# Stability of three-unit-charge systems

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We discuss the binding of three unit charges $q_i=\pm1, \mp1, \mp1$, with various constituent masses $m_i$. It is found that the most convenient variables are the normalized inverse masses defined by $\alpha_i=m_i^{-1}/(m_1^{-1}+m_2^{-1}+m_3^{-1})$. The region where this three-body system is stable against dissociation into a neutral atom and an isolated charge is restricted to a narrow band that is characterized by small values of $|\alpha_2-\alpha_3|$ and includes the well-known cases $H_2^+$, $e^+e^-e^-$, and $H^-$. Some properties of this region of stability are presented. A method of proof of instability of $e^-pe^+$ is proposed.

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## I. INTRODUCTION

There are many regularities associated with series of molecules obtained by replacing an atom by another one of similar valence structure. These regularities, first observed as empirical rules, can be understood from the universality of the binding mechanisms.

We shall illustrate this property in the simplest case one encounters beyond hydrogenic atoms: three charges $q_i=\pm1, \mp1, \mp1$ bound by their Coulomb interaction. At first glance, one is surprised by the observation that systems as different as $Ps^-=e^+e^-e^-$ [1], $H_2^+=e^-pp$, or $H^-=pe^-e^-$ are stable against dissociation, whereas $p\mu^-e^-$ breaks into $(p\mu^-)+e^-$. A more careful investigation reveals an empirical rule: like charges have to be borne by equal or nearly equal masses, in order to get stability.

This rule applies, for instance, in the case of $p\mu^-e^-$, and the observed instability is not surprising: the muon is tightly bound to the nucleon, thus screening the proton charge for the electron. On the other hand, one can consider the especially simple situation where these two positive charges are both much heavier than the negative charge. In this case, the Born-Oppenheimer approximation applies, with the consequence that increasing the mass of one of the positive charge does not ruin stability. Thus, when reading the empirical rule for stability, "nearly equal in mass" cannot be interpreted as $|m_2-m_3|/(m_2+m_3)$ being small. Indeed, $H_2^+$ isotopes such as $HD^+$ $(e^-dp)$ and $HT^+$ $(e^-tp)$ are stable, though $m_2$ is much heavier than $m_3$.

This consideration suggests that the inverse masses $1/m_i$ are more suitable variables than the masses themselves. It is a property of Coulomb interaction that all masses can be scaled. Therefore the most convenient variables are

$$\alpha_i=\frac{1/m_i}{1/m_1+1/m_2+1/m_3}\ , \tag{1}$$

with the normalization

$$\alpha_1+\alpha_2+\alpha_3=1\ . \tag{2}$$

The masses being "nearly equal" in the above stability rule is then naturally interpreted as $|\alpha_2-\alpha_3|$ being small. For example, $|\alpha_2-\alpha_3|\sim0.000\,27$ for $e^-dp$ and $0.000\,36$ for $e^-tp$. It also follows from the normalization (2) that all possible cases can be conveniently plotted inside an equilateral triangle as shown in Fig. 1.

Inside the triangle, at some of the points, a system of charges $q_1=\pm1$ and $q_2=q_3=\mp1$ is stable against dissociation into a neutral atom and an isolated charge, and, at other points, it is not stable. In this paper, on the basis of nonrelativistic three-body quantum mechanics, we study

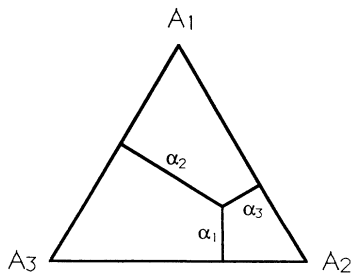


FIG. 1. Domain of possible inverse masses $\alpha_i$, submitted to $\sum_i \alpha_i=1$.

 

the "stability frontier," defined as the lines that separate the "stability domain" from the "instability domains." It turns out that each instability domain is convex, when drawn in the $(\alpha_1,\alpha_2,\alpha_3)$ triangle. The stability domain consists of a narrow band around the symmetry axis $\alpha_2=\alpha_3$, and we shall give a simple bound on the width of this band.

We are aware that there is already a rich literature on the subject. An excellent introduction can be found in the book by Thirring [2] who gives, at the end of Chap. 4 of this reference, a list of difficult problems where the presently investigated problem is number 1. Most papers concern improved calculations of systems with definite masses of interest. Some of the most recent estimates are given in Ref. [3]. There are also some studies where the question of the mass dependence of the binding energy is addressed [4]. In particular, a numerical investigation of the stability frontier was undertaken in Ref. [5].

