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Analytic solutions of Bloch and Maxwell-Bloch equations in the case of arbitrary field amplitude and phase modulation

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The matrix exponent method which we applied recently for the solution of the Bloch equations [Sov. Phys. JETP **62**, 1125 (1985)] allows us to obtain in the present paper the Bloch-equation solutions when the amplitude and the phase of the exciting field are arbitrary functions of time. The solution's validity conditions are derived. It is shown that one can also apply the matrix exponent method to the solution of the Maxwell-Bloch equations. The equation describing the evolution of the area of a pulse with modulated envelope and phase is obtained. This equation has an analytic solution which is a generalization of the well-known McCall-Hahn solution.

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I. INTRODUCTION

The equations that describe a time evolution of a two-level system excited by a resonance radiation, taking into account both collisional and radiation relaxation in the dipole approximation, are a closed set of three linear equations of the first order (for the Bloch equations, see, for example, [1]). If the exciting field amplitude and phase are constant, the Bloch equations are a set of linear equations of the first order with constant coefficients and they can be easily solved (Torrey [2] solved such equations by Laplace transformations). The dependence of the exciting field amplitude and phase on time leads to variable coefficients in the Bloch equations. In the particular case of amplitude modulation when the field amplitude is a periodic function of time (such a field can be presented as a set of monochromatic equidistant fields with the same amplitude), the Bloch-equation coefficients are periodic functions of time. Such an equation can be solved by the so-called Floquet-theorem method [3]. According to this method, the solution of an equation with periodic coefficients can be presented by an infinite series of the sidebands. The Floquet theorem was applied for the first time, we believe, to solution of equations for wave-function amplitudes by Autler and Townes [4], and to solution of equations describing the resonance field propagation through the two-level medium by Stenholm and Lamb [5]. A multimode case (including the two-mode cases) of an intracavity field was treated by Hambenne and Sargent [6]. Feneuille, Schweighofer, and Oliver [7] and Toptygina and Fradkin [8] applied the Floquet theorem to the solution of the Bloch equations for the simplest case of field amplitude modulation (bichromatic field). Recently [9] Toptygina and Fradkin showed the Floquet-theorem method allows one to obtain the solution of the Bloch equations in the case of a polychromatic exciting field, i.e., in the general case of the periodical amplitude modulation. The Floquet-theorem method has two essential deficiencies. First, the amplitudes of sidebands are expressed in terms of continuous fractions which are needed in the numerical calculations. Second, this method is not applicable in the case of nonperiodical modulation of the parameters of the exciting field. There are other methods to obtain the solution of the Bloch equations (see, for example, the Introducton in [11] and [13]). We applied the matrix exponent method which does not have the above-mentioned deficiencies in [10] for the case of stochastic phase modulation, in [11] for the case of periodical amplitude modulation, and in [12] for the case of an arbitrary amplitude of the exciting field. In Sec. II of the present paper we obtain the general solution of the Bloch equations in the case of an arbitrary amplitude and a phase of the exciting field. In Sec. III we use this general solution for the solution of the Maxwell-Bloch equations, i.e., we take into account the propagation of the radiation in the resonant medium. This problem has been solved up to now either by disregarding any phase modulation effects (for example, even in the fundamental work [14]) or with the help of the perturbation theory or by the inverse scattering problem [15–19].

II. THE SOLUTION OF THE BLOCH EQUATIONS BY THE MATRIX EXPONENT METHOD IN THE CASE OF ARBITRARY PARAMETERS OF EXCITING FIELD

Suppose that a two-level system is excited by a field, the electric component of which has the plane-wave form with modulated amplitude and phase:

$$\mathcal{E}(t,z)=E(t,z)\cos[\omega t-kz+\phi(t,z)]\ , \tag{1}$$

where $E(t,z)$, ω, k, and $\phi(t,z)$ are the amplitude, frequency, wave number, and phase of the wave, respectively.

The equations describing the time evolution of the density matrix elements of a two-level system in the dipole approximation without taking into account the propagation (the dependence on z) can be written in the form [20] (we assume the equality of the frequency ω and the transition frequency of a two-level system)

$$\sigma_{21}(t)=-\Gamma_2\sigma_{21}(t)-(i/\hbar)d\mathcal{E}(t)e^{i\omega t}n(t) ,$$
$$n(t)=-[n(t)-n_0]\Gamma_1 -2(i/\hbar)\mathcal{E}(t)[\sigma_{21}(t)e^{-i\omega t}-\sigma_{12}(t)e^{i\omega t}] , \quad (2)$$

where σ_{12} and σ_{21} are the slowly varying components of the density matrix elements ρ_{12} and ρ_{21}, respectively, $n(t)=\rho_{22}(t)-\rho_{11}(t)$ is the level population difference per unit volume, $n_0=n(t=0),d$ is the dipole moment of a two-level system, and Γ_2^{-1} and Γ_1^{-1} are the polarization- and population-relaxation times, respectively.

