Unified expression for Fermi and Bose distributions

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Some theorems relating the Dirac δ function to both Fermi and Bose distributions are presented in this paper. Several potential applications of these theorems are discussed in detail. In particular, an inverse problem for determining the density of states of Fermi systems is solved with a closed-form expression.

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I. EXPRESSION OF THE δ FUNCTION

It is well known that almost every expression of the δ function has important applications in physical science. Our theorem can be expressed as

$$\sum_{m=0}^{\infty}(-1)^m\frac{(\pi kT)^{2m}}{(2m+1)!}\frac{\partial^{2m+1}}{\partial E_F^{2m+1}}F_T^F(E,E_F)=\delta(E-E_F)\ ,\tag{1}$$

where $F_T^F(E,E_F)$ is the Fermi distribution [1],

$$F_T^F(E,E_F)=\frac{1}{1+e^{(E-E_F)/kT}}\ .\tag{2}$$

In fact, Eq. (1) is equivalent to the following expression:

$$A(x,y)=\sum_{m=0}^{\infty}\frac{(-1)^m\pi^{2m}}{(2m+1)!}\frac{\partial^{2m+1}}{\partial y^{2m+1}}(1+e^{x-y})^{-1}$$
$$=\delta(x-y)\ .\tag{3}$$

Proof:
(I) $y<x$:

$$A(x,y)=\sum_{m=0}^{\infty}\frac{(-1)^m\pi^{2m}}{(2m+1)!}\frac{\partial^{2m+1}}{\partial y^{2m+1}}\sum_{n=1}^{\infty}(-1)^{n+1}e^{-n(x-y)}$$
$$=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-n(x-y)}}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(n\pi)^{2m+1}}{(2m+1)!}$$
$$=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-n(x-y)}}{\pi}\sin n\pi=0\ .\tag{4}$$

(II) $y>x$:

$$A(x,y)=\sum_{m=0}^{\infty}\frac{(-1)^m\pi^{2m}}{(2m+1)!}\frac{\partial^{2m+1}}{\partial y^{2m+1}}\sum_{n=0}^{\infty}(-1)^n e^{n(x-y)}$$
$$=\sum_{n=1}^{\infty}(-1)^n\frac{e^{n(x-y)}}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(n\pi)^{2m+1}}{(2m+1)!}$$
$$=\sum_{n=1}^{\infty}(-1)^n\frac{e^{n(x-y)}}{\pi}\sin n\pi=0\ .\tag{5}$$

(II) $y\in(-\infty,\infty)$:

$$\int_{-\infty}^{\infty}A(x,y)dy$$
$$=\sum_{m=0}^{\infty}\frac{(-1)^m\pi^{2m}}{(2m+1)!}\int_{-\infty}^{\infty}\frac{\partial^{2m+1}}{\partial y^{2m+1}}(1+e^{x-y})^{-1}dy$$
$$=\sum_{m=0}^{\infty}\frac{-1)^m\pi^{2m}}{(2m+1)!}\frac{\partial^{2m}}{\partial y^{2m}}(1+e^{x-y})^{-1}\Big|_{y=-\infty}^{y=\infty}$$
$$=\frac{1}{1+e^{x-y}}\Big|_{y=-\infty}^{y=\infty}=1+0=1\ .\tag{6}$$

From (4)–(6), we can conclude that

$$A(x,y)=\delta(x-y)\ .\tag{7}$$

Therefore, Eq. (1) is proved. Equation (1) is a new relation between the Dirac δ function and the Fermi-Dirac distribution.

II. AN INVERSE PROBLEM FOR THE DENSITY OF STATES OF FERMI SYSTEMS

Now we give an example to apply Eq. (1) to a Fermi system. According to Eq. (1), the temperature-dependent density of states near the Fermi level $g(E_F,T)$ can be expressed as