## II. SUMMARY OF KNOWN RESULTS

In this section, we recall some results which have been established in the literature [3,6]. We consider the Hamiltonian

$$H=\frac{\vec{p}_1^{\,2}}{2m_1}+\frac{\vec{p}_2^{\,2}}{2m_2}+\frac{\vec{p}_3^{\,2}}{2m_3}-\frac{1}{r_{12}}-\frac{1}{r_{13}}+\frac{1}{r_{23}}\ , \tag{3}$$

in appropriate units where $\hbar=c=e^2/(4\pi\epsilon_0)=1$.

A selection of the best-known cases are listed below, where $E_0$ is the ground-state energy, $E_{\rm th}$ the threshold energy, and $g=(E_0-E_{\rm th})/E_{\rm th}$ represents the fraction of extra binding.

(a) $e^+e^-e^-$:

$$m_1=m_2=m_3=1,\quad \alpha_1=\alpha_2=\alpha_3=\tfrac{1}{3}\ ,$$

$$E_{\rm th}=-0.25\ \text{a.u.},\quad E_0=-0.261\,995\,6\ \text{a.u.}\ ,$$

$$g=0.047\,982\ . \tag{4}$$

(b) $e^-pp$:

$$m_1=1,\quad m_2=m_3=1836.15\ ,$$

$$\alpha_1=0.998\,912,\quad \alpha_2=\alpha_3=\frac{1-\alpha_1}{2}\ ,$$

$$E_{\rm th}=-0.499\,73\ \text{a.u.},\quad E_0=-0.597\,15\ \text{a.u.}\ ,$$

$$g=0.194\,95\ . \tag{5}$$

(c) $\mu^-pp$:

$$m_1=206.77,\quad m_2=m_3=1836.15\ ,$$

$$\alpha_1=0.8162,\quad \alpha_2=\alpha_3=\frac{1-\alpha_1}{2}\ ,$$

$$E_{\rm th}=-2528.5\ \text{eV},\quad E_0=-2782\ \text{eV},\quad g=0.100\ . \tag{6}$$

(d) $\mu^-dd$:

$$m_1=206.77,\quad m_2=m_3=3670.48\ ,$$

$$\alpha_1=0.8988,\quad \alpha_2=\alpha_3=\frac{1-\alpha_1}{2}\ ,$$

$$E_{\rm th}=-2663\ \text{eV},\quad E_0=-2988\ \text{eV},\quad g=0.122\ . \tag{7}$$

(e) $\mu^+e^-e^-$:

$$m_1=206.77,\quad m_2=m_3=1\ ,$$

$$\alpha_1=0.002\,412,\quad \alpha_2=\alpha_3=\frac{1-\alpha_1}{2}\ ,$$

$$E_{\rm th}=-0.497\,593\ \text{a.u.},\quad E_0=-0.525\,055\ \text{a.u.}\ ,$$

$$g=0.055\,19\ . \tag{8}$$

(f) $pe^-e^-$:

$$m_1=1836.15,\quad m_2=m_3=1\ ,$$

$$\alpha_1=0.000\,272\,23,\quad \alpha_2=\alpha_3=\frac{1-\alpha_1}{2}\ ,$$

$$E_{\rm th}=-0.499\,73\ \text{a.u.},\quad E_0=-0.527\,35\ \text{a.u.}\ ,$$

$$g=0.0553\ . \tag{9}$$

(g) $p_\infty e^-e^-$:

$$m_1=\infty,\quad m_2=m_3=1\ ,$$

$$\alpha_1=0,\quad \alpha_2=\alpha_3=\tfrac{1}{2}\ ,$$

$$E_{\rm th}=-0.5\ \text{a.u.},\quad E_0=-0.527\,751\ \text{a.u.}\ ,$$

$$g=0.0555\ . \tag{10}$$

(h) $\mu^-dt$:

$$m_1=206.77,\quad m_2=5496.9,\quad m_3=3670.48\ ,$$

$$\alpha_1=0.9141,\quad \alpha_2=0.0344,\quad \alpha_3=0.0515\ ,$$

$$E_{\rm th}=-2711.2\ \text{eV},\quad E_0=-3029.3\ \text{eV},\quad g=0.117\ . \tag{11}$$

Some comments and further remarks are in order.

