Reply to "Comments on the amplification of intrinsic fluctuations by chaotic dynamics"

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We believe that the criticisms of Nicolis and Balakrishnan [preceding Comment, Phys. Rev. A **46**, 3569 (1992)] reflect a misunderstanding of the basis of our claims. Here, we repeat a number of points already made in our papers [Phys. Rev. A **43**, 1709 (1991); **42**, 1946 (1990); Phys. Rev. Lett. **64**, 249 (1990)] in order to dispel ambiguity and misunderstanding.

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Nicolis and Balakrishnan [1] have criticized our suggestions (i) that chaotic dynamics can amplify spontaneous fluctuations so that the contraction of a master equation description into a macrovariable description is invalid; and (ii) that a nonlinear Fokker-Planck equation of the sort suggested by a theorem of T. G. Kurtz may be used to accurately describe these large-scale fluctuations. These criticisms seem to reflect a misunderstanding, repeated throughout the Comment, of the basis of our claims. We find this surprising since, in Secs. I–III of Ref. [2], these and related issues are addressed rather thoroughly.

A key point that Nicolis and Balakrishnan seem to have missed is that the magnitude of the correlation of the intrinsic fluctuations is not a free quantity but is uniquely determined by the underlying physics in each specific case. Thus, while the limits noted in Eqs. (1)–(3) of their Comment are, indeed, true for ϵ approaching zero, ϵ is not zero for intrinsic fluctuations. It is well known through numerical simulations by us [2] and others [3–5] that the invariant densities in the deterministic case ($\epsilon=0$) and the stochastic case ($\epsilon\neq 0$) are distinct both for discrete maps and for stochastic differential equations. The degree of difference between the two densities depends on the magnitude of ϵ and the dynamics of the chaotic attractor.

Nicolis and Balakrishnan suggest that we incorrectly identify the macrovariables with the mean values rather than with the most probable values, and that "macroscopic behavior is generally not associated with the mean. . . , but rather with the most probable values." In fact, the most probable values of the variables do not satisfy the usual system of autonomous macrovariable equations when the distribution is broad. They do so only asymptotically in the limit of small fluctuations, in which case the Gaussian form of the conditional probability density implies that the mean and the most probable values are the same. When the macrovariable equations possess a chaotic attractor, conditional fluctuations are small for only a brief period of time, after which neither the mean nor the most probable values satisfy the autonomous macrovariable equations. Nonetheless, our simulations, which employ a stochastic modification of the macrovariable equations, make it clear that the macrovariable equations provide the skeleton that supports the flesh of the molecular fluctuations.

Our views about the utility of the nonlinear Fokker-Planck equation are based on two theorems by Kurtz [6,7]. Kurtz's theorems justify various approximations to sample paths of the master equation in the thermodynamic limit. Kurtz's "first" theorem justifies the use of a linear process with a Gaussian conditional probability density for finite times. For that case the most probable and mean values are identical. As we have shown [8], this approximation has a linearly divergent covariance when the mean has a stable limit cycle and an exponentially divergent covariance when the mean behaves chaotically. In the chaotic case the "finite" time interval, which is not otherwise determined by Kurtz's theorem, is of the order of the reciprocal of the largest Liapunov exponent.

For limit cycles and for chaotic attractors, Kurtz's "second" theorem provides a much improved approximation in the following senses: (i) the conditional covariance no longer diverges, despite the fact that the fluctuations may grow to the size of the attractor; and (ii) in the mathematical limit of very small noise, simulations of the probability density derived from the corresponding nonlinear Langevin equation converge to the invariant measure of the deterministic attractor [2]. In this approximation the nonlinearities succeed in saturating the growth of the fluctuations. While Kurtz's second theorem also has been proved only for finite times, it is obviously a much improved approximation to the thermodynamic limit and would appear to work for times much longer than the reciprocal of the largest Liapunov exponent.

Our confidence in the Fokker-Planck equation (or equivalent nonlinear Langevin equation) that is given by Kurtz's theorem is reinforced by recent work of Mareschal and De Wit [5]. Using a direct simulation of the Boltzmann equation, these authors have simulated bifurcation to a limit cycle in a chemical reaction and "find agreement between the microscopic simulation results and a Langevin description . . . below and beyond the bifurcation point [5]." Three types of Langevin description were used in this work to obtain the asympotic, long-time statistical distribution, including that given by Kurtz's second theorem. All three gave good agreement with the Boltzmann simulations.

In our work, we have used Kurtz's second theorem to obtain an approximate representation of a master equation by a nonlinear Fokker-Planck equation [2,9,10]. This Fokker-Planck equation can be equivalently rendered as a Langevin equation. The Langevin equation happens to be a stochastic version of the macrovariable equation with a well-defined noise correlation determined from this correspondence. Unlike external noise with an arbitrary noise strength, the intrinsic noise is uniquely determined by the underlying physics. In our investigation of the effect of this noise we have considered several classes of examples: two heuristic ones, namely, the Rossler equations [2] and a "master" map for the logistic equation [9], as well as hydrodynamic fluctuations for the Lorenz equations [11], Johnson noise for the Josephson junction [2], spontaneous emission for multimode lasers [10], and chemical noise for mass action kinetics [8]. For each of the examples with a physico-chemical origin, the nonlinear Fokker-Planck equation comes directly from an underlying master equation [12].

We know of no examples in the literature, including those cited by Nicolis and Balakrishnan, which demonstrate that the nonlinear Fokker-Planck equation suggested by Kurtz's second theorem "at best ... may give reasonable results [for] ... a single point attractor although even in this case examples are known for which it can fail badly." While it is true that attempts have been made to formulate other Fokker-Planck equations using completely macroscopic ideas [13], in the thermodynamic limit, even for linear systems, they disagree with the underlying master equation results as one departs from equilibrium (cf. Ref. [12], pp. 173–175).

On the other hand, we agree with Nicolis and Balakrishnan that the growth of molecular fluctuations is "yet another manifestation of the sensitivity to initial conditions of chaotic dynamics." The point, however, is that even in the absence of external noise and uncertainty in the initial conditions, molecular fluctuations destroy the utility of predictions based on the macroscopic kinetic equations within a time of the order of the reciprocal of the largest Liapunov exponent. Furthermore, if the noise is sufficiently large, as seems to be true for the driven Josephson junction [2], the invariant distribution on the chaotic attractor may be modified significantly. Even in cases like the Lorenz system, for which fluctuations have a hydrodynamic origin and are small, one finds an amplification of fluctuations of approximately two orders of magnitude above the values for thermal equilibrium [11]. Such increases in the noise level like that seen near critical points, should be detectable by experiment.

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