)t_c/\omega$ as a function of the threshold parameter $r$ for various values of the decay rate ratio $\gamma_a/\gamma$ in a laser tuned to the line center frequency $\omega=\omega_0$ with $\omega_0/\gamma=10$. For small values of $\gamma_a/\gamma$ the dispersion is close to the rotating-wave approximate result $(\omega^2-\Omega^2)t_c/\omega \approx 0$.

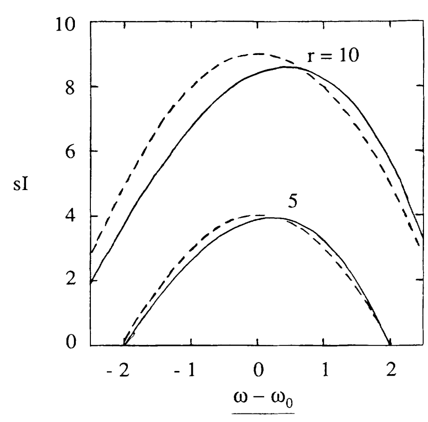


FIG. 3. Normalized intensity $sI$ as a function of frequency detuning $(\omega-\omega_0)/\gamma$ for the line center frequency $\omega_0/\gamma=10$, decay rate ratio $\gamma_a/\gamma=1$, and two values of the threshold parameter. The dashed lines represent the rotating-wave approximate result $sI \approx r-1-(\omega-\omega_0)^2/\gamma^2$. For large values of the threshold parameter, the intensity curve is displaced from the approximation.

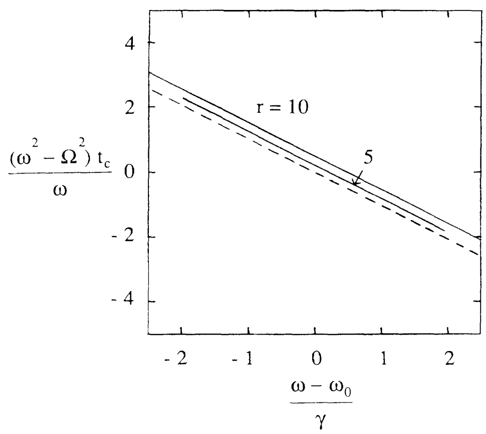


FIG. 4. Dispersion factor $(\omega^2-\Omega^2)t_c/\omega$ as a function of frequency detuning $(\omega-\omega_0)/\gamma$ for the line center frequency $\omega_0/\gamma=10$, decay rate ratio $\gamma_a/\gamma=1$, and two values of the threshold parameter. The dashed lines represent the rotating-wave approximate result $(\omega^2-\Omega^2)t_c/\omega\approx-(\omega-\omega_0)/\gamma$. For large values of the threshold parameter, the dispersion curve is displaced from the approximation.

coherence decay rate and the optical frequency. In this limit only the leading term in the continued fraction is needed, and Eqs. (60) and (67) become

$$\begin{aligned}\frac{1}{r}&=\frac{1}{1+2\alpha_{0r}sI}\text{Im}\left[\frac{\alpha_0}{1-i\omega/\gamma}\right]\Bigg/\text{Im}\left[\frac{\alpha_0}{1-i\omega/\gamma}\right]_{\omega_0}\\&=\frac{1}{1+2\alpha_{0r}sI}\left[\frac{\alpha_{0i}+\alpha_{0r}\omega/\gamma}{1-\omega^2/\gamma^2}\right]\Bigg/\left[\frac{\alpha_{0i}+\alpha_{0r}\omega/\gamma}{1-\omega^2/\gamma^2}\right]_{\omega_0},\end{aligned}\tag{69}$$

$$\begin{aligned}\frac{(\omega^2-\Omega^2)t_c}{\omega}&=\text{Re}\left[\frac{\alpha_0}{1-i\omega/\gamma}\right]\Bigg/\text{Im}\left[\frac{\alpha_0}{1-i\omega/\gamma}\right]\\&=\frac{\alpha_{0r}-\alpha_{0i}\omega/\gamma}{\alpha_{0i}+\alpha_{0r}\omega/\gamma},\end{aligned}\tag{70}$$

where from Eq. (63) $\alpha_0$ is

$$\alpha_0=\frac{1}{2}\left[\frac{1}{1-i(\omega-\omega_0)/\gamma}+\frac{1}{1-i(\omega+\omega_0)/\gamma}\right].\tag{71}$$

Here the intensity dependence is qualitatively the same as for the rotating-wave approximate model given in Eqs. (65) and (68), but the frequency dependence is non-Lorentzian. Also, the frequency of maximum intensity is shifted away from $\omega_0$. One finds that for small values of $\omega/\gamma$ and large values of $r$, the discrepancies between the predictions of the models may amount to several percent.

While the basic model underlying this study includes phenomenological couplings to the nonlasing levels of a laser system, the coherent interactions are assumed to occur only between the lasing states. It has been shown, however, that the rotating-wave approximation is least valid when the optical frequency is not much larger than the coherence decay rate, or in other words when the homogeneous linewidth is almost comparable to the energy-level spacing. Therefore, for quantitative applications it would be necessary that any nonlasing states be well removed from the laser levels. Also, under conditions where substantial field harmonics are generated, there is the further possibility of coupling to energy levels that are very far away from the laser levels. Thus, for lasing in more complicated laser systems, the model discussed here might sometimes be only a qualitative indicator of the laser behavior under conditions where the rotating-wave approximation would fail. For quantitative predictions under these conditions, it would be necessary to include in the model more detailed information about the other laser levels.

## VI. CONCLUSION

Nearly all studies of laser oscillation incorporate a fundamental approximation that is generally known as the rotating-wave approximation. The purpose of this study has been to develop a model which avoids the rotating-wave approximation and can be applied to practical laser oscillators. The model has been developed in detail for the case of a homogeneously broadened ring laser. It has been shown that for wideband highly saturated lasers the intensity and frequency of laser oscillation may differ significantly from the usual approximate results.

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*Present address: Department of Electrical Engineering, Portland State University, P.O. Box 751, Portland, OR 97207-0751.

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