(i) Along the symmetry axis $\alpha_2=\alpha_3$, the binding energy, or the fraction of additional binding, is a smooth function of $\alpha_1$, as it should be. This is illustrated in Fig. 2. As demonstrated by Hill [7], all symmetric systems with $m_2=m_3$ are stable against dissociation. Near $\alpha_1=1$, there is a square-root behavior $g\simeq A-B\sqrt{1-\alpha_1}$. The coefficients can be computed from an

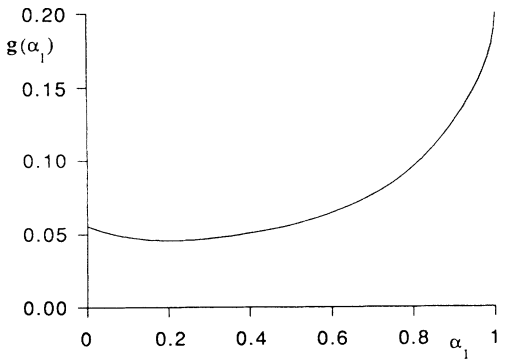


FIG. 2. Fraction of additional binding for symmetric states ($m_2=m_3$), as a function of $\alpha_1$, the inverse mass of the first particle.

harmonic approximation to the Born-Oppenheimer potential near its minimum, with the result $A=0.205\,27$, $B=0.313$.

(ii) The Born-Oppenheimer limit has been investigated in great detail, for obvious reasons. In particular, more results on muonic molecules are available [8]. We have seen in Eq. (5) that $e^-pp$ is amply bound. This deteriorates as $m_3$ becomes smaller, and we believe that $e^-pe^+$ is unbound [9,10]. The critical value for stability, $(m_3)_c$, occurs somewhere between 1 and 1836, if $m_1=1$ fixes the scale. From Rotenberg and Stein [11], $(m_3)_c<2.20$, while from Armour and Schrader [12], $(m_3)_c>1.51$, for $m_2=\infty$.

(iii) Miscellaneous cases. We just mentioned that $e^-pe^+$ is not stable. This result was generalized to finite values of $m_3$, down to $e^-\mu^+e^+$ being unstable [9,10,13]. The system $\mu^-e^+p$ is also unstable [13].

## III. GENERAL PROPERTIES OF THE STABILITY FRONTIER

From the existing investigations listed in the preceding section, we suspect that the stability domain consists of a band around the symmetry axis of the $\{\alpha_i\}$ triangle. The following rigorous properties can be established.

*(i) One crosses the frontier at most once when going along a straight line from a lower corner of the triangle to any point on the symmetry axis.* This property, which is called "star-shape behavior" by mathematicians, is illustrated in Fig. 3. The proof is very simple. If $(m_1,m_2,m_3)$ is stable, with say, $m_2>m_3$, then when $m_3$ increases with $m_1$ and $m_2$ unchanged, the three-body energy $E_0(m_1,m_2,m_3)$ decreases while the threshold energy $E_{\rm th}(m_1,m_2)$ remains constant, so the stability of the system is improved. In terms of the rescaled variables $\alpha_i$, this means that one keeps stability when one moves to the right with $\alpha_1/\alpha_2$ fixed.

*(ii) Each half of the domain of instability is convex.* Assume that $\mathbf{M}'=(\alpha_1',\alpha_2',\alpha_3')$, with $\sum_i\alpha_i'=1$, and $\mathbf{M}''=(\alpha_1'',\alpha_2'',\alpha_3'')$ $(\sum_i\alpha_i''=1)$ both lie on the stability frontier. Then the sets of unscaled inverse masses

$$\mathbf{N}'\left[x_i'=\frac{\alpha_i'}{\alpha_1'+\alpha_2'}\right],\quad \mathbf{N}''\left[x_i''=\frac{\alpha_i''}{\alpha_1''+\alpha_2''}\right] \tag{12}$$

also correspond to the edge between stability and instability. Since these points have the same $x_1'+x_2'$ $=x_1''+x_2''=1$, the threshold energy remains constant for any intermediate point $\mathbf{N}(\lambda)=\lambda\mathbf{N}'+(1-\lambda)\mathbf{N}''$. The Hamiltonian depends linearly on $\lambda$ when one goes from $\mathbf{N}'$ to $\mathbf{N}''$; thus, from a general theorem [2], $E_0(\lambda)\geq\lambda E_0(1)+(1-\lambda)E_0(0)$. If $\mathbf{N}'(\lambda=1)$ and $\mathbf{N}''(\lambda=0)$ both lie on the stability frontier, $\mathbf{M}(\lambda)$ is outside the stability domain. Now, the inverse transformation $\mathbf{N}\to\mathbf{M}$ that rescales the inverse masses $x_i$ into $\alpha_i=x_i/(x_1+x_2+x_3)$ is a conic projection of the space $\mathbb{R}^3$ into the plane $x_1+x_2+x_3=1$. It preserves straight lines, i.e., $\mathbf{M}(\lambda)$ runs on the segment $\mathbf{M}'\mathbf{M}''$. Since $\mathbf{M}(\lambda)$ is in the instability region, the convexity is proved.

