Low-frequency fluctuations in plasma magnetic fields

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It is shown that even a nonmagnetized plasma with temperature T sustains zero-frequency magnetic fluctuations in thermal equilibrium. Fluctuations in electric and magnetic fields, as well as in densities, are computed. Four cases are studied: a cold, gaseous, isotropic, nonmagnetized plasma; a cold, gaseous plasma in a uniform magnetic field; a warm, gaseous plasma described by kinetic theory; and a degenerate electron plasma. For the simple gaseous plasma, the fluctuation strength of the magnetic field as a function of frequency and wave number is calculated with the aid of the fluctuation-dissipation theorem. This calculation is done for both collisional and collisionless plasmas. The magnetic-field fluctuation spectrum of each plasma has a large zero-frequency peak. The peak is a Dirac δ function in the collisionless plasma; it is broadened into a Lorentzian curve in the collisional plasma. The plasma causes a low-frequency cutoff in the typical blackbody radiation spectrum, and the energy under the discovered peak approximates the energy lost in this cutoff. When the imposed magnetic field is weak, the magnetic-field wave-vector fluctuation spectra of the two lowest modes are independent of the strength of the imposed field. Further, these modes contain finite energy even when the imposed field is zero. It is the energy of these modes that forms the zero-frequency peak of the nonmagnetized plasma. In deriving these results, a simple relationship between the dispersion relation and the fluctuation power spectrum of electromagnetic waves is found. The warm plasma is shown, by kinetic theory, to exhibit a zero-frequency peak in its magnetic-field fluctuation spectrum as well. For the degenerate plasma, we find that electric-field fluctuations and number-density fluctuations vanish at zero frequency; however, the magnetic-field power spectrum diverges at zero frequency.

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I. INTRODUCTION AND OUTLINE

In a plasma, quantities such as local electron density, ion density, electron field, and magnetic field are all smoothly varying and well-defined functions of space and time, on some practical or coarse-grained scale. This is only an incomplete definition of a plasma. However, since the constituents of a plasma are discrete particles, these quantities are in a constant state of flux in the most quiescent of plasmas, always rising and falling about their well-defined mean values.

The fluctuations in electromagnetic field are the main concerns of this paper. These fluctuations may be aptly described as random fluctuations for the cases of weakly correlated plasmas such as gaseous plasmas and ideal degenerate plasmas. The statistics of these fluctuations, their root-mean-square amplitudes, for instance, are completely determined by the mean values of the plasma quantities in thermal equilibrium. In particular, the power spectrum of a given quantity's fluctuations is determined completely by (1) the amount of energy needed to produce a fluctuation of a given size in a given mode, (2) the temperature of the plasma, and (3) the dissipation mechanisms at work in the plasma. This determination is expressed for weakly correlated (and nearly linear) plasmas not far from equilibrium by the fluctuation-dissipation theorem [1–3]. In this report we apply the fluctuation-dissipation theorem to weakly correlated plasmas in thermal equilibrium and derive the power spectra of fluctuations in the plasma magnetic field. An alternative method for deriving the spectra, which we will not use here, would be to derive the kinetic theoretic equation applying the superposition principle [4,5].

We begin in Sec. III with a homogeneous, isotropic, nonmagnetized cold plasma. (We will say a few words about Sec. II momentarily). We see that the fluctuation spectra of the magnetic field are particularly interesting because they exhibit a strong zero-frequency component. This zero-frequency component is a Dirac δ function in a nondissipative plasma, and is broadened into a Lorentzian curve in a dissipative plasma. This phenomenon may have implications for the physics of the early Universe. Because of this, much of our calculations in this section are made for an electron-positron plasma whose temperature and density have been chosen so that it, presumably, describes the Universe in the early radiation epoch.

Since we assume, in the beginning, an isotropic, nonmagnetized plasma, it is curious that our mathematics should tell us that we actually have a plasma with a magnetic field which is nearly stationary in time (even though it is far from uniform in space). Does a plasma in a presupposed stationary magnetic field exhibit a similar fluctuation spectrum? As a consistency check, then, we study in Sec. IV the fluctuation spectra of a thermal equilibrium plasma in a uniform, constant magnetic field. In the process of deriving the spectra, a relationship between the dispersion relation of an electromagnetic wave and its fluctuations is found. We find a substantial amount of low-frequency fluctuations, but they are not concentrated in the Dirac δ function we found for the nonmagnetized plasma. Rather, it is seen that the imposed magnetic field effects a transfer of energy away from $\omega=0$ into a range of frequencies running from $\omega=0$ up to the lower hybrid frequency. We find, however, that the limit $\mathbf{B}_0\to 0$ is completely consistent with the results of Sec. III.

We take another look at the isotropic plasma in Sec. V. Here we will derive the magnetic-field fluctuation power spectrum from kinetic theory. We find a broadened zero-frequency peak in the spectrum, but, otherwise, most results found in Sec. III are confirmed here.

In Sec. VI we discuss magnetic-field fluctuation spectra obtained from computer simulations of thermal equilibrium plasmas. In these particle simulations, the fluctuation spectrum of the magnetic field has been recorded. We discuss its size and shape in light of the predictions made in Sec. III.

Throughout our calculations, we need to introduce a phenomenological cutoff in a wave number k. The legitimacy of such a cutoff is established in Sec. VII by way of quantum-mechanical considerations.

In Sec. VIII we address the Bohr–van Leeuwen theorem, namely, that classical statistical mechanics does not allow the magnetization of a physical medium. This might seem to present a contradiction to our result of finite magnetic-field energy at $\omega=0$ in the nonmagnetized plasma. This contradiction is shown to be only apparent.

In Sec. IX we look at electrostatic and electromagnetic fluctuations in a degenerate electron gas. We find that electric-field fluctuations and particle density fluctuations vanish at zero frequency. However, magnetic-field fluctuations diverge at zero frequency. This divergence is proportional to ω^{-1} over a large frequency range.

In Sec. X we examine some of the consequences of our results. The low-frequency magnetic fields we discuss may effect particle transport in plasmas. They may have consequences for structure formation in the radiation epoch of the early Universe. They may, finally, be responsible for anomalous spin relaxation in condensed matter.

Before we discuss these subjects for real plasmas, however, it may be instructive to begin with a model problem which involves the Brownian motion of a system described by a one-dimensional wave equation. In such a system, the resultant fluctuations can be treated as random fluctuations without any correlations. This section should involve some familiar physics, but is intended to elucidate the theoretical foundation of the present paper in a simplified model problem. This is treated in Sec. II.

II. ONE-DIMENSIONAL WAVES WITH BROWNIAN MOTION

We consider a physical system describable by a wave equation:

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}\ , \tag{1}$$

where $y(x,t)$ is the local displacement of some quantity from its equilibrium value and c is the phase velocity of waves in the system. We could be speaking here about sound waves in air or water, longitudinal waves in a compressional spring, transverse waves on a piano wire, etc. What we want to ask is, given that our physical system is in thermal equilibrium, what is the power spectrum of the motion it undergoes because of thermal fluctuations?

We want to make our wave equation a little more realistic, adding two terms to it:

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}-\eta\frac{\partial y}{\partial t}+a(x,t)\ . \tag{2}$$

The first new term is a damping term. It could have as its source some internal friction of the system, or it could be the dissipative effect of thermal fluctuations such as what we see in Brownian motion. $a(x,t)$ is a spatially and temporally random function, describing the fluctuating accelerations imparted to local elements of the system. If we were describing a piano wire here, $a(x,t)$ could represent local fluctuating accelerations from internal thermal fluctuations in the positions of the molecules making up the wire, or it could describe the momentum transferred from air molecules constantly bombarding the wire. In the mathematics that follows, we make the reasonable assumption that $\langle a(x,t)\rangle$ [the ensemble average of $a(x,t)$] is equal to zero and that, therefore, $\langle y(x,t)\rangle$ is equal to zero.

First of all, we Fourier transform our new equation to get

$$(-\omega^2+c^2k^2-i\eta\omega)y(k,\omega)=a(k,\omega)\ . \tag{3}$$

Let $Y(k,\omega)$ be the ensemble-averaged intensity of $y(k,\omega)$ and let $A(k,\omega)$ be the ensemble-averaged intensity of $a(k,\omega)$. Then we see

$$Y(k,\omega)=\frac{A(k,\omega)}{(-\omega^2+c^2k^2)^2+\eta^2\omega^2}\ . \tag{4}$$

The simplest assumption is that $a(x,t)$ is a series of Dirac δ functions randomly distributed in space and time. That is, we assume that the fluctuating acceleration take the form of momentum impluses delivered over very short lengths of the system. The fluctuations are uncorrelated with one another and have a white-noise power spectrum. This being given, $a(x,t)$ will have a correlation function given by

$$\langle a(x_0,t_0)a(x_0+x,t_0+t)\rangle=a^2\delta(t)\delta(x)\ , \tag{5}$$

where a^2 is a number derived from the distribution of the strength of the random impulses and the space and time intervals between them. From this it follows that

$$A(k,\omega)=\int d\omega\, dk\, e^{ikx-i\omega t}\times\langle a(x_0,t_0)a(x_0+x,t_0+t)\rangle=a^2\ . \tag{6}$$

Equation (4) then becomes

$$Y(k,\omega)=\frac{a^2}{(-\omega^2+c^2k^2)^2+\eta^2\omega^2}\ . \tag{7}$$

Now we find $Y(k,t)$ by Fourier transforming $Y(k,\omega)$:

$$Y(k,t)=\frac{1}{2\pi}\int d\omega\, e^{i\omega t}\frac{a^2}{(-\omega^2+c^2k^2)^2+\eta^2\omega^2}$$
$$=\frac{a^2e^{\eta t/2}}{2\eta c^2k^2}\left[\cos(\omega_1 t)+\frac{\eta}{2\omega_1}\sin(\omega_1 t)\right], \qquad (8)$$

where

$$\omega_1=\left[c^2k^2-\frac{\eta^2}{4}\right]^{1/2}.$$

The contour of the integral runs along the real axis in the complex ω plane and is closed in the top half (bottom half) of the plane for $t>0$ ($t<0$). From this, it immediately follows that

$$Y(k,t=0)=\langle y(k)y^*(k)\rangle=\frac{a^2}{2\eta c^2k^2}. \qquad (9)$$

If we assume that the system is in classical thermal equilibrium with some heat bath at temperature T, we are constrained by the equipartition law to say

$$\mu\frac{c^2k^2}{2}\langle y(k)y^*(k)\rangle=\frac{T}{2}, \qquad (10)$$

where μ is a constant representing the inertia of the system. If our system is a wire, μ is the mass per unit length of the wire. It follows from Eqs. (9) and (10) that

$$a^2=\frac{2\eta T}{\mu}$$

and

$$Y(k,\omega)=\frac{2\eta T/\mu}{(\omega^2-c^2k^2)^2+\eta^2\omega^2}. \qquad (11)$$

To find the limit of $Y(k,\omega)$ in the limit $\eta\to0$, we make use of a standard representation of the Dirac δ function to obtain

$$Y(k,\omega)=\frac{2T\pi}{\mu\omega^2}\delta\left[\frac{-\omega^2+c^2k^2}{\omega}\right]. \qquad (12)$$

Note that each mode in k space behaves exactly like a Brownian particle in a harmonic-oscillator potential with a characteristic frequency $\omega_0=ck$. This can be seen most clearly by comparing the current results with Kubo's results for the Brownian motion of a harmonic oscillator [6]. In a sense then, there is nothing new here, And, yet, some interesting results appear. We can content ourselves with examining the simpler form of $Y(k,\omega)$ in Eq. (12). If we want the fluctuation strength as a function of wave number alone, then we integrate $Y(k,\omega)$ over $d\omega$ and divide by 2π. If we want the fluctuation spectrum of frequency alone, we integrate $Y(k,\omega)$ over dk and divide by 2π. We find

$$Y(k)=\frac{T}{\mu c^2k^2} \qquad (13)$$

and

$$Y(\omega)=\frac{T}{\mu c\omega^2}. \qquad (14)$$

If we have a harmonic oscillator of mass m and frequency ω_0, vibrating with an amplitude A, the potential energy of the oscillation $W(\omega_0)$ is, on average, $mA^2\omega_0^2/2$. Each mode k of our oscillator is a harmonic oscillator of mass density μ and frequency ck, vibrating with an amplitude of $\sqrt{Y(k)}$. Therefore, the wave-number power spectrum $W(k)$ is given by

$$W(k)=\frac{\mu c^2k^2}{2}Y(k)=\frac{T}{2}. \qquad (15)$$

Also, since $W(k)dk=W(\omega)d\omega$,

$$W(\omega)=\frac{T}{2c}. \qquad (16)$$

$W(k)$ and $W(\omega)$ both integrate to give the same total-energy density E. The two spectra are consistent with each other and with Percival's theorem. In this purely classical treatment, however, both spectra integrate to give $(1/2\pi)\int T/2dk$, resulting in a one-dimensional ultraviolet divergence.

What happens if we add a "mass" to our system? We may get an equation like

$$\frac{\partial^2y}{\partial t^2}=c^2\frac{\partial^2y}{\partial x^2}-\omega_0^2y-\eta\frac{\partial y}{\partial t}+a(x,t). \qquad (17)$$

We now have an equation more nearly describing a plasma wave, or a massive Klein-Gordan field, or a taut piano wire sitting on top of a set of uncoupled springs. Now, to study this system, we run through all of our above mathematics with the substitution $c^2k^2\to c^2k^2+\omega_0^2$. We find for $Y(k,\omega)$,

$$Y(k,\omega)=\frac{2\eta T/\mu}{(\omega^2-c^2k^2-\omega_0^2)^2+\eta^2\omega^2}. \qquad (18)$$

If we take the limit $\eta\to0$ again, we find

$$Y(k,\omega)=\frac{2\pi T}{\mu\omega^2}\delta\left[\frac{\omega^2-c^2k^2-\omega_0^2}{\omega}\right], \qquad (19)$$

$$Y(k)=\frac{T}{\mu(c^2k^2+\omega_0^2)}, \qquad (20)$$

and

$$Y(\omega)=\frac{T}{\mu c\omega(\omega^2-\omega_0^2)^{1/2}}. \qquad (21)$$

We also have a $Y(\omega)$ which diverges as $(\omega^2-\omega_0^2)^{-1/2}$ at $\omega=0$.

$W(\omega)$ diverges as well. Since $\omega(k)=(c^2k^2+\omega_0^2)^{1/2}$, $W(k)=T/2$, again satisfying the equipartition law. However, since, once again, $W(k)dk=W(\omega)d\omega$, we find

$$W(\omega)=\frac{T}{2}\frac{\omega}{c(\omega^2-\omega_0^2)^{1/2}}. \qquad (22)$$

We have a divergence in the energy contained in frequencies close to ω_0. The integral of $W(\omega)$ over all ω diverges. However, this divergence is caused by contributions to the integral from $\omega\to\infty$, not from $\omega\approx\omega_0$. In other words, this divergence is an ultraviolet divergence. It corresponds to the ultraviolet divergence we get when we integrate $W(k)=T/2$ over all k. In fact, $\int W(\omega)d\omega=\int W(k)dk$. Therefore, once again, the two spectra are consistent with one another and with Percival's theorem.

The ultraviolet divergence we have seen in Eqs. (15), (16), and (22) is known to be removed by the Planck distribution due to the quantum-mechanical effect. The general quantum-mechanical fluctuation-dissipation theorem is spelled out in detail by Sitenko [3]. In fact, the introduction of quantum mechanics was necessary to avoid the ultraviolet divergence of the blackbody photon spectrum. This was Planck's idea.

Suppose we have some kind of quantum-mechanical, Hamiltonian system. It can be a hydrogen atom or a harmonic oscillator or anything describable by quantum mechanics. In this system, we will have a potential energy $V(x,t)$. Let us assume that we can break this potential up into a fairly smoothly varying part and a random part. Let us also assume that the random part of the potential couples to the expectation value of some current in the system so that

$$V(t)=-\int dx\ A(x,t)\langle j(x,t)\rangle \ . \tag{23}$$

We make two more assumptions: First, $V(t)$ is the only explicitly time-dependent part of the Hamiltonian. Then,

$$\frac{\partial V(t)}{\partial t}=-\int dx\ \dot{A}(x,t)\langle j(x,t)\rangle \ . \tag{24}$$

Lastly we assume $A(x,t)$ and $j(x,t)$ are related to one another by some linear operator so that

$$j_i=\hat{\alpha}_{ij}A_j(x,t) \ , \tag{25}$$

or, after Fourier transforming in x and t,

$$j_i(k,\omega)=\alpha_{ij}(k,\omega)A_j(k,\omega) \ . \tag{26}$$

The spectral distribution of the space-time correlation function $\langle j_i j_j\rangle_{rt}$ will be denoted by $\langle j_i j_j\rangle_{\mathbf{k},\omega}$. It is related to the expectation value of the product of the Fourier components of $\mathbf{j}$ by

$$\langle j_i^*(\mathbf{k})j_j(\mathbf{k}')\rangle_\omega=(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\langle j_i j_j\rangle_{\mathbf{k},\omega} \ . \tag{27}$$

A calculation of the transition probabilities arising from the action of $A(x,t)$ on the system shows that the energy absorbed per unit time by the system is

$$Q=\frac{\omega}{4\pi}\sum_{\mathbf{k},\mathbf{k}'}A_i(\mathbf{k}\omega)A_j^*(\mathbf{k}',\omega)\times\{\langle j_i^*(\mathbf{k})j_j(\mathbf{k}')\rangle_\omega^{\hbar\omega}-\langle j_i^*(\mathbf{k})j_j(\mathbf{k}')\rangle_\omega\} \ , \tag{28}$$

where

$$\langle j_i^*(\mathbf{k})j_j(\mathbf{k}')\rangle_\omega=2\pi\sum_{m,n}f(E_n)j_i^*(\mathbf{k})_{nm}j_j(\mathbf{k}')_{mn}\delta(\omega-\omega_{nm})$$

and

$$\langle j_i^*(\mathbf{k})j_j(\mathbf{k}')\rangle_\omega^{\hbar\omega}=2\pi\sum_{m,n}f(E_n-\hbar\omega)j_i^*(\mathbf{k})_{nm}j_j(\mathbf{k}')_{mn}\delta(\omega-\omega_{nm}) \ .$$

In the above expressions, $\omega_{nm}=(E_n-E_m)/\hbar$, and $f(E_n)$ is the statistical distribution of the states of the system.

