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COMMENTS

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Comments on the amplification of intrinsic fluctuations by chaotic dynamics

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The effect of intrinsic fluctuations in systems exhibiting deterministic chaos is analyzed. It is suggested that the conclusions of the recent series of papers by Fox and Keizer [Phys. Rev. Lett. **64**, 249 (1990); Phys. Rev. A **41**, 2969 (1990); **42**, 1946 (1990); **43**, 1709 (1991)] on the breakdown of the deterministic description need to be reassessed.

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In a recent series of papers [1–4] Fox and Keizer have considered the effects of intrinsic (thermodynamic, molecular-level) fluctuations in systems exhibiting deterministic chaos. Their main conclusions are that, in the chaotic regime, "the macrovariable picture breaks down," the macrovariables themselves "cease to be meaningful variables," and "the breakdown of the autonomous macrovariable equations associated with large-scale intrinsic fluctuations forces a reassessment of the meaning of chaos in real physical systems." We feel that these drastic conclusions are unjustified, and give below some arguments aimed at clarifying ideas in this important issue.

To get a clear picture of the issues involved, it is helpful to have a brief summary of the approach of Fox and Keizer [1–4]. Consider a set of deterministic equations for the macrovariables of a dynamical system. Since molecular-level intrinsic fluctuations are always present, the "exact" behavior of the system is presumably described by some underlying master equation at the microscopic (or at least mesoscopic) level. Under normal circumstances, when the relative fluctuations scale like $\epsilon^{1/2}$ (where ϵ is the inverse of the system size N, a very small quantity indeed for a macroscopic system), a standard expansion procedure leads to a Fokker-Planck equation (FPE) for the linearized fluctuations about the mean values of the variables (which are equal, up to corrections of order $\epsilon^{1/2}$, to the deterministic or "macroscopic" values). Now suppose the deterministic equations have a chaotic attractor, signaled by a positive Lyapunov exponent in the Jacobi matrix. Fox and Keizer correctly point out that as the same matrix governs the dynamics of the linearized fluctuations in the FPE, the latter also diverge exponentially. From there on, however, it is claimed that the deterministic equations themselves (the zeroth-order approximation in this procedure) become invalid, since the fluctuation-induced corrections are comparable to the values predicted by these equations. The assertion is thus that while the deterministic equations can serve as guides that help *detect* chaos, they are no longer meaningful in the chaotic regime; one must necessarily go back to the original master equation.

Having come to this conclusion, Fox and Keizer [4] propose the employment of an alternative reduction of the master equation as a possible means of handling the problem. Invoking certain theorems of Kurtz [5], they claim that a Fokker-Planck equation with nonlinear drift *and* diffusion coefficients, frequently referred to as "nonlinear Fokker-Planck equation" (NFPE), Eq. (22) of Ref. [4], is the correct way of dealing with the large intrinsic fluctuations in the presence of chaos. While the mean value of a variable calculated from the FPE differs from the exact mean value (obtained in principle from the master equation) by $O(\epsilon^{1/2})$, the corresponding difference in the case of the NFPE is $O(\epsilon \ln 1/\epsilon)$. This circumstance, it is claimed, allows one "not only to handle the large intrinsic fluctuations," but to do so "with even greater accuracy." Numerical simulations are done for the examples of the Rössler model and the Josephson-junction equation (or the periodically forced, damped, planar pendulum) with additive Gaussian white noise. The changes observed (relative to the noise-free case) in the shape of the attractor and the invariant distribution on it are then claimed to support the above conclusions.

In what follows, we point out that there are flaws in both aspects of the foregoing: the supposed breakdown of the macrovariable description, as well as the use of the NFPE as the way out of the difficulty. Consider a given set of macrovariable evolution equations with a chaotic attractor. An essential point to realize in connection with deterministic chaos is that, because of the sensitivity to initial conditions, long-term predictions can *only* be carried out in a probabilistic sense—even before the inclusion of the effects of noise, intrinsic or external. To formulate this point properly it is useful to consider the asymptotic (time-independent) properties of the system