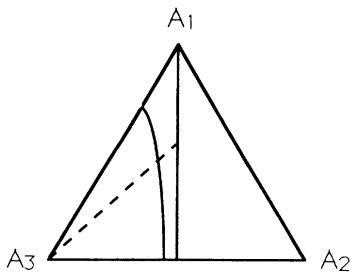


FIG. 3. A straight line crosses the stability frontier at most once between $A_3$ and any point on the symmetry axis.

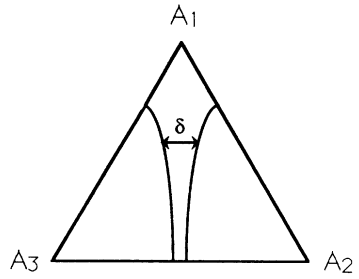


FIG. 4. Topology of the stability and instability domains, from rigorous results and the knowledge that every symmetric system $(m_2=m_3)$ is stable.

*A priori,* from these rigorous results, the stability region could consist either of a connected band in the middle, or of two separated domains, with a hole of instability near the center of the triangle. The latter scenario is excluded by the proof by Hill [7] that every symmetric system $(m_2=m_3)$ is stable. We are thus left with the topology depicted in Fig. 4. In the following sections, we shall study three properties of the stability frontier: (i) its intersection with the legs of the triangle, i.e., where the Born-Oppenheimer approximation is valid, (ii) its intersection with the bottom side, i.e., the asymmetric hydrogen ion $pe'^-e^-$ with $m_3\neq m_2$, and (iii) the width of the band as a function of the vertical coordinate $\alpha_1$.

## IV. THE BORN-OPPENHEIMER APPROXIMATION

Let us consider in this section the region near the upper angle $A_1$ of the triangle. The Born-Oppenheimer method can be used to compute the binding energy when the masses $m_2$ and $m_3$ are large. However, if both $m_2$ and $m_3$ are finite, the Born-Oppenheimer method is not very useful to calculate the stability frontier. Indeed, the threshold is given by

$$E_{\rm th}(m_1,m_2)=-\frac{m_1m_2}{2(m_1+m_2)}\,, \tag{13}$$

if one assumes $m_2\geq m_3$, whereas the Born-Oppenheimer potential has an asymptotic value which corresponds to an electron bound to one of the positive charges and reads

$$V_{\rm BO}(r_{23}=\infty)=-\frac{m_1}{2} \tag{14}$$

or, at best,

$$V_{\mathrm{BO}}(r_{23}=\infty)=-\frac{m_1(m_2+m_3)}{2(m_1+m_2+m_3)}\ , \tag{15}$$

if one uses Jacobi variables to remove the center-of-mass motion. Thus enforcing binding in the Born-Oppenheimer approximation would require more than stability.

We thus restrict ourselves now to $m_2=\infty$, in order to investigate the point where the curve of Fig. 4 reaches the left-hand side of the triangle. We provisionally release the normalization $\sum_i \alpha_i=1$ and fix $m_1=1$, so that we are dealing with the Hamiltonian

$$H=\frac{\vec{p}_3^{\,2}}{2m_3}+\frac{1}{r_3}+\left[\frac{\vec{p}_1^{\,2}}{2}-\frac{1}{r_1}-\frac{1}{r_{13}}\right]\ . \tag{16}$$

Replacing the bracket by its lowest eigenvalue $\varepsilon(r_3)$ leads to the "extreme" Born-Oppenheimer approximation, which corresponds to the one-body operator

$$H_E=\frac{\vec{p}_3^{\,2}}{2m_3}+\frac{1}{r_3}+\varepsilon(r_3)\ , \tag{17}$$

and overbinds the system.

The value of $\varepsilon(r)$ is available in the literature [6]. A simple computation shows that the effective potential $1/r+\varepsilon(r)$ leads to binding down to values of $m_3$ as low as $m_3\simeq 1.4$ (we integrate the radial equation at energy $E=-0.5$ and see whether the wave function exhibits nodes). However, we knew already that the critical value $(m_3)_c$ is larger than 1, since $e^-pe^+$ is believed to be unbound. Armour and Schrader [12] have even shown that $(m_3)_c\geq 1.51$.

