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Hamilton-Jacobi treatment of second-class constraints

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A singular system with primary and secondary constraints of the second kind is investigated by two different methods, and exact agreement is observed.

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I. INTRODUCTION

The Lagrangian formulation of classical physics requires the configuration space formed by n generalized coordinates q_i and generalized velocities $\dot{q}_i$. The passage from the Lagrangian approach to the Hamiltonian approach is achieved by introducing the generalized momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i}\ , \qquad i = 1, ..., n\ . \tag{1}$$

However, a valid phase space is formed if the rank of the Hessian matrix

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ , \qquad j = 1, ..., n \tag{2}$$

is n. Systems which have this property are called regular and their treatments are found in a standard mechanics book. Systems which have the rank less than n are called singular systems.

The well-known method to investigate the Hamiltonian formulation of singular systems was initiated by Dirac [1,2]. In this formulation one defines the total Hamiltonian as

$$H_T = H_0 + v_\mu H'_\mu\ ; \qquad \mu\ = n - p + 1, ..., n \tag{3}$$

where H_0 being the usual Hamiltonian, defined as

$$H_0 = -L + p_i \dot{q}_i, \qquad i = 1, ..., n$$

and v_μ are unknown coefficients. The primary constraints

$$H'_\mu(q, p) \approx 0\ , \tag{4}$$

are obtained using the singular nature of the Lagrangian. (The Einstein summation rule has been used throughout this work.)

The Poisson bracket (PB) of two functions $f(q_i, p_i)$ and $g(q_i, p_i)$ is defined as

$$\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\ . \tag{5}$$

Time variation of any function g, defined in the phase space, is given as,

$$\dot{g} = \{g, H_T\} = \{g, H_0\} + v_\mu\{g, H'_\mu\}\ . \tag{6}$$

The consistency conditions, which mean that the total time derivative of primary constraints should be zero, are given as

$$\dot{H}'_\mu = \{H'_\mu, H_0\} + v_\nu\{H'_\mu, H'_\nu\} \approx 0\ . \tag{7}$$

These equations may be identically satisfied with the help of the primary constraints, or they lead to new relations that are called the secondary constraints. Besides, primary and secondary constraints are divided into two types according to the PB relations: First-class constraints that have vanishing PB's with all other constraints and second-class constraints that have nonvanishing PB's.

Güler [3,4] has recently developed a completely different method to investigate singular systems. He started with the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \qquad i, j = 1, ..., n \tag{8}$$

of rank $(n - p)$. Then p of the momenta are dependent. The equivalent-Lagrangian method [5–7] is used to obtain the set of Hamilton-Jacobi partial-differential equations (HJPDE). The generalized momenta corresponding to generalized coordinates q_i are defined as:

$$p_a = \frac{\partial L}{\partial \dot{q}_a}\ ; \qquad a = 1, ..., n - p \tag{9}$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}. \tag{10}$$

Since the rank of the Hessian matrix is $(n - p)$ one may solve (9) for $\dot{q}_a$ as functions of $t, q_i, \dot{q}_\mu$ and p_a as

$$\dot{q}_a = W_a(t, q_i, \dot{q}_\mu, p_a). \tag{11}$$

Now substituting (11) in (10) one has

$$p_\mu = -H_\mu(t, q_i, \dot{q}_\nu,\ p_a), \qquad \nu = n - p + 1, ..., n \tag{12}$$

The Hamiltonian H_0 is defined as

$$H_0 = -L(t, q_i, \dot{q}_\nu, W_a) + p_a W_a - \dot{q}_\mu H_\mu. \tag{13}$$

Relabeling the coordinates t as t_0 and q_μ as t_μ then the set of HJPDE is expressed as

$$H'_\alpha\left(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_\beta}\right) = 0, \quad \alpha,\beta = 0, n-p+1, ..., n$$
$$a = 1,2,...,n-p, \qquad (14)$$

where

$$H'_\alpha = H_\alpha(t_\beta, q_a, p_a) + p_\alpha. \qquad (15)$$

The equations of motion are written as total differential equations in many variables as follows:

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a}dt_\alpha, \qquad dp_a = -\frac{\partial H'_\alpha}{\partial q_a}dt_\alpha, \qquad (16)$$

$$dz = \left(-H_\alpha + p_a\frac{\partial H'_\alpha}{\partial p_a}\right)dt_\alpha, \qquad (17)$$

where $z = S(t_\alpha, q_a)$. The linear operators X_α corresponding to the set (16) are

$$X_\alpha f(t_\beta, q_a, p_a) = \frac{\partial f}{\partial q_a}\frac{\partial H'_\alpha}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial H'_\alpha}{\partial q_a} + \frac{\partial f}{\partial t_\alpha}. \qquad (18)$$

The system is integrable if the bracket relations

$$(X_\alpha, X_\beta)f = (X_\alpha X_\beta - X_\beta X_\alpha)f = C^\gamma_{\alpha\beta}X_\gamma; \ \forall\alpha,\beta,$$
$$\gamma = 0, n-p+1, ..., n \qquad (19)$$

hold. If the relations (19) are not satisfied identically, one may add bracket relations which cannot be expressible in this form as new operators. So the number of independent operators are increased, and a new complete system can be obtained. Then the new operators can be written in the Jacobi form, and one may find the corresponding integrable system of total differential equations.