However, averaging Eq. (24) over one period of oscillation shows that the energy absorbed per unit time is also equal to

$$Q=\frac{\omega}{4}\sum_{\mathbf{k}}(\alpha_{ij}^*-\alpha_{ji})A_i(\mathbf{k}\omega)A_j^*(\mathbf{k},\omega) \ . \tag{29}$$

Comparing Eqs. (28) and (29) and making use of Eq. (27) shows

$$\langle j_i j_j\rangle_{\mathbf{k}\omega}^{\hbar\omega}-\langle j_i j_j\rangle_{\mathbf{k}\omega}=i\hbar\{\alpha_{ij}^*(\mathbf{k}\omega)-\alpha_{ji}(\mathbf{k}\omega)\} \ . \tag{30}$$

If the system is in thermodynamic equilibrium, immersed in a heat bath with temperature T, $f(E_n)$ is given by the Gibbs distribution

$$f(E_n)=e^{(F-E_n)/T} \ ,$$

where F is the free energy of the system and T is the system temperature. In this case,

$$\langle j_i j_j\rangle_{\mathbf{k}\omega}^{\hbar\omega}=e^{\hbar\omega/T}\langle j_i j_j\rangle_{\mathbf{k}\omega} \ .$$

Therefore,

$$\langle j_i j_j\rangle=\frac{\hbar}{e^{\hbar\omega/T}-1}i(\alpha_{ij}^*-\alpha_{ji}) \ . \tag{31}$$

Now, for our specific problem,

$$\frac{\partial V(x,t)}{\partial t}=\int dx\ \mu a(x,t)\dot{y}(x,t) \ . \tag{32}$$

So then, comparing Eq. (23) with Eq. (32), we see that in our piano-wire system, $-\dot{A}(x,t)=\mu a(x,t)$ and $j(x,t)=\dot{y}(x,t)$. From our equation of motion we can find the factor α:

$$\alpha(k,\omega)=\frac{\omega^2}{\mu(-\omega^2+c^2k^2-i\eta\omega)} \ , \tag{33}$$

and a little bit of algebra will show that Eq. (31) gives

$$\langle|\dot{y}|^2\rangle=\frac{\hbar}{e^{\hbar\omega/T}-1}\,\frac{2\eta\omega^3}{\mu[(\omega^2-c^2k^2)^2+\eta^2\omega^2]} \ . \tag{34}$$

Now, since $\dot{y}=-i\omega y$, $|\dot{y}|^2=\omega^2|y|^2$. So,

$$\langle|y|^2\rangle=\frac{\hbar}{e^{\hbar\omega/T}-1}\,\frac{2\eta\omega}{\mu[(\omega^2-c^2k^2)^2+\eta^2\omega^2]} \ . \tag{35}$$

Note that in the limit $\hbar\to 0$, Eq. (35) will give Eq. (11). The power spectrum is $\mu\omega^2\langle|y|^2\rangle/2$; it equals

$$W(\omega)=\frac{\hbar}{e^{\hbar\omega/T}-1}\,\frac{\eta\omega^3}{[(\omega^2-c^2k^2)^2+\eta^2\omega^2]} \ . \tag{36}$$

As $\hbar\to 0$, this expression also gives the classical limit of Eq. (16). In the quantum-mechanical expression Eq. (36), the ultraviolet divergence ($\omega\to\infty$) is clearly removed due to the Planck distribution factor. In the rest of the paper, our focus is on the lower-frequency behavior of functions corresponding to W in a plasma. However, in many instances we shall see infrared divergences due to the plasma effects.

III. A GASEOUS PLASMA WITH NO EXTERNAL FIELD

Even at or near thermal equilibrium, a plasma has fluctuations. The various fields of a plasma (electromagnetic, electrostatic, density, etc.) fluctuate about their mean values. The strengths of these fluctuations are functions of two characteristics of the plasma: the dissipation mechanisms present in it, and its temperature T. The relation between these quantities can be found by means of the fluctuation-dissipation theorem [2]. In this section we employ the fluctuation-dissipation theorem to derive the power spectra of magnetic-field fluctuations in an isotropic, nonmagnetized plasma which we describe with fluid equations of motion.

The following derivation closely parallels the work of Geary *et al.* [7]. We consider a homogeneous, isotropic plasma in thermal equilibrium. The strength of the electric-field fluctuations as a function of frequency and wave vector, from the fluctuation-dissipation theorem, is

$$\frac{1}{8\pi}(\langle E_iE_j\rangle_{\mathbf{k}\omega}^{\hbar\omega}-\langle E_iE_j\rangle_{\mathbf{k}\omega})=i\hbar\{\Lambda_{ji}^{-1}-\Lambda_{ij}^{-1*}\} ,$$

where

$$\Lambda_{ij}(\omega,\mathbf{k})=\frac{c^2k^2}{\omega^2}\left[\frac{k_ik_j}{k^2}-\delta_{ij}\right]+\epsilon_{ij}(\omega,\mathbf{k}) ,$$

$[\epsilon_{ij}(\omega,\mathbf{k})]$ being the dielectric tensor of the plasma [3]. $\langle E_iE_j\rangle_{\mathbf{k}\omega}^{\hbar\omega}$ and $\langle E_iE_j\rangle_{\mathbf{k}\omega}$ are defined similarly to $\langle j_ij_j\rangle_{\mathbf{k}\omega}^{\hbar\omega}$ and $\langle j_ij_j\rangle_{\mathbf{k}\omega}$ in Sec. II.

If the plasma is in thermal equilibrium, then

$$\langle E_iE_j\rangle_{\mathbf{k}\omega}^{\hbar\omega}=e^{\hbar\omega/T}\langle E_iE_j\rangle_{\mathbf{k}\omega} ,$$

as can be inferred from the results given in Sec. II. Therefore,

$$\frac{1}{8\pi}\langle E_iE_j\rangle_{\mathbf{k}\omega}=\frac{i}{2}\frac{\hbar}{e^{\hbar\omega/T}-1}\{\Lambda_{ji}^{-1}-\Lambda_{ij}^{-1*}\} .$$

Consider an electromagnetic wave in the plasma; call its wave vector ($\mathbf{k}=k\hat{\mathbf{x}}$). Invoking Faraday's law, we find

$$\frac{\langle B_2^2\rangle_{\mathbf{k}\omega}}{8\pi}-\frac{i}{2}\frac{\hbar}{e^{\hbar\omega/T}-1}\frac{c^2k^2}{\omega^2}\{\Lambda_{33}^{-1}-\Lambda_{33}^{-1*}\}$$

and

$$\frac{\langle B_3^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{i}{2}\frac{\hbar}{e^{\hbar\omega/T}-1}\frac{c^2k^2}{\omega^2}\{\Lambda_{22}^{-1}-\Lambda_{22}^{-1*}\} ,$$

where the subscripts 1, 2, and 3 refer to the x, y, and z directions, respectively. So the total magnetic-field fluctuation strength is

$$\frac{\langle B_{\rm tot}^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{i}{2}\frac{\hbar}{e^{\hbar\omega/T}-1}\frac{c^2k^2}{\omega^2}\times\{\Lambda_{22}^{-1}+\Lambda_{33}^{-1}-\Lambda_{22}^{-1*}-\Lambda_{33}^{-1*}\} . \quad (37)$$

We now find $\epsilon_{ij}(\omega,\mathbf{k})$, in order to determine $\Lambda_{ij}(\omega,\mathbf{k})$. First, we specify the equation of motion of the plasma. From the equation of motion, we find a relationship between the electric field and the current. $\epsilon_{ij}(\omega,\mathbf{k})$ will follow from this relationship.

We introduce a simple model of a plasma based on a cold plasma fluid theory, neglecting kinetic effects necessary to adequately describe warm plasmas. (Perhaps the model is too simplistic; a discussion of this point will follow below.) If the velocities and electromagnetic fields are small enough that we can neglect the $\mathbf{v}\times\mathbf{B}$ forces, then

$$m_\alpha\frac{d\mathbf{v}_\alpha}{dt}=e_\alpha\mathbf{E}-\eta_\alpha m_\alpha\mathbf{v}_\alpha , \quad (38)$$

where α is a particle species label and n_α is the effective collision frequency of species α. An equation of motion more accurate than Eq. (38) may lead to an expression for $\langle B^2\rangle_{\mathbf{k}\omega}$ with more realistic mathematical properties. Be that as it may, the equation of motion we have yields

$$(-i\omega+\eta_\alpha)\mathbf{j}_\alpha=\frac{\omega_{p\alpha}^2}{4\pi}\mathbf{E} , \quad (39)$$

where $\mathbf{j}_\alpha$ is the current density of species α. The dielectric tensor $\epsilon_{ij}(\omega,\mathbf{k})$ is given by

$$\epsilon_{ij}(\omega,\mathbf{k})=\delta_{ij}+4\pi\sum_\alpha\chi_{\alpha ij}(\omega,\mathbf{k}) , \quad (40)$$

where the susceptibility tensor $\chi_{\alpha ij}$ is defined by the relation

$$j_{\alpha i}=-i\omega\chi_{\alpha ij}(\omega,\mathbf{k})E_j(\omega,\mathbf{k}) .$$

So, from Eq. (39),

$$4\pi\chi_{\alpha ij}(\omega,\mathbf{k})=\frac{\omega_{p\alpha}^2}{\omega(\omega+i\eta_\alpha)}\delta_{ij}$$

and

$$\epsilon_{ij}(\omega,\mathbf{k})=\delta_{ij}-\sum_\alpha\frac{\omega_{p\alpha}^2}{\omega(\omega+i\eta_\alpha)}\delta_{ij} . \quad (41)$$

It will be seen below that the results of these calculations have a particularly interesting impact on the physics of dense plasmas, such as that of the early Universe. Just prior to cooling below 1 MeV, the Universe was, apparently, an electron-positron plasma [8], and it is this type of plasma that we will discuss in the next several paragraphs. However, the derivations and results are valid, with minor modifications, for more ordinary plasmas as well.

In an electron-positron plasma, $\omega_{pe^+}^2=\omega_{pe^-}^2$ and $\eta_{e^+}=\eta_{e^-}=\eta$. So Eq. (41) becomes

$$\epsilon_{ij}(\omega,\mathbf{k})=\delta_{ij}-\frac{\omega_p^2}{\omega(\omega+i\eta)}\delta_{ij} , \quad (42)$$

where $\omega_p^2=\omega_{pe^+}^2+\omega_{pe^-}^2$. We now obtain

$$\Lambda_{ij}=\begin{pmatrix} 1-\dfrac{\omega_p^2}{\omega(\omega+i\eta)} & & \\ & 1-\dfrac{c^2k^2}{\omega^2}-\dfrac{\omega_p^2}{\omega(\omega+i\eta)} & \\ & & 1-\dfrac{c^2k^2}{\omega^2}-\dfrac{\omega_p^2}{\omega(\omega+i\eta)} \end{pmatrix} . \tag{43}$$

We combine Eqs. (37) and (43) and obtain, after some algebra,

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{2\hbar\omega}{e^{\hbar\omega/T}-1}\eta\omega_p^2\frac{c^2k^2}{\omega^2}\frac{1}{[\omega^2-c^2k^2-\omega_p^2]^2+\eta^2[\omega-c^2k^2/\omega]^2} , \tag{44}$$

or

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{2\hbar\omega}{e^{\hbar\omega/T}-1}\eta\omega_p^2\frac{c^2k^2}{(\omega^2+\eta^2)c^4k^4+2\omega^2c^2k^2(\omega_p^2-\omega^2-\eta^2)+[(\omega^2-\omega_p^2)^2+\eta^2\omega^2]\omega^2} . \tag{45}$$

The first form of $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$, with a pole being clearly offset from the electromagnetic plasma-wave pole, might be more physically understandable, whereas the second form will make integration a less difficult task. Note that if relativistic temperature effects are included, the above formulas are altered by the substitution $\omega_p\rightarrow\omega_p/\sqrt{\gamma}$. Figure 1 shows a contour plot of the natural logarithm of this function weighted with the geometrical factor k^2. The density ($n_e=4.84\times10^{30}/\text{cm}^3$) and temperature ($T=10^{10}$ K) have been chosen to represent the early Universe plasma at 1 s after the Big Bang. The collision frequency η has been set 100 times larger than expected. This smooths out the contours of the graph and gives a better view of the qualitative behavior of the spectrum.

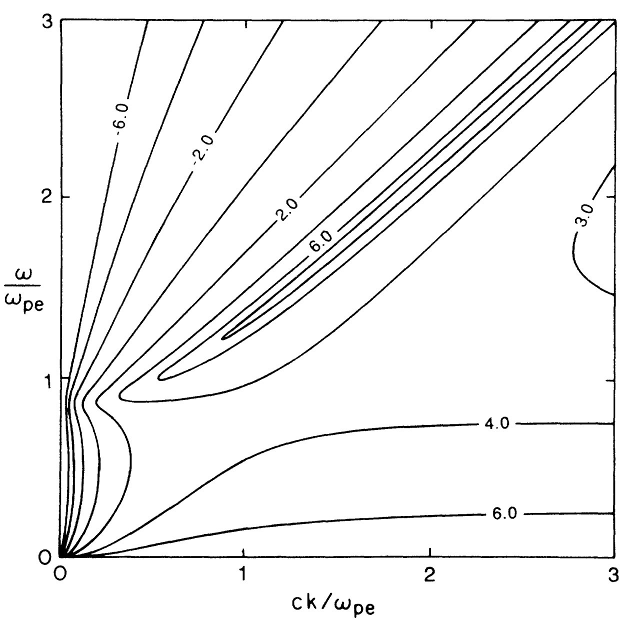

FIG. 1. The natural logarithm of $(\langle B^2\rangle_{\mathbf{k}\omega}/8\pi)(k^2c^2/2\pi^2\omega_{pe}^2\hbar)$. This contour plot corresponds to an electron-positron plasma. Electron density is $n_e=4.84\times10^3/\text{cm}^3$. Temperature is $T=10^{10}$ K. Collision frequency is $2.2\times10^{19}\ \text{s}^{-1}$. The collision frequency has been set at 100 times the expected value. This smooths out the contour ridges without changing their locations drastically, thus giving a better view of the qualitative behavior of the spectrum. The difference in height between adjacent contours is 2.0, except for the solitary contour on the right edge, which has a value of 3.0.

We now want to find the fluctuation power spectrum as a function of frequency: $\langle B^2\rangle_\omega$. We find this spectrum by integrating $\langle B^2\rangle_{\mathbf{k}\omega}$ over wave number k and dividing the result by $(2\pi)^3$. This integration can be done analytically, as shown in Appendix A. We obtain

$$\frac{\langle B^2\rangle_\omega}{8\pi}=\frac{2\hbar\omega}{e^{\hbar\omega/T}-1}\frac{2\eta}{2\pi^2\omega_{pe}^2}\left[\frac{\omega_{pe}}{c}\right]^3 \times\int_0^\infty dx\frac{x^4}{(\omega'^2+\eta'^2)x^4+\cdots} , \tag{46}$$

where $x=ck/\omega_{pe}$ and the primed quantities are normalized by ω_{pe} (e.g., $\eta'=\eta/\omega_{pe}$). However, we are immediately faced with a problem. At large k, which corresponds directly to large x, the integrand of Eq. (46) becomes effectively constant, so the integral diverges.

The divergence occurs at high wave numbers. However, this divergence at high k is different from the one we discussed in Sec. II. As seen in Eq. (46), the Planck factor $(e^{\hbar\omega/T}-1)^{-1}$ is already incorporated and thus no ultraviolet divergence arises as $\omega\rightarrow\infty$. Rather, the divergence resides in the more subtle interaction between matter and radiation. Up to this point, we have based our calculations on classical fluid equations of motion with a model collision term. In these equations the photon fields appear as smooth electromagnetic fields. In this sense, these equations may be regarded as multicomponent fluid equations for electrons. However, at some small enough physical scale (or, equivalently, some large enough wave number), the granular nature of any fluid (photons or electrons) will become apparent and render the continuum fluid equations invalid. Where the fluid "picture" breaks down, we need new equations. We might obtain such equations from a kinetic theory which includes more exact collisional effects, wave-particle interactions, etc. In the interest of tractability, however, we want to continue with the simple model presently before us. How do we manage this?