After the substitution of Eq. (1) into Eqs. (2) and throwing off the items proportional to $\exp(\pm 2i\omega t)$ (it is true at $|\phi| \ll \omega$) and changing to the Bloch variables u,v by the replacement $\sigma_{21}=(u-iv)/2=\sigma_{12}^*$ we obtained the Bloch equations for the exciting field (1):

$$\frac{dX(t)}{dt}=A(t)X(t)+L , \quad (3)$$

where

$$X(t)=\begin{bmatrix} u(t) \\ v(t) \\ n(t) \end{bmatrix} ,$$

$$A(t)=\begin{bmatrix} -\Gamma_2 & 0 & -a(t) \\ 0 & -\Gamma_2 & b(t) \\ a(t) & -b(t) & -\Gamma_1 \end{bmatrix} , \quad L=n_0\Gamma_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ,$$

where $a(t)=\Omega_R(t)\sin\phi(t);b(t)=\Omega_R(t)\cos\phi(t);$ and $\Omega_R(t)=(d/\hbar)E(t)$. Provided the commutator $[A(t),\exp B(t)]$ is equal to zero, where $B(t)=\int_0^t A(t')dt'$, the solutions of Eq. (3) can be written as [21]

$$X(t)=e^{B(t)}\left[\int_0^t e^{-B(t')}L\,dt'+\begin{bmatrix} 0 \\ 0 \\ n_0 \end{bmatrix}\right] . \quad (4)$$

To calculate $\exp B(t)$ we shall make use of Silvester's formula [3]:

$$e^{B(t)}=e^{\lambda_1}\frac{B-\lambda_2 I)(B-\lambda_3 I)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+e^{\lambda_2}\frac{(B-\lambda_1 I)(B-\lambda_3 I)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} +e^{\lambda_3}\frac{(B-\lambda_1 I)(B-\lambda_2 I)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} , \quad (5)$$

where $\lambda_{1,2,3}$ are the eigenvalues of matrix $B(t)$ and I is the identity matrix. Assuming the equality $\Gamma_2=\Gamma_1\equiv\Gamma$, the evident expressions for $\lambda_{1,2,3}$ can be easily obtained without making any assumptions concerning the field amplitude:

$$\lambda_1=-\Gamma t, \quad \lambda_{2,3}=-\Gamma t\pm if(t) , \quad (6)$$

where $f^2(t)=[I_a^2(t)+I_b^2(t)]$; $I_a(t)=\int_0^t a(t')dt'$; and $I_b(t)=\int_0^t b(t')dt'$. After substituting Eqs. (6) into Eq. (5) we obtain

$$e^{\pm B(t)}=\frac{e^{\mp\Gamma t}}{f^2(t)}\begin{bmatrix} I_b^2+I_a^2\cos f(t) & I_aI_b[1-\cos f(t)] & \mp I_af(t)\sin f(t) \\ I_aI_b[1-\cos f(t)] & I_a^2+I_b^2\cos f(t) & \pm I_bf(t)\sin f(t) \\ \pm I_af(t)\sin f(t) & \mp I_bf(t)\sin f(t) & f^2(t)\cos f(t) \end{bmatrix} . \quad (7)$$

The substitution of Eq. (7) into Eq. (4) allows us to obtain the solutions of the Bloch equations for the case of exciting field (1) in the form

$$X(t)-n_0\Gamma\frac{e^{-\Gamma t}}{f^2(t)}\begin{bmatrix} I_1[I_b^2+I_a^2\cos f(t)]-I_2I_aI_b[1-\cos f(t)]-(I_3+\Gamma^{-1})I_af(t)\sin f(t) \\ I_1I_aI_b[1-\cos f(t)]-I_2[I_a^2+I_b^2\cos f(t)]+(I_3+\Gamma^{-1})I_bf(t)\sin f(t) \\ (I_1I_a+I_2I_b)f(t)\sin f(t)+(I_3+\Gamma^{-1})f^2(t)\cos f(t) \end{bmatrix} , \quad (8)$$

where

$$I_1(t)=\int_0^t e^{\Gamma t'}I_a(t')\frac{\sin f(t')}{f(t')}dt' \quad I_2(t)=\int_0^t e^{\Gamma t'}I_b(t')\frac{\sin f(t')}{f(t')}dt', \quad I_3(t)=\int_0^t e^{\Gamma t'}\cos f(t')dt' .$$