The best upper limit, to our knowledge, seems to be $(m_3)_c\leq 2.20$ in Ref. [11]. In an attempt to improve this bound, we have used the so-called "variational Born-Oppenheimer" approximation. It consists of the trial wave function

$$\Psi_{\mathrm{var}}=f(\vec{r}_3)\varphi(\vec{r}_1;\vec{r}_3)\ , \tag{18}$$

where $\varphi$ is the lowest solution of the two-center problem for particle 1. This results in the modified one-body equation

$$H_{\mathrm{VBO}}f\equiv\left[\frac{\vec{p}_3^{\,2}}{2m_3}+\frac{1}{r_3}+\varepsilon(r_3)+\frac{1}{m_3}g_{00}(r_3)\right]f=E_{\mathrm{VBO}}f\ . \tag{19}$$

The correction $g_{00}$ has been tabulated in the literature and already used by several authors [6], for values of $m_3$ corresponding to actual nuclei. We have solved Eq. (19) with various values of $m_3$. Stability requires rather large $m_3$, at least in the simplest version of the variational Born-Oppenheimer approximation. Improvements might consist of replacing Eq. (18) by an expansion on the electron eigenstates, resulting in coupled equations for the nuclear motion, instead of the single equation (19). For the moment, we are left with the result [11,12] $1.51<(m_3)_c<2.20$, or, in terms of normalized inverse masses, $0.60<(\alpha_1)_c<0.69$.

## V. THE ASYMMETRIC HYDROGEN-ION LIMIT

We know that $H^-=p_\infty e^-e^-$ is stable. Variational calculations, and the argument presented in Sec. VI, show that an "asymmetric" ion $p_\infty e'e$ remains stable as long as $m_2\lesssim 1.1$. Glaser *et al.* [14] have shown that stability is lost for $m_2\gtrsim 1.57$. Their proof can be extended to the case of finite $m_1$ as follows.

To return to a situation similar to the infinitely heavy case ($m_1=\infty$) without losing the advantage of the extra kinetic energy, we write the operator inequality

$$\frac{\vec{p}^{\,2}}{2m}-\frac{\lambda}{|\vec{r}-\vec{r}_0|}+\frac{\lambda^2 m}{2}\geq 0\ , \tag{20}$$

valid for all $\vec{r}_0$ and for all $\lambda\geq 0$, whose usefulness arose from discussions with Basdevant. This inequality follows in an obvious way from the Schrödinger equation for one particle in a Coulomb potential. For a two-body system in Coulomb interaction, it leads, after suitable optimization, to the correct energy proportional to the reduced mass. For the three-body Hamiltonian (3), we get the lower bound

$$H\geq\frac{\lambda}{|\vec{r}_1-\vec{r}_2|}-\frac{\lambda^2 m_1}{2}+\frac{\vec{p}_2^{\,2}}{2m_2}+\frac{\vec{p}_3^{\,2}}{2m_3}-\frac{1}{|\vec{r}_1-\vec{r}_2|}-\frac{1}{|\vec{r}_1-\vec{r}_3|}+\frac{1}{|\vec{r}_2-\vec{r}_3|}\ . \tag{21}$$

The optimal $\lambda$ is obtained by requiring that, when we switch off all terms involving particle 3, we get a lower bound which is exactly the binding energy of the $\{1,2\}$ system. This choice is

$$\lambda=\frac{m_2}{m_1+m_2} \tag{22}$$

and we get, setting $\vec{r}_1=0$ (since it is from now on an irrelevant parameter),

$$H\geq\tilde{H}\equiv\frac{\vec{p}_2^{\,2}}{2m_2}+\frac{\vec{p}_3^{\,2}}{2m_3}-\frac{m_2^2 m_1}{2(m_1+m_2)^2}-\frac{m_1}{m_1+m_2}\frac{1}{r_2}-\frac{1}{r_3}+\frac{1}{r_{23}}\ . \tag{23}$$

We can now use the projection method of Ref. [14], as generalized in the Appendix, and we find that there is no binding if

$$\frac{m_1 m_2}{m_3(m_1+m_2)}>1.57,\quad m_3<\frac{3}{4}m_2\left[\frac{m_1}{m_1+m_2}\right]^2\ . \tag{24}$$

Such a condition indicates, for instance, that the system proton-antiproton-muon (neglecting the strong interaction) is not bound. However, we are disappointed from the observation that the extension to finite values of $m_1$ does not provide any useful information. Indeed, one can draw a straight line from the bound already obtained for

$m_1=\infty$ to the bound on the left-hand side of the triangle, corresponding to $m_2$ infinite, and one can exclude any point on the left of this line. One easily checks that this provides a more severe constraint than Eq. (24).