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rather than the transient (time-dependent) behavior. Let $\rho(\mathbf{x})$ be the invariant distribution on the attractor in the absence of fluctuations, where $\mathbf{x}$ denotes the state vector. A distinct attribute of chaos is that such a distribution may indeed exist, and be sufficiently smooth—whereas, for nonchaotic systems, it is bound to be singular. Next, let $\rho_\epsilon(\mathbf{x})$ be the invariant distribution of the system when intrinsic fluctuations are incorporated (ϵ^{-1} being the system size as before). Typically, $\rho_\epsilon(\mathbf{x})$ will be a smooth function even in the nonchaotic regime. A question of obvious relevance is whether $\rho_\epsilon(\mathbf{x})$ is close to $\rho(\mathbf{x})$. The general answer is not known, but some *rigorous* partial results are available. Specifically, for discrete dynamical systems of the form

$$\mathbf{x}_{n+1}=\mathbf{f}(\mathbf{x}_n)+\epsilon\xi_n \ , \tag{1}$$

where $\mathbf{f}$: $R^d \to R^d$, ξ is an uncorrelated stationary stochastic process, and $\epsilon \ll 1$, it can be shown [6] that

$$\lim_{\epsilon\to 0}\|\rho_\epsilon(\mathbf{x})-\rho(\mathbf{x})\|=0 \ , \tag{2}$$

provided that the Frobenius-Perron operator of the deterministic (unperturbed) system $\mathbf{x}_{n+1}=\mathbf{f}(\mathbf{x}_n)$ exists. Here $\| \ \|$ denotes the L^1 norm, $\|u\|=\int_{R^d}|u(\mathbf{x})|\mu(d\mathbf{x})$,$\mu$ being the measure. An even stronger result can be established for hyperbolic attractors in which periodic orbits and fixed points are dense (axiom *A* attractors [7]). Here, because of the smoothing action of the noise, the continuity property of the invariant measure of the attractor (in the absence of noise) along the unstable direction is preserved in the presence of noise, a property reflected by the fact that

$$\lim_{\epsilon\to 0}\rho_\epsilon(\mathbf{x})=\rho(\mathbf{x}) \ . \tag{3}$$

In short, under the effect of small fluctuations the system will visit the phase space with probabilities close to the ones corresponding to the deterministic attractor, provided that the above-mentioned conditions are satisfied. Therefore deterministic chaos fully retains its relevance and its implications in the presence of noise, in that it determines (up to a small correction) the probabilistic structure of the system in the limit of long times. Interestingly, in this perspective it is on *non*chaotic systems that fluctuations seem to have the most drastic effect, since the perturbation of the (now) singular distribution of the deterministic system by a noise, however small, leads to a smooth distribution. An intriguing manifestation of this phenomenon is the noise-induced asymptotic periodicity in the Keener map [8].

A brief comment on the way the invariant measure is attained in the course of time may be in order. Consider first the case of $\epsilon=0$ (fluctuations are discarded). The master equation reduces then to a first-order partial differential equation [Ref. [4], Eq. (10)]. If the initial values of $\mathbf{x}$ are specified exactly $\rho(\mathbf{x},0)=\delta(\mathbf{x}-\mathbf{m}_0)$, then the solution to this equation is simply [Ref. [4], Eq. (11)],

$$\rho(\mathbf{x},t)=\delta(\mathbf{x}-\mathbf{m}(t)) \ , \tag{4}$$

where $\mathbf{m}(t)$ are given by the deterministic equations. This singular distribution generally does not approach the invariant measure at $t\to\infty$. A smooth initial distribution, however, broadens as t evolves and approaches the correct invariant measure if the latter is unique. An interesting case is that of a (unique) limit cycle or of a (unique) chaotic attractor, where $\lim_{t\to\infty}\rho(\mathbf{x},t)$ is a *broad* distribution having the entire attractor as its support.