Our reasoning is as follows: Consider electromagnetic waves propagating through a plasma. The dispersion relation for waves of long wavelength is strongly dependent on the collective effects of the plasma. Waves of shorter wavelength are affected less by the plasma. If a wave has a wavelength much shorter than the collisionless skin depth c/ω_p, it moves through the plasma almost as if it were moving through empty space. It stands to reason that, for wavelengths much smaller than c/ω_p and frequencies much greater than ω_p, the fluctuation spectrum of the magnetic field must be much the same as a blackbody radiation spectrum. In particular, the collision of electrons should not matter. This being the case, a reliable high-frequency, high-wave-number limit should be obtained if we let $\eta \to 0$. A more rigorous quantum-mechanical treatment is presented in Sec. VII. We take the $\eta \to 0$ limit with the aid of a standard representation of the Dirac δ function; we obtain

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{2\hbar\omega}{e^{\hbar\omega/T}-1}\omega_p^2 c^2 k^2 \pi\delta\left[\frac{\omega(\omega^2-c^2k^2-\omega_p^2)}{\omega^2-c^2k^2}\right]\times\frac{1}{(\omega^2-c^2k^2)^2}\ . \tag{47}$$

Integrating Eq. (47) over d^3k and dividing by $(2\pi)^3$ gives

$$\frac{\langle B^2\rangle_{\omega}}{8\pi}=\frac{T}{\pi}\delta(\omega)\int\frac{\omega_p^2k^2}{\omega_p^2+c^2k^2}dk+\frac{1}{2\pi c^3}\frac{\hbar}{e^{\hbar\omega/T}-1}(\omega^2-\omega_p^2)^{3/2}\ . \tag{48}$$

Remembering that the magnetic-field energy will make up roughly half of the total electromagnetic energy in high frequencies, and remembering that the magnetic energy density is found by integrating $\langle B^2\rangle_\omega$ over $d\omega$ and then dividing by 2π, we can see that the second term in this expression closely resembles the blackbody radiation spectrum at frequencies much greater than ω_p. In fact, if $\omega_p \to 0$, the entire expression reduces exactly to the correct blackbody spectrum.

This suggests a possible procedure: We break up the integral in Eq. (46) into two intervals. One interval runs from $|\mathbf{k}|=0$ to $|\mathbf{k}|=k_{\text{cut}}$. The other interval runs from $|\mathbf{k}|=k_{\text{cut}}$ to $|\mathbf{k}|=\infty$. (The choice of k_{cut} will be clarified below.) In the first interval, we keep η finite and treat the integrand exactly. In the second interval, we let $\eta \to 0$ and drop the zero-frequency part of the spectrum. The result, thus approximated, is

$$\frac{\langle B^2\rangle_{\omega}}{8\pi}=\frac{1}{\pi^2}\frac{\hbar\omega'}{e^{(\hbar\omega_{pe}/T)\omega'}-1}2\eta'\left[\frac{\omega_{pe}}{c}\right]^3\times\int_0^{x_{\text{cut}}}dx\frac{x^4}{(\omega'^2+\eta')x^4+\cdots}+\frac{\hbar(\omega'^2-\omega_p'^2)^{3/2}}{2\pi(e^{(\hbar\omega_{pe}/T)\omega'}-1)}\left[\frac{\omega_{pe}}{c}\right]^3\times\Theta[\omega'-(x_{\text{cut}}^2+\omega_p'^2)^{1/2}]\ , \tag{49}$$

where Θ is the Heaviside step function. The second term is the high-frequency and high-wave-number expression we obtained in Eq. (48). Elsewhere, we have referred to these two types of photons as soft and hard photons [9] and have dealt with the matter of the plasticity of photons [10]. The cutoff in integration removes the divergence. At $\omega=0$, we get

$$\lim_{\omega\to 0}\frac{\langle B^2\rangle_{\omega}}{8\pi}=\frac{2T}{\pi^2\eta}\left[\frac{\omega_{pe}}{c}\right]^3 x_{\text{cut}}\ . \tag{50}$$

At relativistic temperatures, this result is altered by a single factor of $1/\gamma$.

In reality, η should vanish smoothly as $k \to \infty$. However, as long as our results do not critically depend on the manner in which η approaches zero, the abrupt cutoff we suggest here should be acceptable as a crude model. We are interested in the contribution to the zero-frequency peak from long-wavelength fluctuations. So we will choose $x_{\text{cut}}=1$. This corresponds to $k_{\text{cut}}=\omega_p/c$, which, as can be inferred from the first term of Eq. (48), is the spatial correlation length of the zero-frequency fluctuations.

Plots of the spectrum expressed in Eq. (49) are shown in Fig. 2. These plots show the fluctuation spectrum in plasmas with parameters approximating the early Universe during the plasma epoch. Figures 2(a) and 2(b) represent the early universe at about 1 s after the Big Bang, Figs. 2(c) and 2(d) at about 10^8 s, and Figs. 2(e) and 2(f) at about 10^{12} s. Note that the rise in the zero-frequency peak is so sharp in the logarithmic-linear graphs [Figs. 2(a), 2(c), and 2(e)] that it is difficult to distinguish the peak from the vertical axes. (A break at the top of the graph indicates the height of the peak.) Note also that the log-log plots of Eq. (49) [Figs. 2(b), 2(d), and 2(f)] clearly show the ω^{-2} behavior at the low-frequency end of the spectrum. This is characteristic of the Lorentzian tail found in Appendix A.

An alternative method exists for ensuring the convergence of the integral in Eq. (46). It will prove to be unsatisfactory, but we mention it here for completeness. We go back to our original equation of motion and include viscosity:

$$\frac{d\mathbf{v}_\alpha}{dt}=e_\alpha\mathbf{E}-\eta_\alpha\mathbf{v}_\alpha+\mu_\alpha\nabla^2\mathbf{v}_\alpha\ . \tag{51}$$

We can now make the substitution $\eta \to \eta+\mu k^2$ in Eqs. (38)–(46). Doing this, we find

$$\frac{\langle B^2\rangle_{\omega}}{8\pi}=\frac{\hbar}{\pi^2}\frac{\omega'}{e^{(\hbar\omega_{pe}/T)\omega'}-1}\left[\frac{\omega_{pe}}{c}\right]^3\frac{\omega_p'^2}{\omega_{pe}}\int_0^\infty\frac{(\eta'+\mu'x^2)x^4}{\omega'^2(\omega'^2-x^2-\omega_p'^2)^2+(\eta'+\mu'x^2)^2(\omega'^2-x^2)^2}dx\ . \tag{52}$$

We now have an integrand which varies as $1/x^2$ as $x \to \infty$. We therefore have a convergent integral. We also have a modified value of the magnetic fluctuation strength at $\omega=0$:

$$\frac{\langle B^2\rangle_\omega}{8\pi}=T^2\frac{\omega_p'^2}{\omega_{pe}}\left[\frac{\omega_{pe}}{c}\right]^3\left[\frac{1}{\mu'\eta'}\right]\left[\mu'=\mu\frac{\omega_{pe}}{c'^2}\right]. \quad (53)$$

If we take $\mu=0.73T/\eta m$ [11], we find

$$\frac{\langle B^2\rangle_\omega}{8\pi}=\frac{0.85}{\pi}\frac{T}{\omega_{pe}}\left[\frac{\omega_{pe}}{c}\right]^3\frac{1}{\eta'}\left[\frac{T}{m_ec^2}\right]^{1/2}. \quad (54)$$

We see that $\langle B^2\rangle_\omega/8\pi$ still has a $1/\eta$ dependence, but its dependence on temperature has changed: It is now proportional to $T^{3/2}$.

As has been stated, however, this reliance on viscosity to produce a convergent integral is unacceptable. What is needed is a means of modifying the integrand of Eq. (46) which does not alter the blackbody spectrum at high frequencies and wave numbers, where the plasma should have less and less effect on the electromagnetic spectrum. Viscosity does not do the job: Including viscosity in the above manner puts terms in $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$ which *increase* in importance as the wave vector increases, thus modifying

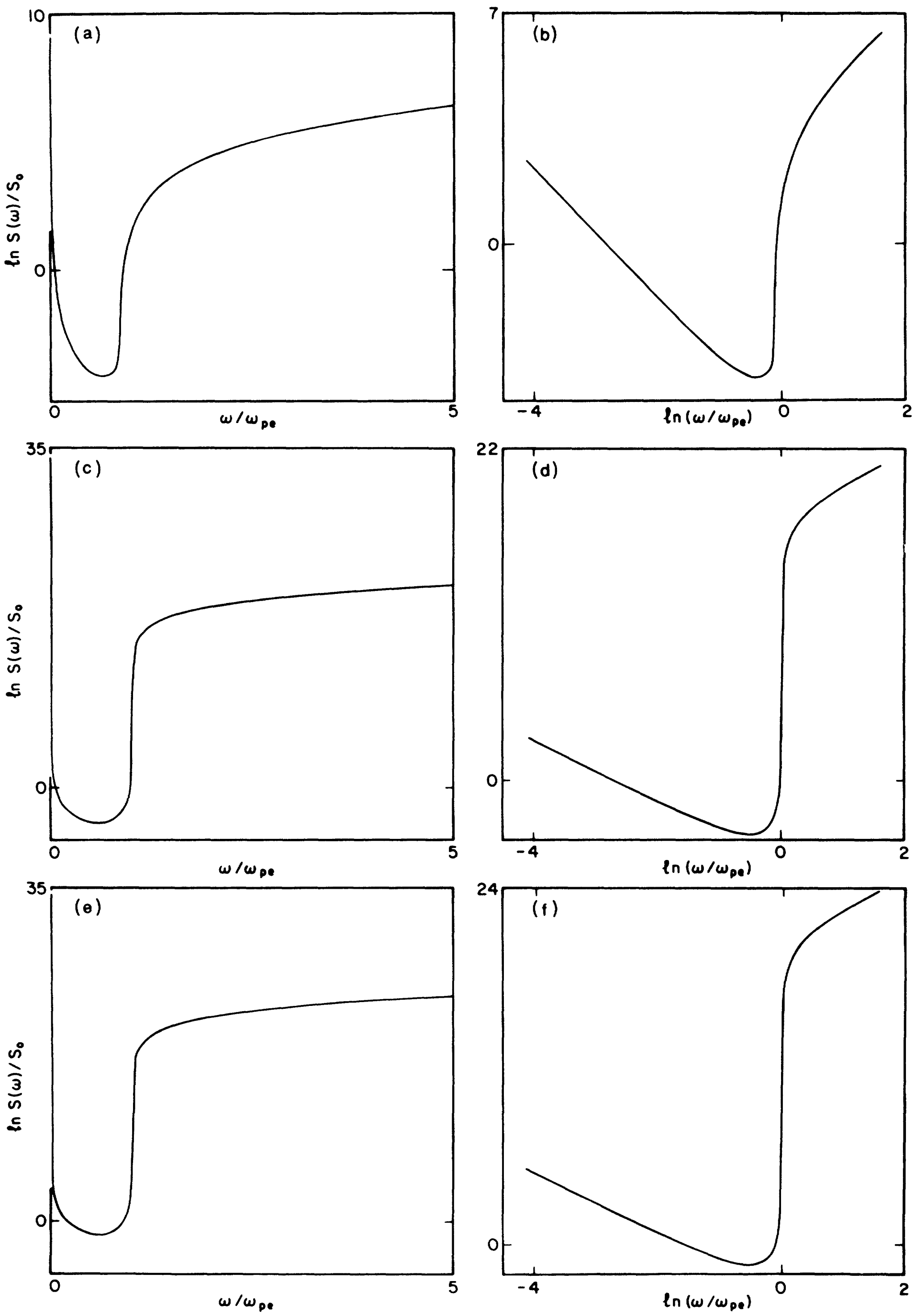

FIG. 2. The spectral intensity of magnetic fields $S(\omega)=\langle B^2\rangle_\omega/8\pi$ in thermal equilibrium, nonmagnetized plasma: (a) Corresponding to the plasma 1 s after the Big Bang. $T=10^{10}$ K; $n_e=4.8\times10^{30}/\text{cm}^3$). $\ln[S(\omega)/S_0]$ is plotted linearly in ω. The zero-frequency peak is at the top of the graph, where S_0 is the normalization. (b) Corresponding to the plasma 1 s after the Big Bang. $T=10^{10}$ K; $n_e=4.8\times10^{30}/\text{cm}^3$. $\ln[S(\omega)/S_0]$ plotted logarithmically in ω. The low-frequency line has a slope around -2, which rises to a peak at $\omega=0$. (c) Corresponding to the plasma 10^8 s after the Big Bang. $T=10^6$ K; $n_e=6.5\times10^9/\text{cm}^3$. $\ln[S(\omega)/S_0]$ plotted linearly in ω. The zero-frequency peak is at the top of the graph. (d) Corresponding to the plasma 10^8 s after the Big Bang. $T=10^6$ K; $n_e=6.\times10^9/\text{cm}^3$. $\ln[S(\omega)/S_0]$ plotted linearly in ω. The slope of the low-frequency line is ~-2. It rises to a peak at $\omega=0$. (e) Corresponding to the plasma 10^{12} s after the Big Bang. $T=10^4$ K; $n_e=6.5\times10^3/\text{cm}^3$. $\ln S(\omega)/S_0$ is plotted linearly in ω. The zero-frequency peak is at the top of the graph. (f) Corresponding to the plasma 10^{12} s after the Big Bang. $T=10^4$ K; $n_e=6.5\times10^3/\text{cm}^3$. $\ln S(\omega)/S_0$ is plotted logarithmically in ω. The slope of the low-frequency line is around -2. It continues to rise until peaking at $\omega=0$.

the blackbody spectrum. Perhaps, at low frequencies and wave numbers, viscosity should be included for higher accuracy. However, it does not solve any basic problems of the theory outlined thus far, nor does it lead to a qualitatively different shape of $\langle B^2\rangle_\omega/8\pi$ at low frequencies, so we will dispense with it from here on.

We have, so far, concentrated our efforts on electron-positron plasmas. We say a few words about plasmas with one major ion species. An analysis similar to what we have done in Eqs. (38)–(42) shows that the dielectric tensor of such a plasma may be given as

$$\epsilon_{ij}(\omega,\mathbf{k})=\delta_{ij}-\frac{\omega_{pe}^2}{\omega(\omega+i\eta_e)}\delta_{ij}-\frac{\omega_{pi}^2}{\omega(\omega+i\eta_i)}\delta_{ij} \ . \tag{55}$$

From this we find that, when $\omega\to 0$,

$$\frac{\langle B^2\rangle_\omega}{8\pi}=\frac{T}{\pi^2c^2}\left[\frac{\omega_{pe}^2}{\eta_e}+\frac{\omega_{pi}^2}{\eta_i}\right]k_{\text{cut}} \ . \tag{56}$$

In an equilibrium hydrogen plasma, $\omega_{pe}^2\approx 200\omega_{pi}^2$. Also,

$$\eta_e=2.91\times 10^{-6}n_e\ln\Lambda T^{-3/2}\ \mathrm{s}^{-1}$$

and

$$\eta_i=4.78\times 10^{-18}n_e\ln\Lambda T^{-3/2}\ \mathrm{s}^{-1}$$

(Ref. [12]). The ratio between the first and second terms in Eq. (56) is approximately 16.4. Therefore, ion motion raises the value of the $\omega=0$ peak by about 6% of the value it would have if the ions were frozen.

We turn our attention to the wave-number spectrum of magnetic fluctuations, i.e., $\langle B^2\rangle_k/8\pi$. This spectrum is found by integrating Eq. (45) over frequency. The Planck factor $(e^{\hbar\omega}-1)^{-1}$ makes the integral difficult. However, we can find an exact result in the limit $\eta\to 0$:

$$\begin{aligned}\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{2\hbar\omega}{e^{\hbar\omega/T}-1}\omega_p^2c^2k^2\pi\delta\left[\frac{\omega(\omega^2-c^2k^2-\omega_p^2)}{\omega^2-c^2k^2}\right]\frac{1}{(\omega^2-c^2k^2)^2}\\&=\int_0^{\infty}\frac{d\omega}{2}\,\frac{2\hbar\omega}{e^{\hbar\omega/T}-1}\omega_p^2c^2k^2\{\delta(\omega)+\delta[\omega\pm(c^2k^2+\omega_p^2)^{1/2}]\}\\&\quad\times\frac{1}{|(\omega^2-c^2k^2)(3\omega^2-c^2k^2-\omega_p^2)-2\omega^2(\omega^2-c^2k^2-\omega_p^2)|} \ .\end{aligned} \tag{57}$$

After integration, we obtain

$$\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}=\frac{\hbar c^2k^2}{(e^{\hbar(\omega_p^2+c^2k^2)^{1/2}/T}-1)}\,\frac{1}{(\omega_p^2+c^2k^2)^{1/2}}+T\frac{\omega_p^2}{\omega_p^2+c^2k^2} \ . \tag{58}$$

The second term of this expression has the same physical source as the first term of the right-hand side of Eq. (48), namely, the zero-frequency fluctuations. The magnetic-field energy contained in these fluctuations can be found by integrating the second term of the present expression over d^3k and dividing by $(2\pi)^3$, or by integrating over the first term of Eq. (48) over $d\omega$ and dividing by 2π. The result given by the two methods will be identical regardless of the value of k_{cut}. (Note that, once again, the limit $\omega_p\to 0$ gives the standard blackbody radiation spectrum.) The first term in this expression is clearly the blackbody spectrum modified by the plasma. The second term was obtained by Geary *et al.* [7]. They obtained this term via the Darwin approximation, i.e., by neglecting radiation. Therefore, our result satisfies both radiative and nonradiative limits.

Notice that, in the classical limit $\hbar(\omega_p^2+c^2k^2)^{1/2}\ll T$, the two terms of Eq. (58) add together to yield the well-known equipartition law

$$\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}\to T \ . \tag{59}$$

Remembering that we have obtained this expression by summing over both polarizations of the magnetic field, we see that we have satisfied the equipartition law of classical statistical mechanics. This raises an interesting point. The first term in Eq. (58) is the contribution to the magnetic fluctuation spectrum from the standard, cold-plasma, electromagnetic waves. The second term, as has been stated above, is a contribution from some kind of nonradiative fluctuation in the electromagnetic field. The standard cold-plasma waves do not satisfy the classical equipartition law. The cold-plasma equations do not allow any other plasma wave. Therefore, it would seem that the only possibility for this "missing energy" would be in $\omega=0$ fluctuations qualitatively similar to those discussed here.