## VI. WIDTH OF THE STABILITY BAND

Let us now comment on the width $\delta$ of the stability band. If the normalization (2) is imposed, the coordinates in the triangle plot are $x=(\alpha_2-\alpha_3)/\sqrt{3}$ and $y=\alpha_1$, so that

$$\delta(\alpha_1)=\frac{2}{\sqrt{3}}|\alpha_2-\alpha_3|_{\text{frontier}} . \tag{25}$$

One can get a simple lower limit on $\delta$ once one knows the binding energy on the symmetry axis accurately. We rewrite the Hamiltonian as

$$\begin{aligned} H(\alpha_1,\alpha_2,\alpha_3)&=H_S+H_A \\ &=\left[\alpha_1\frac{\vec{p}_1^{\,2}}{2}+\frac{\alpha_2+\alpha_3}{4}(\vec{p}_2^{\,2}+\vec{p}_3^{\,2})\right. \\ &\quad\left.-\frac{1}{r_{12}}-\frac{1}{r_{13}}+\frac{1}{r_{23}}\right] \\ &\quad+\left[\frac{\alpha_2-\alpha_3}{4}(\vec{p}_2^{\,2}-\vec{p}_3^{\,2})\right] , \end{aligned} \tag{26}$$

so that the terms which are symmetric under $2\leftrightarrow 3$ exchange are separated from the antisymmetric one. Let $\Psi_S$ be the ground state of $H_S$, which is symmetric. From the variational principle, one obtains

$$\begin{aligned} E_0(\alpha_1,\alpha_2,\alpha_3)&\le\langle\Psi_S|H(\alpha_1,\alpha_2,\alpha_3)|\Psi_S\rangle \\ &=E_0[\alpha_1,\tfrac{1}{2}(\alpha_2+\alpha_3),\tfrac{1}{2}(\alpha_2+\alpha_3)] . \end{aligned} \tag{27}$$

Thus, if

$$E_0[\alpha_1,\tfrac{1}{2}(\alpha_2+\alpha_3),\tfrac{1}{2}(\alpha_2+\alpha_3)]=E_{\text{th}}(\alpha_1)[1+g(\alpha_1)] , \tag{28}$$

we obtain after a little algebra using the explicit expression of $E_{\text{th}}$ that

$$\delta\ge\frac{2}{\sqrt{3}}\frac{g(\alpha_1)}{1+g(\alpha_1)}(1+\alpha_1) . \tag{29}$$

The lower limits on $\delta/2$ are shown in Fig. 5. For $\alpha_1=0$, it corresponds to a mass ratio $m_2/m_3=1.111$. We believe that it is very close to the exact half-width for small values of $\alpha_1$, because the antisymmetric term $H_A$ in Eq. (26) is small and enters at second order only. For larger $\alpha_1$, the width is clearly underestimated by this lower bound: one can see in Fig. 5 that it saturates the width of the triangle only for $\alpha_1\gtrsim 0.82$, while the exact value is between $\alpha_1=0.60$ and $\alpha_1=0.69$, as seen in Sec. IV.

## VII. SUMMARY AND OUTLOOK

In this paper, we have tried to unify the existing information on the stability of systems made of three unit charges. The detailed calculations involve different techniques: Born-Oppenheimer type for $ppe^-$, Helium-atom type for $pe^-e^-$, etc. However, simple patterns emerge: stability requires that the particles with the same charge have nearly equal inverse masses; the stability frontier, in the triangle of normalized inverse masses, is made of two symmetric curves which are convex; the width of the stability band, delimited by the two frontier curves, seems to slightly decrease and then increase regularly as the inverse mass of the first particle increases.

Our findings are summarized in Fig. 6. The dashed curve reproduces the minimal width already shown in Fig. 5 and the solid curve is our guess at the actual frontier. We hope that the present note will stimulate further mathematical and numerical work on these three-body systems.

There are immediate applications. For instance, $\mu^-dp$ and other systems of interest for muon catalyzed fusion [8] are stable. There has been also some discussions on the level of rigor of the proof of the instability of $e^-pe^+$ when the proton mass becomes finite [9,10,13,15]. If one draws a line on our triangle plot, starting very conservatively from the upper limit of Glaser *et al.* [14] ($m_1=\infty$, $m_2=1.57$, and $m_3=1$, converted into normalized $\alpha_i$'s) and ending on the left-hand side of the triangle ($m_1=1$,

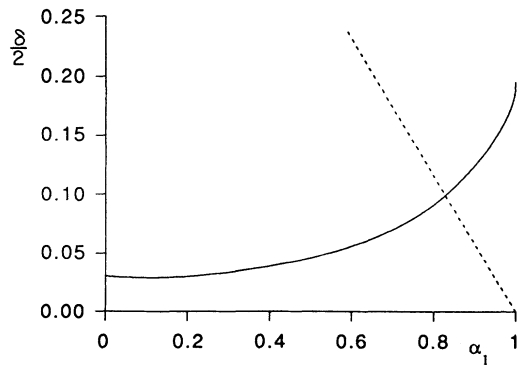


FIG. 5. Lower limit on the half-width $\delta/2$ of the stability band, as a function of $\alpha_1$. Also shown is the half-width of the triangle of the normalized inverse masses.