II. AN EXAMPLE OF A SECOND-CLASS CONSTRAINT

A. The Güler's method

Let us consider the Lagrangian

$$L = \tfrac{1}{2}a_1\dot{q}_1^2 - \tfrac{1}{2}a_2(\dot{q}_2^2 - 2\dot{q}_2\dot{q}_3 + \dot{q}_3^2) + b\dot{q}_2 - c \qquad (20)$$

which was considered in Ref. [3]. Here a_1, a_2, b, c are functions of q_1, q_2, q_3, and t. The generalized momenta read as

$$p_1 = a_1\dot{q}_1, \quad p_2 = a_2(\dot{q}_3 - \dot{q}_2) + b, \quad p_3 = a_2(\dot{q}_2 - \dot{q}_3). \qquad (21)$$

Since the rank of the Hessian matrix is two one of the momenta is depending on the others. Thus, we have

$$\dot{q}_1 = \frac{p_1}{a_1} = w_1, \quad \dot{q}_3 = \dot{q}_2 - \frac{p_3}{a_2} = w_3,$$
$$p_2 = -p_3 + b = -H_2. \qquad (22)$$

The Hamiltonian H_0 is defined as

$$H_0 = -L(q_1, q_2, q_3, \dot{q}_2, w_1, w_3) + p_1w_1 + p_3w_3 + (-p_3 + b)\dot{q}_2 \qquad (23)$$

or

$$H_0 = \frac{1}{2}\left(\frac{p_1^2}{a_1} - \frac{p_3^2}{a_2}\right) + c. \qquad (24)$$

To simplify the problem let us specify the functions a_1, a_2, b, c as

$$a_1 = 1, \quad a_2 = \tfrac{1}{2}, \quad b = q_1 + q_3, \quad c = q_1 + q_2 + q_3^2.$$

Thus, we have

$$H'_0 = p_0 + \tfrac{1}{2}(p_1^2 - 2p_3^2) + q_1 + q_2 + q_3^2 = 0, \qquad (25)$$
$$H'_2 = p_2 + p_3 - q_1 - q_3 = 0. \qquad (26)$$

These calculations lead us to the following equations:

$$dq_1 = p_1dt, \qquad dq_3 = -2p_3dt + dq_2,$$
$$dp_1 = -dt + dq_2, \qquad dp_3 = -2q_3dt + dq_2. \qquad (27)$$

The linear operators defined in (18) read as

$$X_0f = p_1\frac{\partial f}{\partial q_1} - 2p_3\frac{\partial f}{\partial q_3} - \frac{\partial f}{\partial p_1} - 2q_3\frac{\partial f}{\partial p_3} + \frac{\partial f}{\partial t},$$
$$X_2f = \frac{\partial f}{\partial q_3} + \frac{\partial f}{\partial p_1} + \frac{\partial f}{\partial p_3} + \frac{\partial f}{\partial q_2}, \qquad (28)$$

where $f = f(q_1, q_2, q_3, p_1, p_3, t)$. Since the bracket relation

$$(X_0, X_2)f = 2\frac{\partial f}{\partial q_3} - \frac{\partial f}{\partial q_1} + 2\frac{\partial f}{\partial p_3} \qquad (29)$$

cannot be expressible as a linear combination of X_0 and X_2, we define a new operator X_1 as

$$X_1f = -2\left(\frac{\partial f}{\partial p_3} + \frac{\partial f}{\partial q_3}\right) + \frac{\partial f}{\partial q_1} = -(X_0, X_2)f. \qquad (30)$$

So, the extended system is composed of the linear operators X_1, X_0, X_2. To test for the integrability conditions we should check the following bracket relations of the operators X_1, X_0, X_2 :

$$(X_0, X_1)f = -4\left(\frac{\partial f}{\partial p_3} + \frac{\partial f}{\partial q_3}\right), \qquad (31)$$

$$(X_1, X_2)f = 0. \qquad (32)$$

Again, one should notice that the linear operators (31) cannot be expressible as a linear combination of X_1, X_0, X_2. Thus, we add the new operator

$$X_3f = \frac{\partial f}{\partial p_3} + \frac{\partial f}{\partial q_3} = -\frac{1}{4}(X_0, X_1). \qquad (33)$$