Let us clarify the conditions under which the solutions (8) are true. With the help of (7) we obtain

$$[A(t),e^{B(t)}]=e^{-\Gamma t}\frac{(aI_b-bI_a)}{f^2(t)}\begin{bmatrix} 0 & f\sin f & I_b(1-\cos f) \\ -f\sin f & 0 & I_a(1-\cos f) \\ I_b(1-\cos f) & I_a(1-\cos f) & 0 \end{bmatrix} . \quad (9)$$

Hence the solutions of Eqs. (3) in the form (8) are true if any of the following conditions is fulfilled:

$$aI_b=bI_a , \quad (10a)$$

$$t\to\infty . \quad (10b)$$

The condition (10a) means that Eqs. (8) are true at any time t in the range $[0,t]$, but not for arbitrary functions $E(t)$ and $\phi(t)$. With the help of obvious forms of the functions $a(t)$, $b(t)$, $I_a(t)$, and $I_b(t)$ one can easily see the condition (i) is fulfilled first when both amplitude and phase modulations are absent, i.e., when $E=$const and $\phi=$const; second, when $E(t)$ is any arbitrary function but ϕ=const. In the particular case when $\phi=0$ we obtain

$$a(t)=I_a(t)=0,\quad b(t)=\Omega_R(t)\ ;$$
$$I_b(t)=\int_0^t \Omega_R(t')dt'=f(t)\ ,$$
$$I_1(t)=0,\quad I_2(t)=\int_0^t e^{\Gamma t'}\sin I_b(t')dt'\ ,$$
$$I_3(t)=\int_0^t e^{\Gamma t'}\cos I_b(t')dt'\ .$$

Hence the solutions (8) in this case will read as

$$X(t)=n_0\Gamma e^{-\Gamma t}\begin{bmatrix}-I_2(t)\cos I_b(t)+[I_3(t)+\Gamma^{-1}]\sin I_b(t)\\ I_2(t)\sin I_b(t)+[I_3(t)+\Gamma^{-1}]\cos I_b(t)\end{bmatrix}\ . \tag{11}$$

These solutions have been obtained in [12]. In our opinion the solutions (8) were not obtained up to now.

The condition (10b) means the solutions (8) are true for any arbitrary functions $E(t)$ and $\phi(t)$ but for a stationary state only, i.e., when the condition $\Gamma t \gg 1$ is fulfilled. If we assume that the field amplitude $E(t)$ is connected with the phase $\phi(t)$ by such a dependence: $\Omega_R(t)=C\dot{\phi}(t)$, where C is any arbitrary constant, we can easily obtain

$$I_a(t)=-C\cos\phi(t),\quad I_b(t)=C\sin\phi(t)\ , \tag{12}$$

and hence

$$I_1(t)=-\sin C\int_0^t e^{\Gamma t'}\cos\phi(t')dt',$$
$$I_2(t)=\sin C\int_0^t e^{\Gamma t'}\sin\phi(t')dt'\ , \tag{13}$$
$$I_3(t)=\frac{\cos C}{\Gamma}(e^{\Gamma t}-1)\ .$$

In 1985 Nayak and Agarwal [22] solved the Bloch equations by an application of the continued-fraction method in the case of the periodical phase modulation of the exciting field. Now we can obtain a more general solution with the help of Eqs. (12) and (13). So we assume

$$\phi(t)=M\sin\Omega t \tag{14}$$

where M and Ω are the modulation index and modulation frequency, respectively.

Hence

$$\Omega_R(t)=C\dot{\phi}(t)=CM\Omega\cos\Omega t\ , \tag{15}$$

i.e.,

$$E(t)=\frac{\hbar}{d}\Omega_R(t)=\frac{\hbar}{d}CM\Omega\cos\Omega t\ . \tag{16}$$

It is seen from Eq. (16) that the exciting field in this case is the bichromatic field with the modulated phase of each component. This is the generalization of the Nayak-Agarwal case to the amplitude modulation.