The inclusion of fluctuation ($\epsilon\neq 0$) will modify the detailed form of this distribution, but under the conditions discussed above the qualitative characteristics (a broad distribution centered on the attractor) will subsist. As far as the validity of the macrovariable equations is concerned, the important point is not ρ or ρ_ϵ being broad or not, but rather *ρ_ϵ being close to ρ or not.* We note, incidentally, that in the above examples (a) the mean value of the distribution is unrepresentative since it is close to the unstable state from which the attractor has bifurcated and (b) the maxima of the probability distribution occur on the attractor. This illustrates the fact that macroscopic behavior is generally *not* associated with the mean (as claimed by Fox and Keizer), but rather with the most probable value. The latter can be quite different from the mean, notably if the system does not possess a single point attractor.

An interesting situation arises when a system possesses two simultaneously stable steady states disposed symmetrically around an intermediate unstable state. In the absence of fluctuations ($\epsilon=0$) the invariant measure is not unique. At finite ϵ a unique invariant measure is recovered if the limit of $\rho_\epsilon(\mathbf{x},t)$ for $t\to\infty$ is taken before the limit of small ϵ. One finds an invariant distribution in the form of two sharp peaks of equal weight centered on the two stable states. Again, for such a distribution the mean is unrepresentative. The physically relevant states are the most probable ones, and are completely different from the mean.

Next we turn to the claim that the NFPE provides a formulation capable of handling large fluctuations even when the standard reduction of the master equation to the FPE breaks down. The NFPE has in general state-dependent drift *and* diffusion coefficients, and is obtained by using the limit theorems of Kurtz [5]. However, it is important to note that in their general form these theorems hold good for possibly very long, but *finite* times. The crucial $t\to\infty$ limit can *only* be taken in particular circumstances such as for instance, the presence of a single point attractor. Equally significant is the fact, long recognized in the literature [9–11] that an equation such as the NFPE is in general *incompatible* with the master equation. At best, it may give reasonable results when the system possesses a single point attractor, although even in this case examples are known for which it can fail badly [12]. The conclusions based on the analysis of Ref. [4], Eqs. (21)–(31), using the NFPE as the correct equation in the case of a chaotic attractor, are therefore unjustified.

In the light of the foregoing comments, how should one understand the exponential growth of fluctuations and their effect on the Rössler and forced pendulum attractors reported in Refs. [1–4]? We believe that the former phenomenon is nothing but yet another manifestation of the sensitivity to initial conditions of chaotic dynamics, as a result of which individual trajectories lose their significance beyond the Lyapunov time. (Actually, they do so already as a result of external noise and/or numerical round-off errors, independent of thermodynamic fluctuations.) In contrast, statistical properties will as a rule remain robust, at least as long as the invariant measure $\rho(\mathbf{x})$ in the absence of noise has good regularity properties.

Finally, as regards the apparently strong effect of even very small stochastic forcings on the attractors mentioned above [4], we are of the opinion that this mainly reflects the sensitive dependence of these attractors on the *parameter* values [13,14] in the regions explored, and on the fact that these attractors need not be hyperbolic everywhere. For instance, at $\mu > 4.23$ the Rössler attractor is known to be in the regime of reverse bifurcations in which chaotic bands successively merge as μ is increased. Given the fact that noise in general tends to smooth out the measure of the deterministic attractor [7] it is not surprising that the reverse bifurcation at $\mu = 4.3$ is anticipated at $\mu = 4.23$ itself once the noise increases beyond a small but nonzero threshold, and band merging occurs [15]. Similar comments apply, naturally, to the invariant distribution on the (perturbed) attractor. In this regard, it may also be worth stressing that any attractor (of a continuous flow) of dimension larger than zero possesses a phase-like variable, along which the Lyapunov exponent is zero. Phase-like variables are known [16] to be very sensitive to fluctuations even in the *absence* of chaos—for instance, as pointed out already above, the phase variable of a uniform limit cycle exhibits macroscopic fluctuations.

To sum up, deterministic chaos in a system does not imply, by any means, the breakdown of the macroscopic picture itself owing to the amplification of intrinsic fluctuations. The deterministic system of equations continues to describe the behavior of the most probable (or typical) values of the variables and, above all, to generate the system's attractor. However, as is well known and widely accepted, individual phase trajectories lose their significance beyond the Lyapunov time, being supplanted by the properties of the attractor (the basic structure of which is determined by the macrovariable system of equations) and the probability distribution on it in the presence of noise (of whatever origin, intrinsic or external).

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