We end this section with some brief observations. The energy under the $\omega=0$ peak shows itself in the wave-number spectrum by way of the second term in Eq. (58). The total energy under this peak is on the order of $T(\omega_p/c)^3 1/(6\pi)^2$. The energy lost to the blackbody spectrum because of the plasma can be approximated by the Rayleigh-Jeans formula

$$T\int_0^{\omega_p}\frac{d\omega}{2\pi^2}\,\frac{\omega^2}{c^3}=\frac{1}{6\pi^2}T\left[\frac{\omega_p}{c}\right]^3 \ . \tag{60}$$

The energy under the $\omega=0$ peak is approximately equal to the energy cutoff from the blackbody spectrum. Figuratively, we can say the plasma squeezes the fluctuation energy of modes with frequencies less than ω_p into modes with frequencies very close to zero.

A word should be said about the effects of these low-frequency magnetic fields on the equations of motion of the plasma. Specifically, is it justifiable to neglect the nonlinear $(\mathbf{v}/c)\times\mathbf{B}$ force in the equation of motion, as has been done in Eq. (38)? Towards answering this question, we will examine the magnetic power spectra of a plasma with an imposed magnetic field in Sec. IV. First, though, it can be shown that the ratio of the $(\mathbf{v}/c)\times\mathbf{B}$ to the electric-field force is typically of the order of $v/v_{\rm ph}\approx v/c_s$, where v is the fluid velocity, $v_{\rm ph}$ is the phase velocity, and c_s is the sound speed. This is usually much less than unity.

We also make more detailed estimates about particle motion in the isotropic plasma. A typical value of the spontaneously generated magnetic field has been found to be $B\approx[8\pi T(\omega_p/c)^3]^{1/2}$. The correlation length of these fields is c/ω_p. If a particle in a nonrelativistic plasma has a velocity which is some fraction ζ of the thermal velocity, i.e., $v=\zeta\sqrt{T/m_\alpha}$ (α being the particle species label), then the Larmor radius of the particle is given by

$$\rho_L=v/(eB/mc)=\zeta\frac{c}{\omega_p}\frac{1}{(8\pi)^{1/2}}\frac{m_\alpha^{1/2}c^{3/2}}{e\omega_p^{1/2}} .$$

In the late radiation epoch, and in most astrophysical and laboratory plasmas in existence today, the Larmor radius of a typical plasma particle (i.e., $\zeta\approx1$) is much larger than the length of a typical region of constant magnetic field, i.e., c/ω_p. Only for the coldest of the particles would it be necessary to consider the $\mathbf{v}\times\mathbf{B}$ force. It would seem, then, that Eq. (38) and results following from it are valid in these contexts.

The plasma of the early radiation epoch may be a different story, however. The Larmor radius of a typical electron will be

$$\rho_L\approx c/(eB/m\gamma c)=\frac{c}{\omega_p}\frac{1}{(8\pi)^{1/2}}\frac{m\gamma c^{5/2}}{T^{1/2}e\omega_p^{1/2}} .$$

The factor multiplying the collisionless skin depth is now on the order of 10. It would seem that the equation of motion which has been used suffices for a crude estimate of particle motion, but the introduction of the nonlinear $(\mathbf{v}/c)\times\mathbf{B}$ force would have non-negligible effects.

IV. FLUCTUATIONS WITH AN IMPOSED MAGNETIC FIELD

We began our study of magnetic-field fluctuations in an electron-positron plasma by assuming a cold-plasma equation of motion, Eq. (38). In adopting this equation, we assumed that the effects of magnetic fields *on* plasma motion were small. Yet, when the fluctuation spectrum of the magnetic field is calculated, a zero-frequency magnetic "fluctuation" is found. This "fluctuation" can be quite large, depending on the parameters of the plasma. We begin with an equation of motion which takes no account of magnetic fields, and we end with a plasma which has a temporally fairly constant (though far from spatially uniform) magnetic field which the plasma has "imposed on itself." This may be looked upon as an example of spontaneous breakdown of symmetry. Should this (nearly) constant (but tangled) magnetic field have been included in the original equation of motion? If it had been included, would the fluctuation spectra turn out to be much the same, or do we have a contradiction here? Toward resolving this quandary, we now attempt a calculation of interest in its own right. Namely, we find the magnetic fluctuation spectrum of a plasma with an imposed, temporally constant, spatially uniform magnetic field.

We take the equation of motion of our plasma to be

$$m_\alpha\frac{d\mathbf{v}_\alpha}{dt}=e_\alpha\mathbf{E}+e_\alpha\frac{\mathbf{v}}{c}\times\mathbf{B}_0-\eta_\alpha m_\alpha\mathbf{v}_\alpha , \tag{61}$$

where $\mathbf{B}_0=B_0\hat{\mathbf{z}}$. Admittedly, this constant magnetic field will not capture the complexity of the spontaneous zero-frequency field that was calculated in Sec. III. So it is best to regard the following analysis as a qualitative, rather than thoroughly quantitative, consistency check of the calculations which we have already completed. We continue on with this proviso in mind.

The simultaneous presence of collisions and magnetic field in the equation of motion will complicate our algebra. For the time being, we drop the collisional term from the equation of motion. As long as we are dealing with plasma waves in which $\mathbf{v}_j$ is largely perpendicular to $\mathbf{B}_0$, this approximation amounts to ignoring η in favor of $\Omega_j=|e_jB_0/m_jc|$. However, when we deal with modes in which $\mathbf{v}_j$ is purely parallel to $\mathbf{B}_0$, collisionality must be reintroduced.

Our first step in calculating the magnetic-field fluctuation spectrum is to set $\mathbf{k}=k_y\hat{\mathbf{y}}+k_z\hat{\mathbf{z}}$. We then find the dielectric-permittivity tensor to be

$$\Lambda=\begin{vmatrix} K_\perp-\dfrac{c^2k^2}{\omega^2} & -iK_\times & 0 \\ iK_\times & K_\perp-\dfrac{c^2k^2}{\omega^2}\cos^2\theta & \dfrac{c^2k^2}{\omega^2}\sin\theta\cos\theta \\ 0 & \dfrac{c^2k^2}{\omega^2}\sin\theta\cos\theta & K_\parallel-\dfrac{c^2k^2}{\omega^2}\sin^2\theta \end{vmatrix} , \tag{62}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$, and

$$K_\perp=1-\frac{\omega_{pi}^2}{\omega^2-\Omega_i^2}-\frac{\omega_{pe}^2}{\omega^2-\Omega_e^2} ,$$

$$K_\times=-\frac{\omega_{pi}^2}{\omega^2-\Omega_i^2}\frac{\Omega_i}{\omega}+\frac{\omega_{pe}^2}{\omega^2-\Omega_e^2}\frac{\Omega_e}{\omega} ,$$

$$K_\parallel=1-\frac{\omega_{pi}^2}{\omega^2}-\frac{\omega_{pe}^2}{\omega^2} .$$

(Here, Ω_e has been taken equal to $|\Omega_e|$.) The results of the damped equation of motion can be recovered by making the substitutions [13]

$$\omega_{p\alpha}\to\omega_{p\alpha}\frac{\omega}{\omega+i\eta},\quad \Omega_\alpha\to\Omega_\alpha\frac{\omega}{\omega+i\eta} .$$

As in the previous sections, we use the inverse of this tensor, i.e., Λ^{-1}, to calculate the fluctuation spectra of the magnetic field:

$$\frac{\langle B_x^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{i}{2}\frac{\hbar}{e^{\hbar\omega/T}-1}\frac{c^2}{\omega^2}[k_y^2\Lambda_{33}^{-1}+k_z^2\Lambda_{22}^{-1}-k_yk_z(\Lambda_{23}^{-1}+\Lambda_{32}^{-1})-\text{c.c.}]\ , \tag{63}$$

$$\frac{\langle B_y^2\rangle_{\mathbf{k}\omega}}{8\pi}-\frac{i}{2}\frac{\hbar}{e^{\hbar\omega/T}-1}\frac{c^2}{\omega^2}(k_z^2\Lambda_{11}^{-1}-\text{c.c.})\ , \tag{64}$$

$$\frac{\langle B_z^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{i}{2}\frac{\hbar}{e^{\hbar\omega/T}-1}\frac{c^2}{\omega^2}(k_y^2\Lambda_{11}^{-1}-\text{c.c.})\ . \tag{65}$$

Before we calculate the spectra, we will make several observations which will make the calculations much simpler. First, we note that the sum of all of the magnetic-field energy can be written as follows:

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{i}{2}\frac{\hbar}{e^{\hbar\omega/T}-1}\frac{c^2}{\omega^2}\epsilon_{inp}\epsilon_{ilm}k_nk_l(\Lambda_{pm}^{-1}-\Lambda_{mp}^{-1*})$$
$$=\frac{i}{2}\frac{\hbar}{e^{\hbar\omega/T}-1}\frac{c^2}{\omega^2}(k^2\delta_{pm}-k_pk_m)\times(\Lambda_{pm}^{-1}-\Lambda_{mp}^{-1*})\ , \tag{66}$$

where ϵ_{ilm} is the fully antisymmetric tensor. Next, let us define the tensor λ_{ij} by the relationship

$$\lambda_{ij}=\det(\Lambda)\Lambda_{ij}^{-1}\ . \tag{67}$$

Then,

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{i}{2}\frac{\hbar}{e^{\hbar\omega/T}-1}\frac{c^2}{\omega^2}(k^2\delta_{pm}-k_pk_m)\left[\left[\frac{\lambda_{pm}}{\det(\Lambda)}\right]-\left[\frac{\lambda_{mp}}{\det(\Lambda)}\right]^*\right]$$
$$=\frac{\pi\hbar}{e^{\hbar\omega/T}-1}\frac{c^2}{\omega^2}(k^2\delta_{pm}-k_pk_m)\lambda_{pm}\delta(\det(\Lambda))\ ,$$
$$=\frac{\pi\hbar}{e^{\hbar\omega/T}-1}\frac{c^2}{\omega^2}(k^2\delta_{pm}-k_pk_m)\lambda_{pm}\frac{\sum_i\delta(\omega-\omega_i(\mathbf{k}))}{|(\partial/\partial\omega)\det(\Lambda)|}\ , \tag{68}$$

where $\{\omega_i(\mathbf{k})\}$ is the set of roots of the equation $\det[\Lambda(\mathbf{k},\omega)]=0$.

The cold-plasma-model dielectric tensor is independent of $\mathbf{k}$: $\epsilon_{ij}=\epsilon_{ij}(\omega)$. Therefore,

$$\Lambda_{ij}=\left[1-\frac{c^2}{\omega^2}\right]\delta_{ij}+\frac{c^2}{\omega^2}k_ik_j+\epsilon_{ij}(\omega)\ . \tag{69}$$

It follows that

$$\frac{\partial\Lambda_{ij}}{\partial k}=-2\frac{c^2}{\omega^2}k\delta_{ij}+2\frac{c^2}{\omega^2}\frac{k_ik_j}{k}\ , \tag{70}$$

where $k=|\mathbf{k}|$. When we substitute this result into Eq. (68), we find

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{\pi\hbar}{e^{\hbar\omega/T}-1}\left[-\frac{k}{2}\right]\frac{\partial_k(\Lambda_{ij})\lambda_{ij}}{|(\partial/\partial\omega)\det(\Lambda)|}\times\sum_i\delta(\omega-\omega_i(\mathbf{k}))\ . \tag{71}$$

A straightforward calculation shows that, given any 3×3 matrix A, whose elements depend on some set of parameters $x,y,\ldots$, the derivative of the determinant of A with respect to any one of these parameters is

$$\partial_x\{\det[A(x,y,\ldots)]\}=\partial_xA_{ij}(x,y,\ldots)a_{ji}(x,y,\ldots)\ , \tag{72}$$

where $a_{ij}(x,\ldots)$ is the matrix whose elements are made up of the cofactors of $A_{ij}(x,\ldots)$ and repeated indices represent summation. This result holds true for $\Lambda_{ij}(\omega,\mathbf{k})$, yielding from Eq. (71), because of the symmetry of $\partial_k(\Lambda_{ij}(\omega,\mathbf{k}))$,

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{\pi\hbar}{e^{\hbar\omega/T}-1}\left[-\frac{k}{2}\right]\frac{\partial_k[\det(\Lambda)]}{|(\partial/\partial\omega)\det(\Lambda)|}\times\sum_i\delta(\omega-\omega_i(\mathbf{k}))\ . \tag{73}$$

What we have here is a sum of the magnetic fluctuation intensities of all modes. The magnetic fluctuations of a given particular mode are characterized by the dispersion relation

$$\det[\Lambda(\mathbf{k},\omega)]=0\ .$$

Where this function is equal to zero in $\mathbf{k}$-ω space, we have the wave vector and accompanying frequency of a propagating wave. If we infinitesimally vary $\mathbf{k}$ and ω so as to remain on a given surface in $\mathbf{k}$-ω space where $\det(\Lambda)=0$, we must vary $\mathbf{k}$ and ω such that

$$0=\Delta\det[\Lambda(\mathbf{k},\omega)]$$
$$=\Delta\mathbf{k}\cdot\partial_{\mathbf{k}}[\det(\Lambda)]+\Delta\omega\partial_\omega[\det(\Lambda)]\ ,$$

which implies

$$\frac{\partial\omega}{\partial\mathbf{k}}=-\frac{\partial_{\mathbf{k}}[\det(\Lambda)]|_{\omega=\omega(\mathbf{k})}}{\partial\omega[\det(\Lambda)]|_{\omega=\omega(\mathbf{k})}}\ . \tag{74}$$

This implies, further, that

$$\frac{\partial\omega}{\partial k}=-\frac{\partial_k[\det(\Lambda)]|_{\omega=\omega(\mathbf{k})}}{\partial_\omega[\det(\Lambda)]|_{\omega=\omega(\mathbf{k})}}\ . \tag{75}$$

This result will be independently true for each surface in $\mathbf{k}$-ω space on which $\det[\Lambda(\mathbf{k},\omega)]$ equals zero. This means it will be true independently for each propagating mode.

We can substitute this result into Eq. (73), finding that the total magnetic fluctuation strength for a given mode is

$$\left.\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}\right|_{\text{mode } i}=\delta(\omega-\omega_i(\mathbf{k}))\frac{\pi\hbar}{e^{\hbar\omega_i/T}-1}\left[\frac{k}{2}\right]\left[\frac{\partial\omega_i}{\partial k}\right] . \tag{76}$$

$\langle B^2\rangle_{\mathbf{k}}/8\pi$ is found by integrating $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$ over $d\omega$ and dividing by 2π. For a given mode, it is

$$\left.\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}\right|_{\text{mode } i}=\frac{\hbar}{e^{\hbar\omega_i/T}-1}\left[\frac{k}{2}\right]\left[\frac{\partial\omega_i}{\partial k}\right] . \tag{77}$$

[In deriving this expression, one must remember that, wherever ω enters $\det(\Lambda)$, it enters in an even power. This means $\det(\Lambda)$ is even in ω and, for a given mode i, the contribution to the sum from frequency ω_i is matched by the contribution from $-\omega_i$.] In the limit $\hbar\rightarrow 0$, this becomes

$$\left.\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}\right|_{\text{mode } i}=\frac{T}{2}\frac{k}{\omega_i(\mathbf{k})}\left[\frac{\partial\omega_i}{\partial k}\right] . \tag{78}$$

We can write this as

$$\left.\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}\right|_{\text{mode } i}=\frac{T}{2}\frac{\mathbf{v}_{\text{ph } i}(\mathbf{k})\cdot\mathbf{v}_{g\ i}(\mathbf{k})}{v^2_{\text{ph } i}(\mathbf{k})} , \tag{79}$$

where $\mathbf{v}_{\text{ph } i}(\mathbf{k})$ is the phase velocity of a wave of mode i with wave vector $\mathbf{k}$, and $\mathbf{v}_{g\ i}(\mathbf{k})$ is the group velocity of the wave.

Because of Faraday's law, it must also be true that the fluctuation spectrum of transverse electric fields is given for each mode by

$$\left.\frac{\langle E^2_\perp\rangle_{\mathbf{k}}}{8\pi}\right|_{\text{mode } i}=\frac{1}{2}\frac{\hbar}{e^{\hbar\omega_i/T}-1}\frac{[\omega_i(\mathbf{k})]^2}{c^2k}\left[\frac{\partial\omega_i}{\partial k}\right] . \tag{80}$$

In the limit $\hbar\rightarrow 0$, this gives

$$\left.\frac{\langle E^2_\perp\rangle_{\mathbf{k}}}{8\pi}\right|_{\text{mode } i}=\frac{T}{2}\frac{\omega_i(\mathbf{k})/k}{c^2}\left[\frac{\partial\omega_i}{\partial k}\right]=\frac{T}{2}\frac{\mathbf{v}_{\text{ph } i}(\mathbf{k})\cdot\mathbf{v}_{g\ i}(\mathbf{k})}{c^2} . \tag{81}$$

We should note that these results are not valid without limit. If there is a zero-frequency mode in the plasma, $\partial\omega/\partial k$ will be zero, while k/ω will be infinite. In this case, we cannot use Eq. (78) to calculate the magnetic fluctuation spectrum of this mode, unless we can make use of some sort of limiting procedure to take care of the product of zero with infinity. However, on the other hand, Eq. (81) tells us unequivocally that the transverse electric-field intensity of a zero-frequency mode will be zero.

We have, in general, made the task of calculating the magnetic fluctuation spectrum much simpler: If we have the functional form of the dispersion relation of a given mode, we can easily calculate the magnetic and transverse electric fluctuation spectra.