By use of the well-known expansions

$$\sin\phi(t)=\sin(M\sin\Omega t)=2\sum_{k=0}^{\infty}J_{2k+1}(M)\sin[(2k+1)\Omega t]\ , \tag{17}$$
$$\cos\phi(t)=\cos(M\sin\Omega t)=2\sum_{k=0}^{\infty}c_kJ_{2k}(M)\cos(2k\Omega t)\ ,$$

where J_m is the Bessel function of the first kind, $c_0=\frac{1}{2}$; $c_1=c_2=\cdots=c_m=1$, we can easily obtain

$$I_1(t)=-2(\sin C)\sum_{k=0}^{\infty}c_kJ_{2k}(M)\left[\frac{e^{\Gamma t}}{\Gamma^2+4k^2\Omega^2}[\Gamma\cos(2k\Omega t)+2k\Omega\sin(2k\Omega t)]-\frac{\Gamma}{\Gamma^2+4k^2\Omega^2}\right]\ , \tag{18}$$

$$I_2(t)=2(\sin C)\sum_{k=0}^{\infty}J_{2k+1}(M)\left[\frac{e^{\Gamma t}}{\Gamma^2+(2k+1)^2\Omega^2}\{\Gamma\sin[(2k+1)\Omega t]-(2k+1)\Omega\cos[(2k+1)\Omega t]\}+\frac{(2k+1)\Omega}{\Gamma^2+(2k+1)^2\Omega^2}\right]\ . \tag{19}$$

After substitution of Eqs. (18), (19), (17), (12), and (13) into the general solution (8) and making the assumption $t\to\infty$, we obtain the steady-state solutions of the Bloch equations in the case of periodical phase and amplitude modulation of the exciting field [we write for the sake of simplicity the solution for $n(t)$ only]:

$$\begin{aligned}n(t\to\infty)=&\ n_0\cos^2C\\&+2n_0\Gamma(\sin^2C)\sum_{k,l=0}^{\infty}\Bigg[c_kc_lJ_{2k}(M)J_{2l}(M)\frac{1}{\Gamma^2+4k^2\Omega^2}\\&\times\{2k\Omega\sin[2(k+l)\Omega t]\\&+\Gamma\cos[2(k+l)\Omega t]+2k\Omega\sin[2(k-l)\Omega t]+\Gamma\cos[2(k-l)\Omega t]\}\\&+J_{2k+1}(M)J_{2l+1}(M)\frac{1}{\Gamma^2+(2k+1)^2\Omega^2}\\&\times\{(2k+1)\Omega\sin[2(k-l)\Omega t]+\Gamma\cos[2(k-l)\Omega t]\\&-(2k+1)\Omega\sin[2(k+1+l)\Omega t]-\Gamma\cos[2(k+l+1)\Omega t]\}\Bigg]\ .\end{aligned} \tag{20}$$

So, we showed that the application of the matrix exponent method to the solution of the Bloch equations allows us to obtain the analytical solutions in the case of arbitrary amplitude and phase of exciting field subject to the conditions of Eq. (10) (of course, both the amplitude and phase must be slowly varying functions in comparison with the field frequency ω in order that the rotating-wave approximation can be applied).

III. THE SOLUTIONS OF THE MAXWELL-BLOCH EQUATIONS IN THE CASE OF AN ARBITRARY SPATIAL PHASE MODULATION OF THE PULSE

When propagating in a resonance two-level medium the field of the form (1) induces the polarization $P(t,z)$ in this medium [1]:

$$P(t,z)=Nd\int g(\Delta)\{u(t,z,\Delta)\cos[\omega t-Kz+\phi(t,z)] -v(t,z,\Delta) \times\sin[\omega t-Kz+\phi(t,z)]\}d\Delta \,, \quad (21)$$

where N is the density of a resonance atom, $g(\Delta)$ is the nonhomogeneous broadening line shape, and u and v are the Bloch components. The substitution of Eq. (21) into the Maxwell equation $(\partial^2/\partial z^2-c^{-2}\partial^2/\partial t^2)\mathcal{E}(t,z)=(4\pi/c^2)\partial^2P(t,z)/\partial z^2$ and assuming that the amplitude $E(t,z)$ and the phase $\phi(t,z)$ are the slowly varying functions lead to the equation [1]

$$\left[K\frac{\partial}{\partial z}+\frac{k}{c}\frac{\partial}{\partial t}\right]E(t,z)=\pi k^2Nd\int v(t,z,\Delta)g(\Delta)d\Delta \,, \quad (22)$$

where $k=\omega/c$ is the vacuum wave vector.