Then, the linearly independent operators of the extended system are

$$X_0 f = (2p_3 - 2q_3)\frac{\partial f}{\partial p_3} - \frac{\partial f}{\partial p_1} + \frac{\partial f}{\partial t},$$
$$X_1 f = \frac{\partial f}{\partial q_1},$$
$$X_2 f = \frac{\partial f}{\partial p_1} + \frac{\partial f}{\partial q_2}, \quad (34)$$
$$X_3 f = \frac{\partial f}{\partial p_3} + \frac{\partial f}{\partial q_3}.$$

Since all the bracket relations vanish, the system is integrable.

Now, the problem is reduced to get the solutions of the extended system. The total differential equations corresponding to the system (34) are obtained as

$$dp_1 = -dt + dq_2, \quad (35)$$
$$dp_3 = (2p_3 - 2q_3)dt + dq_3. \quad (36)$$

Obviously, the solution of Eq. (35) is

$$p_1 = -t + q_2 + C_1. \quad (37)$$

Expressing Eq. (36) as

$$\frac{d(p_3 - q_3)}{(p_3 - q_3)} = 2dt \quad (38)$$

and integrating it, we have

$$p_3 = q_3 + A\exp(2t), \quad (39)$$

where C_1 and A are arbitrary constants. Using Eq. (26) also one has

$$p_2 = q_1 - A\exp(2t). \quad (40)$$

Coordinates q_1 and q_3 are determined inserting solutions (37) and (39) in (27). Hence,

$$\dot{q}_1 = p_1 = -t + q_2 + C_1, \quad (41)$$
$$\dot{q}_3 = -2p_3 + \dot{q}_2 = -2q_3 - 2A\exp(2t) + \dot{q}_2. \quad (42)$$

It is only by chance that Eq. (35) is in Eq. (27) and Eq. (36) follows simply by subtraction in (27). Since the set (27) is not integrable there is no way to solve this set. Thus, one should extend the system to get an integrable system of equations. In other words, the intermediate work is necessary to solve the problem.

B. The Dirac method

The same problem can also be solved by the Dirac method. The primary constraint (26) leads us to the total Hamiltonian

$$H_T = \tfrac{1}{2}(p_1^2 - 2p_3^2) + q_1 + q_2 + q_3^2 + vH_2'. \quad (43)$$

The consistency condition (7) gives the secondary constraint as

$$\dot{H}_2' = \{H_2', H_T\} = -p_1 + 2p_3 - 2q_3 - 1 = H_1' \approx 0. \quad (44)$$

Imposing the condition

$$\dot{H}_1' = 0 \quad (45)$$

we arrive at the result

$$v = -(4q_3 - 4p_3 - 1). \quad (46)$$

Thus, the equations of motion are

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{q}_2 &= -4q_3 + 4p_3 + 1 = v, \\ \dot{q}_3 &= 6p_3 + 4p_2 - 4q_1 - 8q_3 + 1, \\ \dot{p}_1 &= -4q_3 + 4p_3, \\ \dot{p}_2 &= -1, \\ \dot{p}_3 &= -10q_3 + 4p_2 + 8p_3 - 4q_1 + 1. \end{aligned} \quad (47)$$

Making use of the primary constraint (26) and the secondary constraint (44) we obtain the general solutions of Eqs. (47) as

$$q_1 = A\exp(2t) - t, \quad (48)$$
$$q_2 = 2A\exp(2t) + t + C_2, \quad (49)$$
$$q_3 = B\exp(-2t) + \tfrac{1}{2}A\exp(2t) + \tfrac{1}{2}, \quad (50)$$
$$p_1 = 2A\exp(2t) - 1, \quad (51)$$
$$p_2 = -t, \quad (52)$$
$$p_3 = B\exp(-2t) + \tfrac{3}{2}A\exp(2t) + \tfrac{1}{2}, \quad (53)$$

where B and C_2 are arbitrary constants.

III. CONCLUSION

A singular system with primary and secondary constraints of the second kind is investigated by two different methods. It is observed that solutions are in complete agreement. In fact, substituting (49) in (41) and (42) one obtains solutions (48) and (50). Solutions (51), (52), and (53) are obtained from (37), (39), and (40) in the same manner.

The Güler's method predicts the phase-space coordinates q_1, q_3, p_1, p_2, p_3 in terms of q_2. In other words the integrability conditions are imposed as additional equations. It seems that the equations of motion (47) of the Dirac method predicts six coordinates though, since there are only five linearly independent equations of (47), one can only determine five of the coordinates in terms of q_2. Besides, the parameter v of the Dirac method corresponds to $\dot{q}_2$.

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