We proceed to calculate the magnetic fluctuation spectrum for an electron-positron plasma. Since the masses of the particles are equal and the charges are exactly opposite,

$$\omega_{e+}=\omega_{e-}=\omega,\quad \omega_{pe-}=\omega_{pe+}=\omega_p/\sqrt{2}$$

and

$$K_\perp=1-\frac{\omega_p^2}{\omega^2-\Omega^2},\quad K_\parallel=1-\frac{\omega_p^2}{\omega^2},\quad K_\times=0 .$$

This all means

$$\Lambda=\begin{bmatrix} K_\perp-\dfrac{c^2k^2}{\omega^2} & 0 & 0 \\ 0 & K_\perp-\dfrac{c^2k^2}{\omega^2}\cos^2\theta & \dfrac{c^2k^2}{\omega^2}\sin\theta\cos\theta \\ 0 & \dfrac{c^2k^2}{\omega^2}\sin\theta\cos\theta & K_\parallel-\dfrac{c^2k^2}{\omega^2}\sin^2\theta \end{bmatrix} . \tag{82}$$

There are five distinct modes in the plasma. Two of them solve the dispersion relation

$$K_\perp-\frac{c^2k^2}{\omega^2}=0 . \tag{83}$$

The other three solve

$$\left[K_\perp-\frac{c^2k^2}{\omega^2}\cos^2\theta\right]\left[K_\parallel-\frac{c^2k^2}{\omega^2}\sin^2\theta\right]-\left[\frac{c^2k^2}{\omega^2}\right]^2\sin^2\theta\cos^2\theta=0 . \tag{84}$$

The first two modes have their electric fields polarized purely in the x direction (that is, perpendicular to both $\mathbf{B}_0$ and the direction of propagation). This is evident from the requirement that $\Lambda_{ij}E_j=0$. The magnetic fields of these modes lie in the plane common to $\mathbf{B}_0$ and the wave vector. Note that the dispersion relations of these waves are dependent on the magnitude of $\mathbf{B}_0$ but are not dependent on the direction of propagation. The dispersion relations of these modes are plotted in Fig. 3(a). $\langle B^2\rangle_{\mathbf{k}}/8\pi$ is plotted for each mode in Fig. 3(b).

The other three modes have their electric fields polarized in the plane common to $\mathbf{B}_0$ and the wave vector. Their magnetic fields lie in the x direction, that is, perpendicular to both $\mathbf{B}_0$ and the wave vector. The dispersion relations and fluctuation spectra of these modes are plotted in Figs. 4(a1)–4(d2) for various directions of propagation relative to $\mathbf{B}_0$.

Are there any zero-frequency modes which we may have overlooked because of our neglect of damping in the equation of motion? Also, even if these modes do not exist, is there a finite amount of fluctuation energy in the magnetic field when ω is very small or even equal to zero?

A glance at Fig. 3(b) will show that the total-energy density per $\mathbf{k}$-space volume deposited in the first two modes we mentioned is $T/2$. We thus conclude that, if there is a "hidden" zero-frequency mode, it must be among the second set of modes we have mentioned. That is, it must be polarized so that its magnetic field lies perpendicular to both $\mathbf{B}_0$ and $\mathbf{k}$. We look at the remaining three modes, all thus polarized. We can see from Fig. 6 that, for almost all angles, $\langle B_x^2\rangle_{\mathbf{k}}/8\pi$ added over all three modes gives $T/2$, regardless of $\mathbf{k}$. We conclude again, that, in general, there are no hidden $\omega=0$ modes.

There is one exception to this rule: It occurs when $\theta=\pi/2$, that is, when the wave is propagating perpendicularly to $\mathbf{B}_0$. In this case, we find

$$\Lambda=\begin{bmatrix} 1-\dfrac{c^2k^2}{\omega^2}-\dfrac{\omega_p^2}{\omega^2-\Omega^2} & & \\ & 1-\dfrac{\omega_p^2}{\omega^2-\Omega^2} & \\ & & 1-\dfrac{c^2k^2}{\omega^2}-\dfrac{\omega_p^2}{\omega^2} \end{bmatrix} . \tag{85}$$

$\langle B_x^2\rangle_{\mathbf{k}\omega}/8\pi$ is calculated from Λ_{33}^{-1}:

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{i}{2}\frac{\hbar}{e^{\hbar\omega/T}-1}\frac{c^2k^2}{\omega^2}(\Lambda_{33}^{-1}-\text{c.c.})=\frac{\hbar}{e^{\hbar\omega/T}-1}\frac{c^2k^2}{\omega^2}\pi\delta\left[1-\frac{c^2k^2}{\omega^2}-\omega_p^2\omega^2\right]$$
$$=\frac{\pi\hbar}{e^{\hbar\omega/T}-1}\frac{c^2k^2|\omega|}{2(c^2k^2+\omega_p^2)}\{\delta[\omega-(c^2k^2+\omega_p^2)]+\delta[\omega+(c^2k^2+\omega_p^2)]\} . \tag{86}$$

$\langle B_x^2\rangle_{\mathbf{k}}/8\pi$ is found by integrating this expression over $d\omega$ and dividing by 2π:

$$\frac{\langle B_x^2\rangle_{\mathbf{k}}}{8\pi}=\frac{\hbar}{e^{(\hbar\omega_p^2+c^2k^2)^{1/2}/T}-1}\frac{1}{2}\frac{c^2k^2}{(c^2k^2+\omega_p^2)^{1/2}} . \tag{87}$$

In the limit $\hbar\to 0$, this becomes

$$\frac{\langle B_x^2\rangle_{\mathbf{k}}}{8\pi}=\frac{T}{2}\frac{c^2k^2}{c^2k^2+\omega_p^2} , \tag{88}$$

which is less energy than required by the equipartition law. We then look for a mode at $\omega=0$ which can be derived when damping is considered.

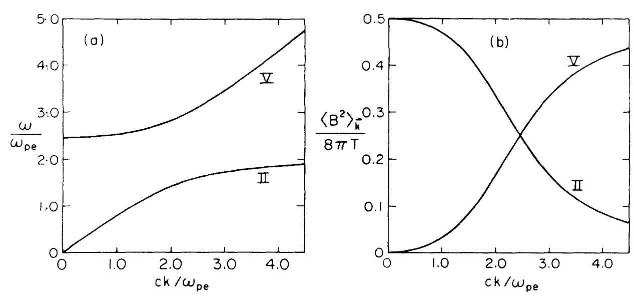

FIG. 3. The dispersion relations and magnetic-field fluctuation strengths for the two direction-independent modes of the electron-positron plasma in a uniform magnetic field: (a) Dispersion relations of the two direction-independent modes of the electron-positron plasma in a uniform magnetic field. Roman numerals label modes in increasing value of frequency. The modes shown here are labeled II and V. Modes I, III, and IV, being dependent on propagation direction, are shown in Fig. 4. (b) $\langle B^2\rangle_{\mathbf{k}}/8\pi$ of the two direction-independent modes. Roman numerals label corresponding modes in (a).

If we revive damping in the equation of motion, we find that

$$\Lambda_{33}=1-\frac{c^2k^2}{\omega^2}-\frac{\omega_p^2}{\omega(\omega+i\eta)} . \tag{89}$$

This implies that

$$\frac{\langle B_x^2\rangle_{\mathbf{k}}}{8\pi}=\frac{T}{2}\frac{\omega_p^2}{c^2k^2+\omega_p^2}+\frac{\hbar}{e^{(\hbar\omega_p^2+c^2k^2)^{1/2}/T}-1}\frac{1}{2}\frac{c^2k^2}{(c^2k^2+\omega_p^2)^{1/2}} . \tag{90}$$

The $\hbar\to 0$ limit is now

$$\frac{\langle B_x^2\rangle_{\mathbf{k}}}{8\pi}=\frac{T}{2}\left[\frac{\omega_p^2}{c^2k^2+\omega_p^2}+\frac{c^2k^2}{c^2k^2+\omega_p^2}\right] , \tag{91}$$

which satisfies the equipartition law.

Notice that the propagating mode represented in the second term is the ordinary mode. Its electric field is polarized in the direction of $\mathbf{B}_0$. This implies that the motion of the plasma itself is, in the linear regime, purely parallel to $\mathbf{B}_0$. This is the exceptional case which we noted at the beginning of this section. Using satisfaction of the equipartition law as our criterion, we have decided that this is the one case in need of special consideration of damping effects. Note also, as we can see from Fig. 4(d2), that the zero-frequency mode represented in the first term in the shear-Alfvén mode. We can see this by following the changes in the shear-Alfvén-wave dispersion relation and magnetic-field spectrum as the angle of propagation changes from $\theta=0$ to $\theta\to\pi/2$. The frequency of the Alfvén mode goes to zero for all k when

$\theta=\pi/2$. This is why it is necessary to consider the dissipative effect in this case; the wave's energy density per k-space volume remains finite. Agim and Prager [14] have found a similar effect in magnetized electron-ion plasmas.

Suppose that in a plasma in a uniform magnetic field, we turn the magnetic-field strength down. The dispersion relations of the various modes change. Some modes merge into each other, and the shear and compressional Alfvén waves become lower and lower in frequency. We examine the compressional Alfvén wave first. The dispersion relation for this wave is contained in the equation

$$K_\perp-\frac{c^2k^2}{\omega^2}=0\ ,$$

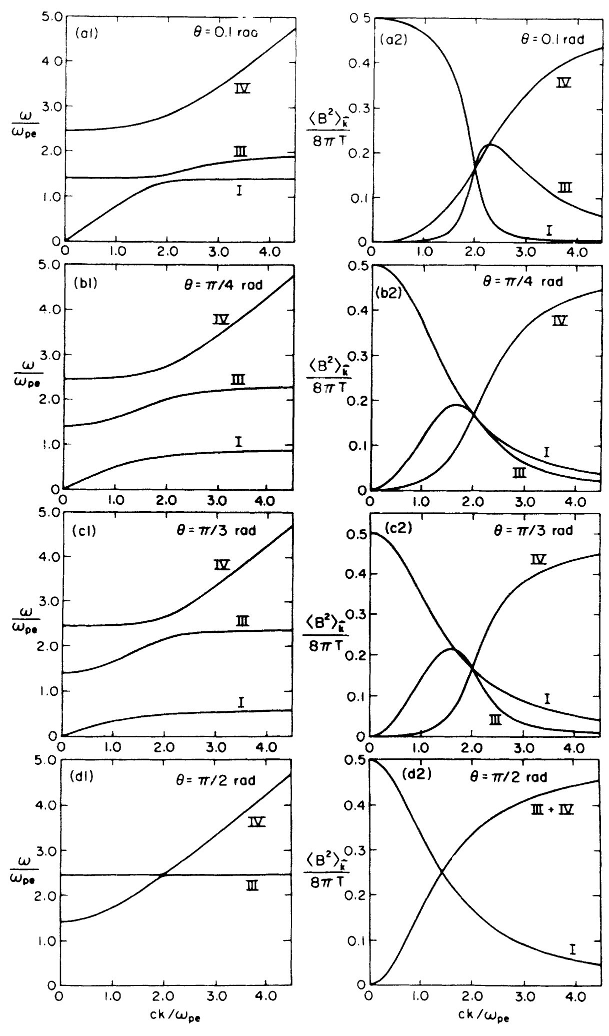

FIG. 4. Dispersion relations and magnetic-field fluctuation strengths for the three direction-dependent modes of the electron-positron plasma in a uniform magnetic field. Roman numerals label modes in order of increasing frequency. θ indicates angle between imposed magnetic field and angle of propagation. Note that the lowest-frequency branch (the shear Alfvén branch) is not plotted in d_1, since its frequency is identically zero when propagating perpendicularly to the magnetic field. However, as shown in d_2, it retains a finite amount of energy.

which can be written as follows:

$$\omega^4+\omega^2(-\Omega^2-\omega_p^2-c^2k^2)+\Omega^2c^2k^2=0\ . \tag{92}$$

To obtain the low-k dispersion relation, we assume ck and ω are small compared to Ω and ω_p. We find

$$\omega^2=\frac{\Omega^2}{\omega_p^2+\omega^2}c^2k^2\ . \tag{93}$$

To obtain the high-k dispersion relation, we assume ck and ω_p are much larger than ω and Ω. We then find

$$\omega^2=\frac{\Omega^2c^2k^2}{\omega_p^2+c^2k^2}\ . \tag{94}$$

The energy density per k volume contained in the magnetic field is, from Eq. (78),

$$\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}=\frac{T}{2}\ , \tag{95}$$

for small k, and

$$\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}=\frac{T}{2}\frac{\omega_p^2}{\omega_p^2+c^2k^2}\ , \tag{96}$$

for large k. Notice that $\langle B^2\rangle_{\mathbf{k}}/8\pi$ is independent of Ω. In the limit $\Omega\to 0$, $\langle B^2\rangle_{\mathbf{k}}/8\pi$ remains finite. Notice also that the spectrum is exactly equal to that of the zero-frequency mode of Sec. III.

Now we examine the shear Alfvén wave. At low k, the dispersion relation is

$$\omega^2=\frac{\Omega^2c^2k^2\cos^2\theta}{\Omega^2+\omega_p^2}\ . \tag{97}$$

At high k, it is

$$\omega^2=\frac{\Omega^2c^2k^2\cos^2\theta}{\omega_p^2+c^2k^2}\ . \tag{98}$$

The magnetic-energy spectrum at low k is obtained from Eq. (78) as

$$\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}=\frac{T}{2} \tag{99}$$

and, at high k, as

$$\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}=\frac{T}{2}\frac{\omega_p^2}{\omega_p^2+c^2k^2}\ . \tag{100}$$

The shear-Alfvén-wave-vector spectrum has exactly the same behavior as that of the compressional Alfvén wave. In particular, the spectrum is independent of Ω and, therefore, finite even if $\Omega=0$. Further, notice that the wave-vector spectra of both Alfvén modes are identical to that of the zero-frequency mode at high k.

Therefore, it may be possible to interpret the zero-frequency mode as a composite of the two Alfvén modes, which are static in the absence of an imposed magnetic field. The zero-frequency mode is a virtual Alfvén wave excitation that is spontaneously generated as the virtual excitation itself creates a magnetic field over a short period of lifetime. In quantum-mechanical terminology, this is usually called virtual particle creation. That is, the virtual Alfvén wave quantum (or magnon) is created in the absence of an external magnetic field, while in the presence of an external magnetic field, the (real) Alfvén wave quanta are excited.

Now we have to answer our second question: Is the fluctuation energy of the magnetic field finite even when ω is small or even zero? We answer this question by calculating $\langle B^2\rangle_\omega/8\pi$ for the two Alfvén modes. We could do this by going back to $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$ and integrating over k. But we use another method here. Since there is no damping, $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$ is made up of functions of ω and $\mathbf{k}$ multiplying a sum of Dirac δ functions, the arguments of which are also functions of ω and $\mathbf{k}$. This means that the energy density in a particular frequency interval $d\omega$ is distributed over a few well-defined, distinct surfaces in k space. This implies the following: We measure the energy of a given mode in a particular frequency band $d\omega$, centered on frequency ω_0. It has some value $(\langle B^2\rangle_\omega/8\pi)d\omega/2\pi$. This mode will have a single surface in k space for which $\omega(\mathbf{k})=\omega_0$. We study a differential volume surrounding this surface, a volume contained within the two surfaces defined by $\omega(\mathbf{k})=\omega_0-d\omega/2$ and $\omega(\mathbf{k})=\omega_0+d\omega/2$. The energy density contained in this differential volume must be equal to the energy density contained in the interval $d\omega$. Mathematically,

$$\frac{\langle B^2\rangle_{\omega=\omega_0}}{8\pi}\frac{d\omega}{2\pi}=d\omega\int\frac{dS}{(2\pi)^3}\frac{1}{|\nabla_{\mathbf{k}}\omega|}\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}\ , \qquad (101)$$

where the integral is performed over the k surface given by $\omega(\mathbf{k})=\omega_0$.

In this cylindrically symmetric system, by Fig. 5 and the accompanying caption we obtain

$$\frac{dS}{|\nabla_{\mathbf{k}}\omega|}=2\pi\frac{k^2\sin\theta\, d\theta}{|\partial_k\omega|}\ . \qquad (102)$$

Substituting this into Eq. (101) and dividing common factors out of both sides of the equation gives

$$\left.\frac{\langle B^2\rangle_\omega}{8\pi}\right|_{\text{mode } i}=\frac{1}{2\pi}\int d\theta(\sin\theta)\frac{k^2}{|\partial_k\omega|}\left.\frac{\langle B^2\rangle_{k(\theta,\omega)}}{8\pi}\right|_{\text{mode } i}. \qquad (103)$$

Combining this with Eq. (78), we see

$$\left.\frac{\langle B^2\rangle_\omega}{8\pi}\right|_{\text{mode } i}=\frac{1}{2\pi}\int d\theta(\sin\theta)\frac{T}{2}\left.\frac{k^3(\omega,\theta)}{\omega}\right|_{\text{mode } i}$$

$$=\frac{T}{4\pi\omega}\int d\theta(\sin\theta)k^3_{\text{mode } i}(\omega,\theta)\ . \qquad (104)$$

In a similar way, the perpendicular electric-field power spectrum can be shown to be

$$\left.\frac{\langle E_\perp^2\rangle_\omega}{8\pi}\right|_{\text{mode } i}=\frac{T\omega}{4\pi c^2}\int d\theta(\sin\theta)k_{\text{mode } i}(\omega,\theta)\ . \qquad (105)$$

(The integral is not necessarily performed over the full range $0\le\theta\le\pi$. For instance, it might turn out that waves propagating at angles greater than some angle θ_0 always have frequencies smaller than some frequency ω_0. In this case, if we wanted the fluctuation spectrum for $\omega=\omega_0$, we could integrate only from $\theta=0$ to $\theta=\theta_0$.)