Integrating on time t in the range $[-\infty,t]$ and taking into account for sufficiently long time t that one can assume $E=0$, we obtain

$$\frac{\partial\Phi(t,z)}{\partial z}=\frac{\pi k^2Nd}{2K\hbar^2}\int_{-\infty}^{\infty}g(\Delta)\int_{-\infty}^{t}v(t',z,\Delta)dt'd\Delta \,, \quad (23)$$

where $\Phi(t,z)=(d/\hbar)\int_{-\infty}^{t}E(t',z)dt'$.

Carrying out the calculations as in [1], we obtain

$$\frac{\partial\Phi(t,z)}{\partial z}=-\frac{\alpha_n}{2}v(t,z,0) \,, \quad (24)$$

where α_n is the resonance-absorption coefficient.

Assuming that the period of the field force on a medium is much less than the relaxation time Γ^{-1}, we can set $\Gamma=0$. In addition we shall assume the pulse phase ϕ is a function of z only (a spatial phase modulation). In this case we can use the condition (10a).

So we obtain under the above-mentioned conditions from Eqs. (8)

$$v(t,z,0)=n_0\cos\phi(z)\sin\Phi(t,z) \,. \quad (25)$$

Then Eq. (24) gives us an equation for the pulse area $\Theta(t)=\Phi(t\to\infty,z)$:

$$\frac{d\Theta(z)}{dz}=-\frac{\alpha_n}{2}\cos\phi(z)\sin\Theta(z) \,. \quad (26)$$

Equation (26) is the generalization of the well-known McCall-Hahn area theorem [14] to the case of the arbitrary spatial phase modulation of an exciting field. The analytic solution of this equation has the form

$$\Theta(z)=2\arctan\left[\exp\left[(-\alpha_n/2)\int_0^z\cos\phi(z)dz\right]\right] \,. \quad (27)$$

Equation (27) in the case of the absence of the phase modulation ($\phi=0$) coincides with the McCall-Hahn solution. One can find some examples of the concrete forms of the function $\phi(z)$ when Eq. (27) has the obvious form:

$$\text{(i) } \phi(z)=\arctan\{[1-(\alpha z)^2]^{1/2}/\alpha z\} \,, \quad (28)$$

where α is an arbitrary parameter.

In this case Eq. (27) gives

$$\Theta(z)=2\arctan[\exp(-\alpha_n\alpha z^2/4)] \,. \quad (29)$$

This solution shows that the phase modulation in form (28) just as in the case of absence of the phase modulation does not prevent the pulse area stabilization. Moreover this stabilization comes more rapidly in our case because the argument in (29) decreases as $\exp(-z^2)$ while at $\phi=0$, as $\exp(-z)$. However, if we assume the following modulation of phase:

$$\text{(ii) } \phi(z)=\arctan\alpha z \quad (30)$$

then we obtain for $\Theta(z)$

$$\Theta(z)=2\arctan(1/\{\alpha z+[1+(\alpha z)^2]^{1/2}\}^{\alpha_n/2\alpha}) \,. \quad (31)$$

In this case the area is stablized more slowly than in the McCall-Hahn solution.

These effects of the phase modulation which were not taken into account in [14] may give an explanation (in addition to the mechanisms mentioned in [14]) to the deviation of the experimental results from the McCall-Hahn solution.

Now we shall attempt to clear up how the spatial phase modulation changes the shape of a pulse. Assuming that $g(\Delta)=\delta(\Delta)$ and making the substitution of variables t and z in Eq. (13) for the runing variable $\tau=t-z/V$, where V is the pulse speed in the medium, we obtain, for the pulse envelope $E(\tau)$,

$$\frac{dE(\tau)}{d\tau}\left[\frac{K}{V}-\frac{k}{c}\right]=-\pi k^2Ndv(t,z,0) =-\frac{\alpha_n}{2}\cos\phi(z)\sin\Phi(t,z) \,. \quad (32)$$

Since $dE/d\tau=(\hbar/d)(d^2\Phi/d\tau)$, we have

$$\frac{d^2\Phi(\tau)}{d\tau^2}=-\frac{\alpha_n\alpha}{\hbar}-\cos\phi(z)\sin\Phi(\tau) \,. \quad (33)$$

When the phase modulation is absent ($\phi=0$) this equation gives the well-known McCall-Hahn hyperbolic-secant solution. Equation (24) cannot be solved in an analytic form and needs a numerical calculation.

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