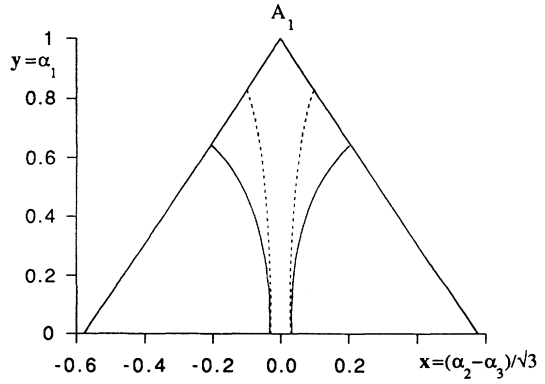


FIG. 6. Shape of the stability border (solid line) vs the minimal width derived from the binding energy on the symmetry axis.

$m_2=\infty$, and $m_3=1.5$ if the result of Armour and Schrader [12] can be considered as rigorous, i.e., if one forgets uncertainties in the numerical calculation of integrals), one easily proves that $e^-pe^+$ is unstable for at least $m_p \gtrsim 4.2 m_e$.

A challenging extension would consist of the study of the hydrogen-molecule-like configurations $q_i=\pm 1,\pm 1, \mp 1, \mp 1$ with arbitrary masses. One can easily show, for instance, that the stability of the positronium molecule $e^+e^+e^-e^-$ [16] against dissociation into two positronia implies the stability of the asymmetric configurations $M^+M^+m^-m^-$. We are sure that many more results await to be revealed. Understanding why these systems, for some mass distributions, prefer a collective binding rather than a splitting into two subsystems is a central issue in few-body physics: we know that there is more nuclear binding in the $\alpha$ particle than in two deuteron atoms, while, in hadron physics, stable multiquarks $q\,q\,\bar{q}\,\bar{q}$ have so far escaped detection.

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## APPENDIX

We derive here a sufficient condition for the absence of binding of the hydrogen-ion-like Hamiltonian corresponding to a fixed center $\vec{r}_1=\mathbf{0}$,

$$\tilde{H}=\frac{\vec{p}_2^{\,2}}{2m_2}+\frac{\vec{p}_3^{\,2}}{2m_3}-\frac{q_2}{r_2}-\frac{q_3}{r_3}+\frac{1}{r_{23}}\ , \tag{A1}$$

with $q_2$, $q_3>0$, and, for instance, $m_3q_3^2<m_2q_2^2$. We follow very closely Ref. [14]. The wave function is written as

$$\Psi(\vec{r}_2,\vec{r}_3)=\sum_{n=1}^{\infty}\xi_n(\vec{r}_3)\varphi_n(\vec{r}_2)\ , \tag{A2}$$

where the $\varphi_n$'s are eigenstates of the Hamiltonian

$$h=\frac{\vec{p}_2^{\,2}}{2m_2}-\frac{q_2}{r_2}\ , \tag{A3}$$

with eigenenergies

$$\epsilon_{n=1}=-\frac{m_2q_2^2}{2},\quad \epsilon_{n>1}\geq -\frac{m_2q_2^2}{8}\ . \tag{A4}$$

We adopt the normalization

$$\|\varphi_n\|_2^2\equiv\int|\varphi_n(\vec{x})|^2d^3x=1,\quad \sum_n\|\xi_n\|_2^2=1\ . \tag{A5}$$

When calculating the expectation value of $\tilde{H}$, we use Schwarz's inequality for the term in $r_{23}^{-1}$,

$$\int\frac{|\Psi(\vec{r}_2,\vec{r}_3)|^2}{r_{23}}d^3r_2>\frac{\left[\int\Psi(\vec{r}_2,\vec{r}_3)\varphi_1(\vec{r}_2)d^3r_2\right]^2}{\int|\varphi_1(\vec{r}_2)|^2r_{23}d^3\vec{r}_2}$$
$$\equiv|\xi_1(\vec{r}_3)|^2W(r_3)\ . \tag{A6}$$