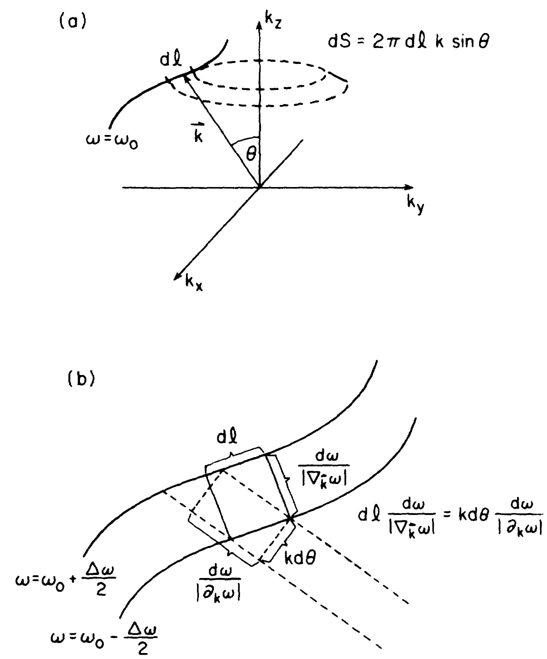

FIG. 5. Elements of integration: (a) The differential surface obtained from rotating a line element dl about the k_z axis is $dS=2\pi dlk\sin\theta$. (b) The areas of the two rectangles are equal for infinitesimal $d\theta$. Therefore, $dl/|\nabla_{\mathbf{k}}\omega|=kd\theta/|\partial_k\omega|$. So, the differential volume

$$dV=dSd\omega/|\nabla_{\mathbf{k}}\omega|=2\pi k^2 d\omega\sin\theta\, d\theta/|\partial_k\omega|\ .$$

We write down $\langle B^2\rangle_\omega/8\pi$ for the compressional Alfvén mode by making use of its dispersion relations, Eqs. (93) and (94) and Eq. (104). In the range of frequencies corresponding to small k,

$$\frac{\langle B^2\rangle_\omega}{8\pi}=\frac{T}{2\pi}\omega^2\left[\frac{(\omega_p^2+\Omega^2)^{1/2}}{c\Omega}\right]^3\ . \qquad (106)$$

In the range of frequencies corresponding to high k,

$$\frac{\langle B^2\rangle_\omega}{8\pi}=\frac{T}{2\pi}\left[\frac{\omega_p}{c}\right]^3\frac{\omega^2}{(\Omega^2-\omega^2)^{3/2}}\ , \qquad (107)$$

for the electron-positron case. Once again, in order to avoid a divergence $\langle B^2\rangle_\omega$ at $\omega=\Omega$, we need to use a cutoff in k.

For ω identically zero, the magnetic fluctuation energy is zero. However, there is a finite amount of energy density per frequency in low frequencies and, if we are justified in considering Ω a low frequency, then we have an infinite amount of low-frequency energy. Once again, we have run into a divergence problem. $\langle B^2\rangle_\omega/8\pi$ will diverge at $\omega=\Omega$. This is similar to the divergence in Eq. (22), though the degree of divergence in Eq. (110) is stronger. We will naively handle this problem by, again, introducing a cutoff in k. To estimate the total energy contained in shear Alfvén waves, Agim and Prager [14] used a cutoff of $k_{\rm cut}=\Omega_i/v_A=\omega_{pi}/c$ in an electron-ion plasma, where ω_{pi} is the ion plasma frequency.

We consider $\langle B^2\rangle_\omega/8\pi$ of the shear Alfvén mode. The directional dependence of the dispersion relation makes the calculation of $\langle B^2\rangle_\omega/8\pi$ a bit more difficult. In particular, we cannot divide up the frequency range into low-k and high-k ranges. For instance, looking at Eqs. (97) and (98), we can see that ω can become zero, no matter how large or small k is. However, in the case where $\omega^2 \ll \omega_p^2$, the dispersion relation Eq. (98) can take the place of Eq. (97) at low k without too great a loss of accuracy. We find $\langle B^2\rangle_\omega/8\pi$ by substituting $k(\omega,\theta)$ from Eq. (98) into Eq. (104). The result is

$$\frac{\langle B^2\rangle_\omega}{8\pi}=\frac{T}{2\pi}\left[\frac{\omega_p}{c}\right]^3\frac{1}{\omega}\left.\frac{-u}{(u^2-\omega^2/\Omega^2)^{1/2}}\right|_{\omega/\Omega}^{1}, \tag{108}$$

where u is a dummy variable which represented $\cos\theta$ in the $d\theta$ integral of Eq. (104). There is actually an infinite amount of energy at low frequencies. $\langle B^2\rangle_\omega/8\pi$ diverges at all frequencies less than ω.

Let us compare $\langle B^2\rangle_\omega/8\pi$ of the two Alfvén waves and the zero-frequency mode. First of all, they all have the same divergence problem, in differing degrees of severity. Secondly, they all scale by the factor $(\omega_p/c)^3$. It may be said that the imposed magnetic field creates Alfvén waves by taking energy out of a small frequency range enclosing $\omega=0$ and spreading it over a frequency range extending from $\omega=0$ to $\omega=\Omega$.

The effects of imposing a cutoff magnitude in wave vector $k_{\rm cut}$ are considered. We begin with the compressional Alfvén mode. From the dispersion relation, Eq. (94), we see that a cutoff in wave-vector magnitude implies a cutoff in frequency:

$$\omega^2<\omega_{\rm cut}^2=\frac{\Omega^2c^2k_{\rm cut}^2}{\omega_p^2+c^2k_{\rm cut}^2}\ . \tag{109}$$

As long as ω is slightly less than $\Omega\cos\theta$ for the shear Alfvén wave and lower than the lower hybrid frequency for the compressional wave, this divergence does not arise. In this sense, the choice of $k_{\rm cut}$ is not sensitive to the divergence. The choice of $k_{\rm cut}$ may be made on various considerations which have not been mentioned in this simple treatment. These considerations may include finite Larmor radius effects, kinetic effects, the discreteness of plasma particles, and quantum effects.

As for the shear Alfvén wave, its directional dependence makes calculations more complicated once again, but it is still tractable. The dispersion relation, Eq. (98), indicates that, given a value of ω, a cutoff in k implies a cutoff in θ:

$$\cos^2\theta>\cos^2\theta_0=\omega^2\frac{\omega_p^2+c^2k^2}{\Omega^2c^2k^2}\ . \tag{110}$$

This means we need to change the lower limit of integration in Eq. (108) to $\cos\theta_0$. We then find

$$\frac{\langle B^2\rangle_\omega}{8\pi}=\frac{T}{4\pi}\frac{\omega_p^3}{\omega c^3}\left[\frac{(\omega_p^2+c^2k_{\rm cut}^2)^{1/2}}{\omega_p}-\frac{1}{(1-\omega^2/\Omega^2)^{1/2}}\right]. \tag{111}$$

The maximum value of ω occurs at $k=k_{\rm cut}$ and $\theta=0$; it is the same as the cutoff frequency of the compressional Alfvén wave. Therefore, once again, we have headed off the divergence.

We now ask how much energy density is contained in the Alfvén modes. We answer by taking the expressions for $\langle B^2\rangle_{\mathbf{k}}/8\pi$ of each mode, adding them together, integrating the sum over d^3k, and dividing by $(2\pi)^3$. The result is

$$\frac{\langle B^2\rangle^0}{8\pi}=T\int_0^{k_{\rm cut}}\frac{d^3k}{(2\pi)^3}4\pi k^2\frac{\omega_p^2}{\omega_p^2+c^2k^2}\ . \tag{112}$$

We stress that the introduction of a cutoff in $\mathbf{k}$ in a magnetized plasma needs further investigation.

In this section we have found, first of all, that the zero-frequency mode, derived from an equation of motion for a nonmagnetized plasma, is consistent with the limit $\mathbf{B}_0\to 0$ for a plasma with an imposed uniform magnetic field. We have found that the zero-frequency mode is a composite of compressional and shear Alfvén waves, along with cyclotron waves at higher frequencies. When an external magnetic field is imposed on the plasma, however, the energy which was stored in $\omega=0$ in the nonmagnetized plasma is spread out in a range of frequency up to the cyclotron frequency. But in a progressively weak enough magnetic field, this spread will be correspondingly small compared to the other relevant frequencies.

The fluctuations associated with the cyclotron waves are reminiscent of the Bernstein wave paradox: In a plasma with an imposed magnetic field, the Bernstein wave is not damped. This is true no matter how small $\mathbf{B}_0$ may be. But then where does Landau damping come from in the limit $\mathbf{B}_0\to 0$? The resolution of this problem is that, as $\mathbf{B}_0\to 0$ in a thermal plasma, more cyclotron resonances become important. The effects of these resonances are added to the particle orbit; the net effect is a damping of the particle motion. This damping reduces to Landau damping when $\mathbf{B}_0\to 0$ [15,16]. Dealing with frequencies near the cyclotron resonance, then, may require accounting for subtle effects which we have not taken into account.

We briefly consider the magnetic-energy spectra of an electron-ion plasma. From Eq. (78) we can obtain numerical results for the spectra and make some valid qualitative comparisons to the electron-positron plasma.

The cold electron-ion plasma has five propagating electromagnetic modes. The dispersion relations of these modes, propagating at various angles with respect to $\mathbf{B}_0$, are shown in Fig. 6. These plots were obtained by making contour plots of the determinant of the dielectric-permittivity tensor, which we last saw explicitly in Eq. (62), and removing the contours of all values of $\det(\Lambda)\neq 0$.

In Fig. 6 are plotted the wave-vector fluctuation spectra of the magnetic field, $\langle B^2 \rangle_{\mathbf{k}}/8\pi$, for each of the five modes of the electron-ion plasma. The spectrum of each mode has been calculated by numerically approximating Eq. (78). Note that for all values of $\mathbf{k}$, the total magnetic fluctuation energy summed over all modes is equal to T. Since both independent polarizations of the magnetic field are included in this sum, this is consistent with the equipartition law.

Note that the two branches of the dispersion relation associated with the Alfvén waves have spectra qualitatively similar to those of the Alfvén branches of the electron-positron plasma. In the low-k limit, it is easy to show that the spectra are independent of the magnetic-field strength. The low-k dispersion relation for the compressional Alfvén wave is

$$\omega^2 = \frac{1}{1+\dfrac{4\pi n_0 c^2}{B_0}} c^2 k^2 , \tag{113}$$

where n_0 is the plasma mass density. Equation (78) tells us that

$$\frac{\langle B^2 \rangle_{\mathbf{k}}}{8\pi} = \frac{T}{2} . \tag{114}$$

The proof is the same for the shear Alfvén mode, for which the dispersion relation is

$$\omega^2 = \frac{1}{1+\dfrac{4\pi n_0 c^2}{B_0}} c^2 k^2 \cos^2\theta , \tag{115}$$

and, once again,

$$\frac{\langle B^2 \rangle_{\mathbf{k}}}{8\pi} = \frac{T}{2} . \tag{116}$$

We see, then, that both Alfvén waves in the electron-ion plasma carry energy, even if the strength of the imposed magnetic field is brought down to zero. We can see from Eqs. (113) and (115) that, if $B_0 = 0$, the frequency of these modes is zero for all $\mathbf{k}$. An isotropic electron-ion plasma will have a finite amount of magnetic-field energy in a narrow frequency band surrounding $\omega = 0$.

Phenomenological wave-number cutoffs can be given for the Alfvén modes in a weakly magnetized plasma. As stated above, Agim and Prager [14] used a cutoff of $k = \omega_{pi}/c$ when calculating the energy contained in shear Alfvén waves. The compressional Alfvén waves, on the other hand, exist in the frequency range of $\omega = 0$ to $\omega \le \omega_{\rm LH}$, where $\omega_{\rm LH}$ is the lower hybrid frequency given by

$$\frac{1}{\omega_{\rm LH}^2} = \frac{1}{\omega_{ci}^2 + \omega_{pi}^2} + \frac{1}{\omega_{ci}\omega_{ce}} ,$$

and ω_{ci} and ω_{ce} are the ion and electron cyclotron frequencies, respectively [13]. In a weakly magnetized plasma, $\omega_{\rm LH} \approx \sqrt{\omega_{ci}\omega_{ce}}$. The dispersion relation of these waves is $\omega = v_A k$. Therefore, we choose $k_{\rm cut}$ so that

$$k_{\rm cut} = \frac{\sqrt{\omega_{ci}\omega_{ce}}}{v_A} = \frac{\omega_{pe}}{c} \approx \frac{\omega_p}{c} .$$

Our cutoff value for the nonmagnetized plasma seems to be a good choice for the compressional waves of the weakly magnetized plasma.

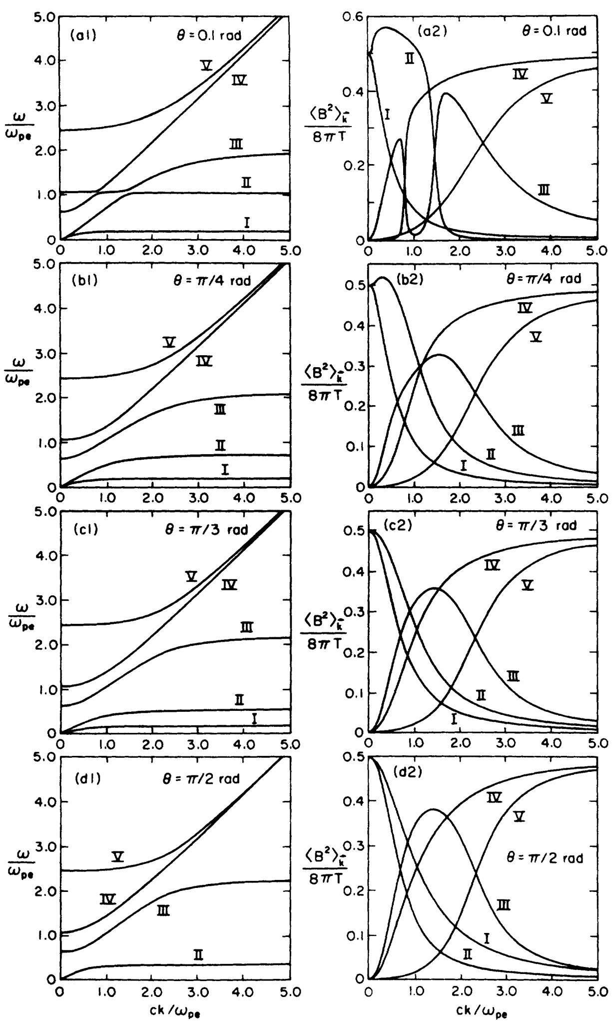

FIG. 6. Dispersion relations and magnetic-field fluctuation strengths for the modes of the electron-ion plasma in a uniform magnetic field. Roman numerals label modes in order of increasing frequency. θ indicates angle between imposed magnetic field and angle of propagation. Note that the lowest-frequency branch (the shear Alfvén branch) is not plotted in d_1, since its frequency is identically zero when propagating perpendicularly to the magnetic field. However, as shown in d_2, it retains a finite amount of energy. Close to $\omega = 0$, the power spectra of modes I and II should be regarded as qualitatively, not quantitatively, correct because of limitations in our numerical analysis.

V. KINETIC-THEORETIC ANALYSIS

Up to this point, we have derived all our results on fluctuations from a simple model with equations of motion describing a cold fluid plasma with a constant collision frequency. This model is good for studying, for example, propagating waves whose electromagnetic fields are largely transverse. Such waves have phase velocities usually exceeding the speed of light; therefore, such thermal effects as Landau damping have no effect on them. But, when we deal with low frequencies, that is, when we study frequency and wave-vector regimes where ω/k is less than or close to the thermal speed of the plasma constituents, we ignore kinetic effects at our peril. It is incumbent on us to attempt a kinetic-theory treatment of low-frequency magnetic-field fluctuations. We will find that a kinetic-theory treatment of the problem returns results which agree qualitatively with what we found in Sec. III.

We assume a homogeneous, isotropic, nonmagnetized hydrogen plasma. We take the electrons and ions to be in equilibrium with one another, having Maxwellian velocity distributions with a temperature T. In this case, the transverse part of the dielectric permittivity is given by

$$\Lambda(\omega,\mathbf{k})=1-\frac{c^2k^2}{\omega^2}+\left[\frac{m}{2\pi T}\right]^{1/2}\frac{\omega_{pe}^2}{\omega^2}\int_L\frac{e^{-mv^2/2T}}{-\omega+kv}+\left[\frac{M}{2\pi T}\right]^{1/2}\frac{\omega_{pi}^2}{\omega^2}\int_L\frac{e^{-Mv^2/2T}}{-\omega+kv}\,, \tag{117}$$

where m is the electron mass and M is the hydrogen ion mass. The subscript "L" attached to each integral sign is meant to specify the contour taken in each integral, namely, each integral is performed over the Landau contour [16]. A problem with this treatment should be mentioned here. We are using straight-line particle orbits to calculate the dielectric function. However, we will be applying our results to frequencies below typical collision frequencies of a plasma, where the straight-line approximation no longer holds. A thoroughly rigorous treatment here would include some consideration of particle collisions. This topic is deferred to future investigation.

As indicated above, we are interested in fluctuations at frequencies and wave vectors in the regime

$$\frac{\omega}{k}\le v_e,v_i\,,$$

where $v_e=\sqrt{T/m}$ and $v_i=\sqrt{T/M}$. In this regime, we can approximate Λ by

$$\Lambda(\omega,\mathbf{k})=1-\frac{c^2k^2}{\omega^2}-\frac{\omega_{pe}^2}{v_e^2k^2}-\frac{\omega_{pi}^2}{v_i^2k^2}+i\left[\frac{\pi}{2}\right]^{1/2}\frac{\omega_{pe}^2}{v_e\omega k}+i\left[\frac{\pi}{2}\right]^{1/2}\frac{\omega_{pi}^2}{v_i\omega k}\,. \tag{118}$$

Since

$$v_i^2/v_e^2=\omega_{pi}^2/\omega_{pe}^2=m/M\,,$$

we can write this as

$$\Lambda(\omega,\mathbf{k})=1-\frac{c^2k^2}{\omega^2}-\frac{2\omega_{pe}^2}{v_e^2k^2}+i\left[\frac{\pi}{2}\right]^{1/2}\alpha\frac{\omega_{pe}^2}{v_e\omega k}\,, \tag{119}$$

where $\alpha=1+\sqrt{m/M}$.