$$g(E_F,T)=\int_{-\infty}^{\infty}dE\,g(E,T)\sum_{m=0}^{\infty}(-1)^m\frac{(\pi kT)^{2m}}{(2m+1)!}\frac{\partial^{2m+1}}{\partial E_F^{2m+1}}F_T^F(E,E_F)$$
$$=\sum_{m=0}^{\infty}(-1)^m\frac{(\pi kT)^{2m}}{(2m+1)!}\frac{\partial^{2m+1}}{\partial E_F^{2m+1}}\int_{-\infty}^{\infty}dE\,g(E,T)F_T^F(E,E_F)$$
$$=\sum_{m=0}^{\infty}(-1)^m\frac{(\pi kT)^{2m}}{(2m+1)!}\frac{\partial^{2m+1}}{\partial E_F^{2m+1}}n(E_F,T)\ ,\tag{8}$$

where $n(E_F,T)$ represents the carrier density, i.e.,

$$n(E_F,T)=\int_{-\infty}^{\infty} dE\, g(E,T)\frac{1}{(1+e^{(E-E_F)/kT})} . \quad (9)$$

Therefore, Eq. (8) is a closed-form solution of the integral equation (9). In other words, one can determine the density of states of a fermion system based on the measurable carrier density and Fermi level. In fact, the first three approximate solutions can be expressed as

$$g_0(E_F,T)=\frac{\partial n(E_F,T)}{\partial E_F} , \quad (10)$$

$$g_1(E_F,T)=\frac{\partial n(E_F,T)}{\partial E_F}-\frac{\pi^2}{6}(kT)^2\frac{\partial^3 n(E_F,T)}{\partial E_F^3} , \quad (11)$$

and

$$g_2(E_F,T)=\frac{\partial n(E_F,T)}{\partial E_F}-\frac{\pi^2}{6}(kT)^2\frac{\partial^3 n(E_F,T)}{\partial E_F^3}+\frac{\pi^4}{120}(kT)^4\frac{\partial^5 n(E_F,T)}{\partial E_F^5} . \quad (12)$$

In principle, the above expressions is available for metals controlled by doping impurities. Although only the first approximation [Eq. (11)] is taken into account, the result is much better than previous work [2]. This method might also be important to improve the solution for different physical systems, such as to determine the density of states in a space-charge-limited current case [3–5]. It should be mentioned that the author has discussed another case for intrinsic semiconductors based on Möbius transform [6].

III. CONCLUSIONS AND DISCUSSION

It can be expected that these relations (1) and (8) might be useful for different physical problems, such as for the semiconductor or nuclear systems. Notice that all the differentiations in Eq. (8) are taken at the original Fermi level of the system, so one only needs the data near the initial Fermi level, and one can only obtain the density of states near the Fermi level in practice. It is not a complete solution for the integrand equation (9) since essentially it is an ill-posed equation; however, what we have obtained is the most important information one needs.

It should also be indicated that Eqs. (1) and (8) can be modified for application to the Bose system with variable chemical potential $\mu(n,T)$, which obeys the distribution as

$$F_T^B(E,E_F)=\frac{1}{1-e^{(E-E_F)/kT}} , \quad (13)$$

or some degenerate Fermi system which obeys

$$F_T^{FG}(E,E_F)=\frac{1}{1+\lambda e^{(E-E_F)/kT}} . \quad (14)$$

The corresponding expressions should be

$$\sum_{m=0}^{\infty}(-1)^m\frac{(\pi kT)^{2m}}{(2m+1)!}\frac{\partial^{2m+1}}{\partial E_F^{2m+1}}\frac{1}{1-e^{(E-E_F)/kT}}=\delta(E-E_F) \quad (15)$$

and

$$\sum_{m=0}^{\infty}(-1)^m\frac{(\pi kT)^{2m}}{(2m+1)!}\frac{\partial^{2m+1}}{\partial E_F^{2m+1}}\frac{1}{1+\lambda e^{(E-E_F)/kT}}=\delta(E+kT\ln\lambda-E_F) , \quad (16)$$

respectively.

All the above expressions have important applications to some fundamental problems in statistical physics.

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[1] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, London, 1958).
[2] G. P. Cerofolini, Bull. C. Sci. **66**, 499 (1980).
[3] F. Stöckmann, Phys. Status Solidi A **64**, 475 (1981).
[4] R. L. Weisfield, J. Appl. Phys. **54**, 6401 (1983).
[5] J. Sworakowski and S. Nespurek, J. Appl. Phys. **65**, 1559 (1989).
[6] N. X. Chen and G. B. Ren, Phys. Lett. A **160**, 319 (1991).