This defines $W(r_3)$ and leads to a lower bound on the energy of the ground state,

$$E>-\frac{m_2q_2^2}{2}\|\xi_1\|_2^2+\int\xi_1^*(r_3)\left[\frac{\vec{p}_3^{\,2}}{2m_3}-\frac{q_3}{r_3}+W(r_3)\right]\times\xi_1(r_3)d^3r_3$$
$$+(1-\|\xi_1\|_2^2)\left[-\frac{m_3q_3^2}{2}-\frac{m_2q_2^2}{8}\right]\ . \tag{A7}$$

Therefore we get

$$E>\inf\left\{\left[-\frac{m_2q_2^2}{2}+\inf\left[\frac{\vec{p}_3^{\,2}}{2m_3}-\frac{q_3}{r_3}+W(r_3)\right]\right],\left[-\frac{m_3q_3^2}{2}-\frac{m_2q_2^2}{8}\right]\right\}. \tag{A8}$$

By rescaling the calculations of Ref. [14], we find that

$$\inf\left[\frac{\vec{p}_3^{\,2}}{2m_3}-\frac{q_3}{r_3}+W(r_3)\right]\geq 0 \tag{A9}$$

if

$$q_3\leq 1$$

and

$$\frac{m_2q_2}{m_3q_3}>1.57\ , \tag{A10}$$

and so we conclude that there is no binding if

$$q_3\leq 1\ ,\quad \frac{m_2q_2}{m_3q_3}>1.57,\quad m_3q_3^2<\tfrac{3}{4}m_2q_2^2\ . \tag{A11}$$

[1] This system, first predicted by J. A. Wheeler, Ann. N.Y. Acad. Sci. **48**, 219 (1946), has actually been observed by A. P. Mills, Phys. Rev. Lett. **46**, 717 (1981).

[2] W. Thirring, *A Course in Mathematical Physics* (Springer-Verlag, Berlin, 1981), Vol. 3.

[3] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms*, 2nd ed. (Plenum, New York, 1977); W. Kolos, C. C. J. Roothaan, and R. A. Sack, Rev. Mod. Phys. **32**, 178 (1960); C. L. Pekeiris, Phys. Rev. **126**, 1470 (1962); Phys. Rev. A **12**, 493 (1975).

[4] A. K. Bhatia and R. J. Drachman, Phys. Rev. A **35**, 4051 (1987); Z. S. Popović and F. R. Vukajlović, *ibid.* **36**, 1936 (1987); A. V. Gur'yanov and T. K. Rebane, Zh. Eksp. Teor. Fiz. **98**, 17 (1990) [Sov. Phys. JETP **71**, 34 (1990)].

[5] R. D. Poshusta, J. Phys. B **18**, 1887 (1985).

[6] S. Cohen, D. L. Judd, and R. J. Riddell, University of California, Lawrence Radiation Laboratory Report No. UCRL-8802, June 1959 (unpublished); S. Cohen, J. R. Hiskes, and R. J. Riddell, Phys. Rev. **119**, 1025 (1960); H. Wind, J. Chem. Phys. **42**, 2371 (1965); **43**, 2956 (1965); W.

Kolos, Acta Phys. Acad. Sci. Hung. **27**, 241 (1969); C. L. Beckel, B. D. Hausen, and J. M. Peek, J. Chem. Phys. **53**, 3681 (1970); M. C. Struensee, J. S. Cohen, and R. T. Pack, Phys. Rev. A **34**, 3605 (1986); A. C. Fonseca and A. Peña, *ibid.* **38**, 4967 (1988).

[7] R. N. Hill, J. Math. Phys. **18**, 2316 (1977).

[8] See for instance, L. Bracci and G. Fiorentini, Phys. Rep. **86**, 169 (1982).

[9] I. Aronson, J. Kleinman, and L. Spruch, Phys. Rev. A **4**, 841 (1971); **12**, 349 (1975).

[10] E. A. G. Armour, J. Phys. B **11**, 2803 (1978).

[11] M. Rotenberg and J. Stein, Phys. Rev. **182**, 1 (1969).

[12] E. A. G. Armour and D. M. Schrader, Can. J. Phys. **60**, 581 (1982).

[13] E. A. G. Armour, J. Phys. B **16**, 1295 (1983); Z. Chen and L. Spruch, Phys. Rev. A **42**, 133 (1990).

[14] V. Glaser, H. Grosse, A. Martin and W. Thirring, in *Mathematical Problems in Theoretical Physics,* Proceedings of the International Conference on Mathematical Physics, Lausanne, 1979, edited by K. Osterwalder, Lectures Notes in Physics Vol. 116 (Springer-Verlag, Berlin, 1980).

[15] R. N. Hill, in *Mathematical Problems in Theoretical Physics* (Ref. [14]).

[16] E. A. Hylleraas and A. Ore, Phys. Rev. **71**, 493 (1947).