It is still true that, in the limit $\hbar\to 0$,

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}=-\frac{2T}{\omega}\frac{c^2k^2}{\omega^2}\,\mathrm{Im}(\Lambda^{-1})\,. \tag{120}$$

(The factor of 2 is included to account for both **B**-field polarizations.) From Eq. (119), $\mathrm{Im}(\Lambda^{-1})$ is readily found to be

$$\mathrm{Im}(\Lambda^{-1})=\frac{-(\pi/2)^{1/2}\alpha\omega_{pe}^2\omega^3k^3/v_e}{(\omega^2k^2-c^2k^4-2\omega_{pe}^2\omega^2/v_e^2)^2+\frac{\pi}{2}\alpha^2\omega_{pe}^4\omega^2k^2/v_e^2}\,. \tag{121}$$

Therefore, from Eqs. (120) and (121),

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}=T\frac{2\left[\frac{\pi}{2}\right]^{1/2}\alpha\omega_{pe}^2c^2k^5/v_e}{(\omega^2k^2-c^2k^4-2\omega_{pe}^2\omega^2/v_e^2)^2+\frac{\pi}{2}\alpha^2\omega_{pe}^4\omega^2k^2/v_e^2}\,. \tag{122}$$

Here we see qualitative confirmation of our earlier results: $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$ has a finite maximum at $\omega=0$ as long as $k^2<2\omega_{pe}^2/v_e^2$. True enough, $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$ had a maximum at $\omega=0$ in the cold plasma, *regardless* of the size of k. However, this restriction on k is a very loose restriction: The value of k at which $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$ no longer has a zero-frequency maximum is several times larger than the wave-number cutoff we chose in Sec. III. The kinetic plasma effect smears out, but does not destroy, the zero-frequency fluctuations we found in the cold-plasma theory. There is another similarity between the spectrum we have found here and that which we found for the cold plasma, namely, the problem that, if we integrate $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$ over d^3k to get $\langle B^2\rangle_\omega/8\pi$, the integral diverges at high k. (The situation has improved a bit. The divergence here is logarithmic, whereas the divergence we faced with the cold plasma was linear.)

We examine the low-frequency behavior of $\langle B^2\rangle_\omega/8\pi$. If we ignore terms in the denominator of $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$ which are of order ω^4, we can approximate $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$ as

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}=T\frac{2(\pi/2)^{1/2}\alpha\omega_{pe}^2c^2k^3/v_e}{c^4k^6+4\omega_{pe}^2\omega^2c^2k^2/v_e^2-2c^2\omega^2k^4+\frac{\pi}{2}\alpha^2\omega_{pe}^4\omega^2/v_e^2}\,. \tag{123}$$

Normalizing frequencies by ω_{pe} and wave numbers by ω_{pe}/c, we rewrite this as

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi}=\frac{T}{\omega_{pe}}\frac{2(\pi/2)^{1/2}\alpha x^3/\beta_e}{x^6+4\omega'^2x^2/\beta_e^2-2\omega'^2x^4+\frac{\pi}{2}\alpha^2\omega'^2/\beta_e^2}, \tag{124}$$

where x and ω' have the same meanings as in Sec. III and $\beta_e=v_e/c$. Note that $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$ scales as T/ω_{pe}. To find $\langle B^2\rangle_\omega/8\pi$, we integrate this expression over d^3k and divide by $(2\pi)^3$. This integral can be carried out exactly, as shown in Appendix B. As stated above, we must impose a wave-number cutoff on the integral. As in Sec. III, the cutoff will come sooner or later through the quantum-mechanical effect, which will be discussed in more detail in Sec. VII. In deriving the results of Fig. 7, we have used the same cutoff as in the previous sections, namely, ω_p/c. This does not cause any inconsistency as long as $\omega/\omega_p<v_i/c$. $\langle B^2\rangle_\omega/8\pi/(T\omega_{pe}^2/c^3)$ of plasmas at temperatures of 10^5, 10^6, and 10^7 K are shown in Fig 7. $\langle B^2\rangle_\omega/8\pi$ scales as ω_p^2/c^3, so the Fig. 7 results are independent of plasma density. Also, for $\omega>0.01\times\omega_{pe}$, $\langle B^2\rangle_\omega/8\pi$ exhibits an ω^{-2} behavior, whereas for very small frequencies, it diverges more slowly, growing approximately as $\omega^{-1/3}$.

We can obtain an expression for $\langle B^2\rangle_{\mathbf{k}}/8\pi$ by integrating $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$ over $d\omega$ and dividing by 2π. Considering that the contribution to the integral from high ω is ignorable, we integrate over all ω. When $k\le\omega_{pe}/c$, the result is

$$\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}=T\left[\frac{\pi}{2}\right]^{1/2}\alpha\frac{\omega_{pe}}{\left[4c^2k^2+\frac{\pi}{2}\alpha^2\omega_{pe}^2\right]^{1/2}}. \tag{125}$$

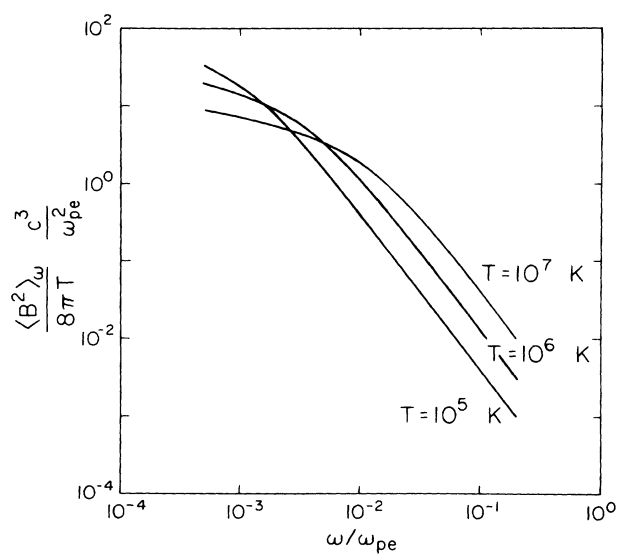

FIG. 7. Kinetic-theory results for magnetic-field fluctuation frequency power spectra of thermal plasmas. Shown are results for electron-ion plasmas at temperatures $T=10^5$, 10^6, and 10^7 K.

The cold-plasma approximation should still hold rather well for the electromagnetic plasma wave. As we go through our standard calculations, we find that its magnetic-field energy density per wave-vector volume closely approximates what we found in Sec. III. Specifically, $\langle B^2\rangle_{\mathbf{k}}/8\pi$ of the propagating electromagnetic plasma wave is very close to

$$\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}=T\frac{c^2k^2}{c^2k^2+\omega_p^2}. \tag{126}$$

We obtain the total $\langle B^2\rangle_{\mathbf{k}}/8\pi$ by adding this to the zero-frequency spectrum given in Eq. (125). We find that $\langle B^2\rangle_{\mathbf{k}}/8\pi$ is generally not equal to T.

Two points should be considered here. First, at $k=0$, $\langle B^2\rangle_{\mathbf{k}}/8\pi$, as given by Eq. (125), is exactly equal to T. Second, the deviation from T is very small for small k. At small k, Eq. (125) is approximately

$$\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}=T\left[1-\frac{4c^2k^2}{\pi\alpha^2\omega_{pe}^2}\right]. \tag{127}$$

The spectrum we might have expected to find in place of that given by Eq. (125) is

$$\langle B^2\rangle_{\mathbf{k}}/8\pi=T\omega_p^2/(c^2k^2+\omega_p^2).$$

At small k, this is approximately

$$\frac{\langle B^2\rangle_{\mathbf{k}}}{8\pi}=T\left[1-\frac{c^2k^2}{\omega_p^2}\right]. \tag{128}$$

The leading terms in these two expansions are certainly of the same order of magnitude. They differ by a ratio of essentially $4/\pi$ in a hydrogen plasma and by a ratio of $2/\pi$ in an electron-positron plasma. This small deviation may arise from the expansion of the plasma dispersion function in Eq. (118). However, it would seem that there is no problem with the fundamental physics here. The zero-frequency peak does exist. The problem is that, in regimes where ω is larger than the thermal velocity times k, the value of the peak falls off faster than predicted by the hot-plasma approximation which we used to obtain Eq. (118). When we integrated over ω to obtain Eq. (125), the high-ω contribution was not quite "ignorable enough."

In any event, $\langle B^2\rangle_{\mathbf{k}}/8\pi$ in the zero-frequency peak is on the order of T for small enough k. A rough approximation of the total-energy density contained in this peak is then T times the k-space volume contained within $k=\omega_p/c$, divided by $(2\pi)^3$, yielding, once again,

$$\frac{\langle B^2\rangle^0}{8\pi}=T\frac{1}{6\pi^2}\left[\frac{\omega_p}{c}\right]^3. \tag{129}$$

VI. PARTICLE SIMULATION

We look for our low-frequency magnetic-field fluctuations in kinetic computer simulations of plasmas. We discuss the results of these simulations in this section.

Earlier, Geary *et al.* [7] discussed the low-frequency magnetic-field fluctuations in numerical plasma simulations. The authors of [7] developed a magnetoinductive code to examine low-frequency behavior in magnetized plasmas. They took as their starting point the Darwin approximation to Maxwell's equations, i.e., they dropped the displacement current from the $\nabla \times \mathbf{B}$ equation. They then made use of the fluctuation-dissipation theorem to derive the magnetic-field fluctuation spectrum $\langle B^2 \rangle_{\mathbf{k}}/8\pi$. The fluctuation spectrum $\langle B^2 \rangle_{\mathbf{k}}/8\pi$ which they found for electromagnetic waves propagating perpendicularly to their imposed magnetic field $\mathbf{B}_0$ are exactly the same as our low-frequency result in our Eq. (58). A comparison of fluctuation spectra for waves in other directions is not useful because the present plasma is nonmagnetized, while that in Geary *et al.* was magnetized, and the electron motion was treated by the guiding-center approximation.

We have carried out particle simulations of thermal equilibrium plasmas employing both one-dimensional (1D) and two-dimensional (2D) fully electromagnetic, fully relativistic particle simulation codes (see, for example, [17]). We have recorded the magnetic-field frequency spectra arising from these simulations. The particles were given initial uniform distributions in space, and Maxwellian velocity distributions. In all cases the computational space boundary conditions were periodic. We ran the simulations for several thousand time steps; at each time step we stored the series of spatial Fourier components of the z component of the magnetic field, i.e., $B_z(\mathbf{k},t)$. At the end of the simulation run, the B_z Fourier components were input into an autocorrelation function, whose frequency spectrum yields the spectral intensity $B_z^2(\mathbf{k},\omega)$. Lastly, the $B_z^2(\mathbf{k},\omega)$ were summed over $\mathbf{k}$ to give $B_z^2(\omega)$.

In the (1D) simulations, we used the following parameters: the number of cells, $L_x=256$; electrons and positrons per cell are 10 each; $\Delta t=0.1/\omega_{pe}$; number of time steps, $N_t=2048$; and speed of light, $c=5\Delta\omega_{pe}$, where Δ is the grid spacing. Simulation runs were made with three different temperatures: $\gamma_{\text{therm}}=1.05$, 1.22, and 34.7 (corresponding to $T\sim 3\times 10^8$, 1.3×10^9, and 2×10^{11} K), where γ_{therm} is the relativistic factor corresponding to the thermal velocity of the plasma. To test the performance of these codes, we examined the dispersion relation produced for B_z for electromagnetic waves in a plasma and compared it with the standard result

$$\omega^2=c^2k^2+\frac{\omega_{p0}^2}{\gamma_{\text{therm}}} ,$$

where

$$\omega_{p0}^2=4\pi ne^2/m_e+4\pi ne^2/m_i .$$

The dispersion relation comparisons were excellent. We also examined B_z fluctuation strengths as functions of wave vector, i.e., $B_z^2(\mathbf{k})$. The fluctuation strengths compared fairly well with theory [Eq. (58)].

The results of the $B_z^2(\omega)$ measurements are shown in Figs. 8–10. In each of these three cases, a strong B_z fluctuation peak is seen at $\omega=0$. In Fig. 8 the 2×10^{11} K result is shown, while Fig. 9 shows the 10^9 K result and Fig. 10 shows the 3×10^8 K case.

We made an additional test on the 1D code by running the nonrelativistic ($\gamma_{\text{therm}}=1.05$) simulation for twice as long, i.e., 4096 time steps. Again, the $\omega=0$ peak appeared in the $B_z^2(\omega)$ spectrum and its width did not change from the width it had in the $N_t=2048$ time-step simulation. This indicates that the presence of the peak is *not* due to the finite window width in the correlation function restricted by the length of the run (i.e., not the Nyquist frequency width $\Delta\omega=1/\Delta t$), but is rather due to the intrinsic physics. In fact, traditionally, such a zero-frequency peak has been observed routinely in particle

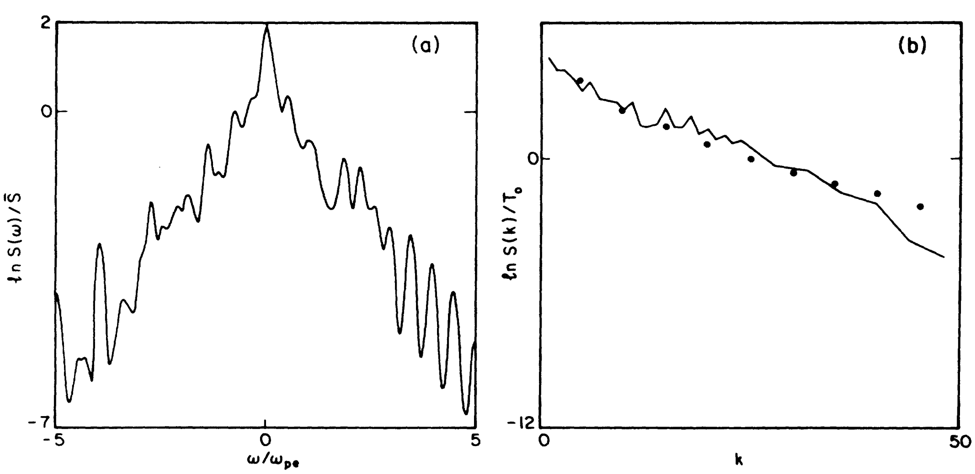

FIG. 8. Spectral intensities $S(\omega)=\langle B^2 \rangle_\omega/8\pi$ and $S(k)=\langle B^2 \rangle_{\mathbf{k}}/8\pi$ from a 1D simulation of an electron-positron plasma. $\gamma_{\text{therm}}=34.7(T=2\times 10^{11}$ K). (a) $\ln[S(\omega)/\bar{S}]$. Note the peak at $\omega=0$, where $\bar{S}$ is the normalization. (b) $\ln[S(k)/T_0]$. The line is from simulation results. The dots represent theoretical values: $6.7-\ln[1+c^2(\gamma/\omega_p^2)k^2e^{k^2a^2}]$, where 6.7 is obtained from least-squares fitting. $k=2\pi m/L_x\Delta$ with m being an integer. The finite-size effect of the code is taken into consideration.

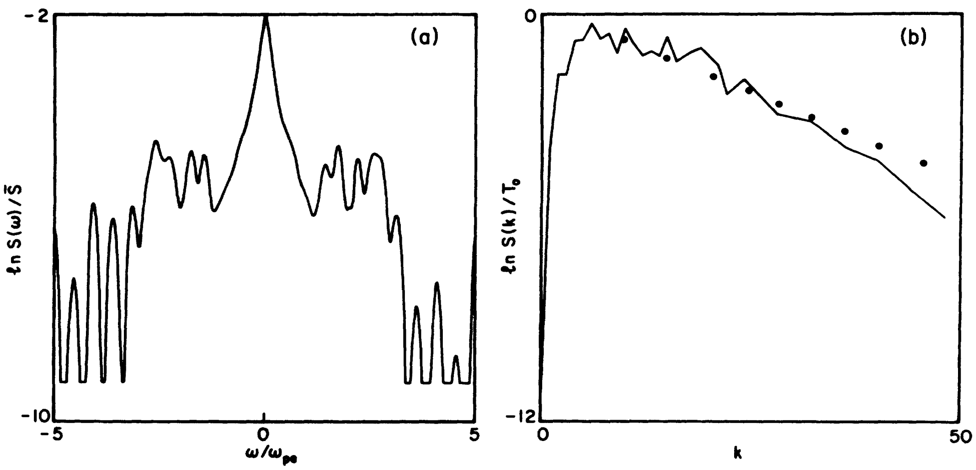

FIG. 9. Spectral intensities $S(\omega)$ and $S(k)=\langle B^2\rangle_k/8\pi$ from a 1D e^+-e^- simulation. $\gamma_{\text{therm}}=1.2(T=1.3\times10^9$ K). (a) $\ln S(\omega)/\bar{S}$. Note the zero-frequency peak. (b) $\ln S(k)/T_0$. The solid line is from simulation results. Dots represent theoretical values: $-0.21-\ln[1+c^2(\gamma/\omega_p^2)k^2e^{k^2a^2}]$, where -0.21 is obtained from least-squares fitting.

simulation, but has not been well understood in its origin.

In the 2D simulation, the parameters were the following: computational area, 32×32 cells; $9e^-+9e^+$ per cell; $\Delta t=0.1/\omega_{pe}$; $N_t=2048$; and $\gamma_{\text{therm}}=1.05$. Again, a strong B_z fluctuation peak is seen at $\omega=0$. The 2D results, together with the results of the 1D run of the same temperature, are shown in Fig. 10.

Our simulation results for the magnetic-field wave-number spectral intensity $S(\mathbf{k})$ follow $1/(\omega_p^2+c^2k^2)$ [the second term in Eq. (58)] more closely than our low-wave-number expansion [Eq. (59)]. See the frames (b) of Figs. 8, 9, and 10. This is explained by the conditions of the simulation. First of all, the grid nature of the simulation puts a cap on the maximum k at π/Δ. Second, as can be seen from our derivation of $S(\mathbf{k})$, the first term in Eq. (42) comes from the energy contained in the radiation. The results shown were obtained by summing $S(\mathbf{k},\omega)$ over frequencies ranging from 0 to $+5\omega_p\sim1/\Delta t$, the Nyquist frequency. When the wave frequency of a given mode is higher than this range, the high-frequency energy of the radiation mode will not enter into the sum. These reduction factors of the radiation branch, plus sharing of energy between nonradiative modes, account for the closer agreement with the expression with only the second term in Eq. (58) rather than Eq. (59) in our simulation. After we take these factors into consideration, the agreement between our kinetic simulation and the theory is good.

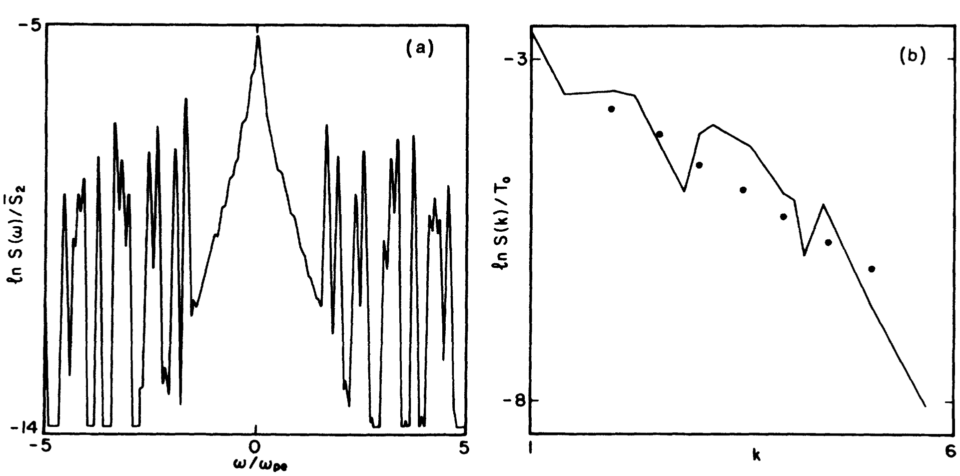

FIG. 10. Spectral intensities $S(\omega)$ and $S(k)=\langle B^2\rangle_k/8\pi$ from a 2D e^+-e^- simulation. $\gamma_{\text{therm}}=1.05(T=3\times10^8$ K). (a) ln $S(\omega)/\bar{S}$. The zero-frequency peak is still present in 2D. (b) $\ln S(k)/S_0$. The line is from simulation results. Dots represent $-2.6-\ln[1+c^2(\gamma/\omega_p^2)k^2e^{k^2a^2}]$, where -2.6 was obtained from least-squares fitting.

VII. INTERACTION BETWEEN PLASMA PARTICLES AND ELECTROMAGNETIC WAVES WITH HIGH MOMENTA

We have deferred a definitive resolution of the treatment of high wave numbers in the integral in Eq. (46). In Secs. III, IV, and V, we phenomenologically introduced a cutoff wave number in k space. Without this cutoff, we obtain an infinity of energy in the magnetic-field power spectrum at $\omega=0$. Such high-k divergences are a common problem with fluid theories which take no account of the granularity of a fluid at some scale. In this section we offer quantum-mechanical justification for the introduction of a cutoff by showing qualitatively the handling of this problem.

The longitudinal dielectric function of a plasma, calculated from kinetic theory based on classical mechanics, is

$$\epsilon_{\parallel}(\mathbf{k},\omega)=1+\sum_j \frac{\omega_{pj}^2}{\omega^2}\int d^3v \frac{\mathbf{k}\cdot\partial_{\mathbf{v}} f_{0j}(\mathbf{v})}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta} , \tag{130}$$

where the index j indicates species of the plasma constituents and $f_{0j}(\mathbf{v})$ is the unperturbed velocity distribution function of species j. (The end results we obtain in this section will be applicable to the transverse dielectric function as well.) If $f_{0j}(\mathbf{v})$ is a Maxwellian, the result is

$$\epsilon_{\parallel}(\mathbf{k},\omega)=1+\sum_j \frac{k_{Dj}^2}{k^2} W\left[\frac{\omega}{kv_{thj}}\right] , \tag{131}$$

where k_{Dj} is the Debye wave number of the species j and v_{thj} is the thermal velocity of the species j. $W(z)$ is the plasma dispersion function.

Equation (129) is, however, a classical approximation to a quantum-mechanical expression, when the photon momentum transfer $\hbar\mathbf{k}$ is small. When we treat the plasma quantum mechanically, the derivative term $\mathbf{k}\cdot\partial_{\mathbf{v}} f_0(\mathbf{v})$ is replaced by a difference in f_0, and the $\mathbf{k}\cdot\mathbf{v}$ term in the denominator is replaced by a difference in energies of momentum states:

$$\mathbf{k}\cdot\partial_{\mathbf{v}} f_0(\mathbf{v})\to\frac{m}{\hbar}[f_0(\mathbf{p}+\hbar\mathbf{k}/2)-f_0(\mathbf{p}-\hbar\mathbf{k}/2)] , \tag{132}$$

$$\mathbf{k}\cdot\mathbf{v}\to\frac{1}{\hbar}[\epsilon(\mathbf{p}+\hbar\mathbf{k}/2)-\epsilon(\mathbf{p}-\hbar\mathbf{k}/2)] , \tag{133}$$

yielding

$$\epsilon_{\parallel}(\mathbf{k},\omega)=1+\sum_j \frac{\omega_{pj}^2}{\omega^2}\int \frac{d^3p}{m^2}\frac{f_0(\mathbf{p}+\hbar\mathbf{k}/2)-f_0(\mathbf{p}-\hbar\mathbf{k}/2)}{\hbar(\omega+i\eta)-[\epsilon(\mathbf{p}+\hbar\mathbf{k}/2)-\epsilon(\mathbf{p}-\hbar\mathbf{k}/2)]} . \tag{134}$$

If the plasma particles are free-particle fermions,

$$f_0(\mathbf{p})=(e^{\beta(p^2/2m-\mu)}+1)^{-1} , \tag{135}$$

with $\beta=1/T$, and μ is the chemical potential. This means

$$\epsilon_{\parallel}(\mathbf{k},\omega)=1+\sum_j \frac{\omega_{pj}^2}{\omega^2}\int \frac{d^3p}{m^2}\frac{(e^{\beta(\mathbf{p}+\hbar\mathbf{k}/2)^2/2m-\beta\mu}+1)^{-1}-(e^{\beta(\mathbf{p}-\hbar\mathbf{k}/2)^2/2m-\beta\mu}+1)^{-1}}{\hbar(\omega+i\eta)-[\epsilon(\mathbf{p}+\hbar\mathbf{k}/2)-\epsilon(\mathbf{p}-\hbar\mathbf{k}/2)]} . \tag{136}$$

If $(\hbar k)^2/2m >> k_B T$, we can approximate this expression by

$$\epsilon_{\parallel}(\mathbf{k},\omega)=1+\sum_j \frac{\omega_{pj}^2}{\omega^2} e^{-\beta(\hbar k)^2/2m}\int \frac{d^3p}{m^2}\frac{e^{-\beta(\mathbf{p}+\hbar\mathbf{k}/2)^2/2m-\beta\mu}-e^{-\beta(\mathbf{p}-\hbar\mathbf{k}/2)^2/2m-\beta\mu}}{\hbar(\omega+i\eta)-[\epsilon(\mathbf{p}+\hbar\mathbf{k}/2)-\epsilon(\mathbf{p}-\hbar\mathbf{k}/2)]} . \tag{137}$$

For very large k, therefore, $\epsilon_{\parallel}(\mathbf{k},\omega)$ falls off exponentially as $e^{-\beta\hbar^2k^2/2m}$.

When $(\hbar k)^2/2m >> T$, the second term in our expression for $\epsilon_{\parallel}(\mathbf{k},\omega)$ becomes very small. This happens because the functions $f_0(\mathbf{p}+\hbar\mathbf{k}/2)$ and $f_0(\mathbf{p}-\hbar\mathbf{k}/2)$ are never sizeable in the same region of $\mathbf{p}$ space. The terms containing these two functions will independently integrate to give very small numbers. This being the case, the effect the plasma has on the electromagnetic spectrum is, indeed, negligible at high enough k.

VIII. THE BOHR–van LEEUWEN THEOREM

Those readers familiar with the Bohr–van Leeuwen theorem might object that, when a permanent magnetic moment exists, it is always a quantum-mechanical effect. In 1911, Niels Bohr [18] demonstrated that a strict, rigorous application of statistical mechanics ruled out the possibility of macroscopic magnetization in classical physical systems. This result, among others, was independently discovered by van Leeuwen and presented in her dissertation in 1919 [19]. We consider here the apparent contradiction between the Bohr–van Leeuwen theorem and the present theory after giving a short review of the theorem. We give a proof of this which closely follows one given by van Vleck [20]. His proof, in turn, is based on that given by van Leeuwen.

We wish to calculate the magnetization of a macroscopic body. Suppose that it is made up of molecules, perhaps possessing permanent or induced magnetic dipole moments. From a classical viewpoint, the magnetic moment of one of the molecules is $e/2m_ec$ times the total angular momentum of the electrons orbiting the molecule. The z component of the magnetic moment is

$$m_z=\frac{e}{2c}\sum_i (x_i\dot{y}_i-y_i\dot{x}_i) . \tag{138}$$

We can write this more generally, thereby economizing on notation and showing the power of the theorem more fully:

$$m_z = \sum_k a_k \dot{q}_k \ , \qquad (139)$$

where the q_k's can be a set of generalized coordinates describing the system (in this case, the positions of a molecule's electrons), the $\dot{q}_k$'s are the corresponding generalized velocities, and the a_k's are functions of the q_k's but not of the $\dot{q}_k$'s.

Magnetization is found by taking an ensemble average of this magnetic moment:

$$M_z = N \frac{\int \sum_k a_k \dot{q}_k e^{-H/kT} dq_1 \cdots dp_1 \cdots}{\int e^{-H/kT} dq_1 \cdots dp_1 \cdots} \ , \qquad (140)$$

where N is the average molecular density, T is the temperature, and $H = H(\{q\},\{p\})$ is the Hamiltonian of the system.

We note that $\dot{q}_k = \partial H / \partial p_k$ and obtain

$$M_z = -NkT \frac{\int \sum_k a_k \frac{\partial}{\partial p_k} e^{-H/kT} \prod_i dq_i \prod_i dp_i}{\int e^{-H/kT} \prod_i dq_i \prod_i dp_i} \qquad (141)$$

$$= -NkT \frac{\int \sum_k (a_k e^{-H/kT})_{p_k=-\infty}^{p_k=+\infty} \prod_i dq_i \prod_{i\,(\neq k)} dp_i}{\int e^{-H/kT} \prod_i dq_i \prod_i dp_i} \ . \qquad (142)$$

We make the reasonable assumption that if any one of the p_k approaches $\pm\infty$, then $H(\{q\},\{p\})$ becomes infinite. This being the case, we find

$$(a_k e^{-H/kT})_{p_k=-\infty}^{p_k=+\infty} = 0 \ . \qquad (143)$$

The magnetization is therefore identically zero.

The result is the same for a plasma. We pick some point which is stationary with respect to the center of mass of the plasma. We find the magnetic moment about this point,

$$\mathbf{m} = \sum_i \frac{e_i}{2cm_i} (\mathbf{x}_i \times \mathbf{p}_i) \ , \qquad (144)$$

where the sum extends over all charges in the plasma. We find magnetization by taking an ensemble average of this sum. The argument proceeds exactly as above and we find that the magnetization is, again, zero.

The question we address is whether this result contradicts the zero-frequency (i.e., permanent) magnetic fields we have found in plasmas or not. The answer is that it does not. The contradiction is only apparent. Van Leeuwen's proof deals with the *ensemble average* of the magnetic moment $\mathbf{m}$. This average is zero. However, the theorem says nothing about *fluctuations about* this average. We take an ensemble of macroscopically identical plasmas. We measure the magnetic field at some particular point in each plasma, and average the measurements. We will, indeed, get a value of zero. However, in each plasma, the magnetic field at the particular point we have chosen will deviate from this zero average. What we have found is that this deviation in each element of the ensemble has, generally speaking, a *time average* different from zero. This result is surprising, but is not in contradiction with any well-established results of electromagnetism or statistical mechanics.

IX. FLUCTUATIONS IN DEGENERATE ELECTRON PLASMAS

Our aim in this section is to obtain expressions for the frequency spectra of electrical-current and magnetic-field fluctuations in completely degenerate electron plasmas. We take a simple model of the degenerate plasma: a completely degenerate gas of fermionic electrons in a uniform background of neutralizing positive charge. When the degeneracy is nearly complete, the Fermi distribution can be taken at its $T=0$ form for the purpose of computing the dielectric function, while this does not imply that the temperature or its associated fluctuations vanish. In this model, the wave functions of the electrons are simple plane waves, rather than the more complicated (and more realistic) Bloch functions associated with a periodic lattice. We can at least expect that our results hold for metallic crystals with a small number of conduction electrons filling the lowest portion of a single conduction band, where the electron Hamiltonian approximates that of a free particle.

Given this model of the degenerate electron gas, the longitudinal dielectric function is [21]

$$\epsilon_l'(\omega,\mathbf{k}) = 1 + \frac{3}{8}\frac{\omega_{pe}^2}{E_F^2}\frac{1}{q^2}\left\{1 + \frac{1}{2q}\left[1 - \frac{1}{4}\left[q - \frac{u}{q}\right]^2\right]\ln\frac{|q(q+2)-u|}{|q(q-2)-u|} + \frac{1}{2q}\left[1 - \frac{1}{4}\left[q + \frac{u}{q}\right]^2\right]\ln\frac{|q(q+2)+u|}{|q(q-2)+u|}\right\} \ ,$$

$$\epsilon_l''(\omega,\mathbf{k}) = \frac{3\pi}{16}\frac{\omega_{pe}^2}{E_F^2}\frac{1}{q^3}\begin{cases} u, & 0 < u < q(2-q) \\ 1 - \frac{1}{4}\left[q - \frac{u}{q}\right]^2, & q|q-2| < u < q(q+2) \\ 0, & 0 < u < q(q-2),\ u > q(q+2) \ , \end{cases} \qquad (145)$$

where $\epsilon_l' = \mathrm{Re}\{\epsilon_l\}$ and $\epsilon_l'' = \mathrm{Im}\{\epsilon_l\}$, q is k/p_F, u is $|\omega|/E_F$, p_F is the Fermi momentum and E_F is the Fermi energy. In this section, we set $\hbar = 1$. The regions in $\mathbf{k}$-ω space in which ϵ_l'' is nonzero are shown in Fig. 11(a).

The number-density fluctuation spectrum $\langle \delta n^2 \rangle_{\mathbf{k}\omega}$ is given by

$$\langle \delta n^2 \rangle_{\mathbf{k}\omega} = \frac{k^2}{2\pi e^2} \frac{1}{e^{\omega/T}-1} \frac{\epsilon_l''}{|\epsilon_l|^2} . \tag{146}$$

As we remarked above, the temperature T appears in the Bose-Einstein distribution, while it does not appear in ϵ. When $T \to 0$, the factor $[\exp(\omega/T)-1]^{-1}$ approaches $-\Theta(-\omega)$, $\Theta(x)$ being the Heaviside step function. The last factor in this expression is $-\mathrm{Im}\{1/\epsilon\}$. It should be noted that this takes on nonzero values when (1) $\mathbf{k}$ and ω lie within the regions specified by Eq. (145) and (2) $\mathbf{k}$ and ω satisfy the dispersion relation for the electrostatic plasma wave propagating through the degenerate plasma. The dispersion relation for the electrostatic plasma wave in the completely degenerate electron gas is [3]

$$\omega^2 = \omega_{pe}^2 + \tfrac{3}{5} k^2 v_F^2 + \frac{k^4}{4m^2} , \tag{147}$$

where v_F is the Fermi velocity of the degenerate gas.

The equation of charge continuity, together with Eq. (146), tells us that the longitudinal current fluctuations are given by

$$\langle \delta j_\parallel^2 \rangle_{\mathbf{k}\omega} = \frac{\omega^2}{2\pi} \frac{1}{e^{\omega/T}-1} \frac{\epsilon_l''}{|\epsilon_l|^2} . \tag{148}$$

Figure 11(b) shows a contour plot of this function weighted by the geometrical factor $4\pi k^2$. We find the frequency spectrum of longitudinal current fluctuations $\langle \delta j_\parallel^2 \rangle_\omega$ by integrating this expression over k and dividing by $(2\pi)^3$. We have performed this numerically. Figure 12 shows the result obtained when the plasmon energy divided by the Fermi energy is 1.49. For very low frequencies, $\langle \delta j_\parallel^2 \rangle_\omega$ varies as ω^3. There is a kink in the spectrum at $\hbar\omega = 1.49 E_F$. This corresponds to the appearance of the electrostatic plasma wave, which exists in the frequency range of $u = 1.49$ to $u \approx 2.3$. Above $u \approx 2.3$, $\langle \delta j_\parallel^2 \rangle_\omega$ rises approximately as $\omega^{1.45}$.

The transverse current fluctuation spectrum in a completely degenerate electron plasma is derived from the transverse part of the dielectric function [3]

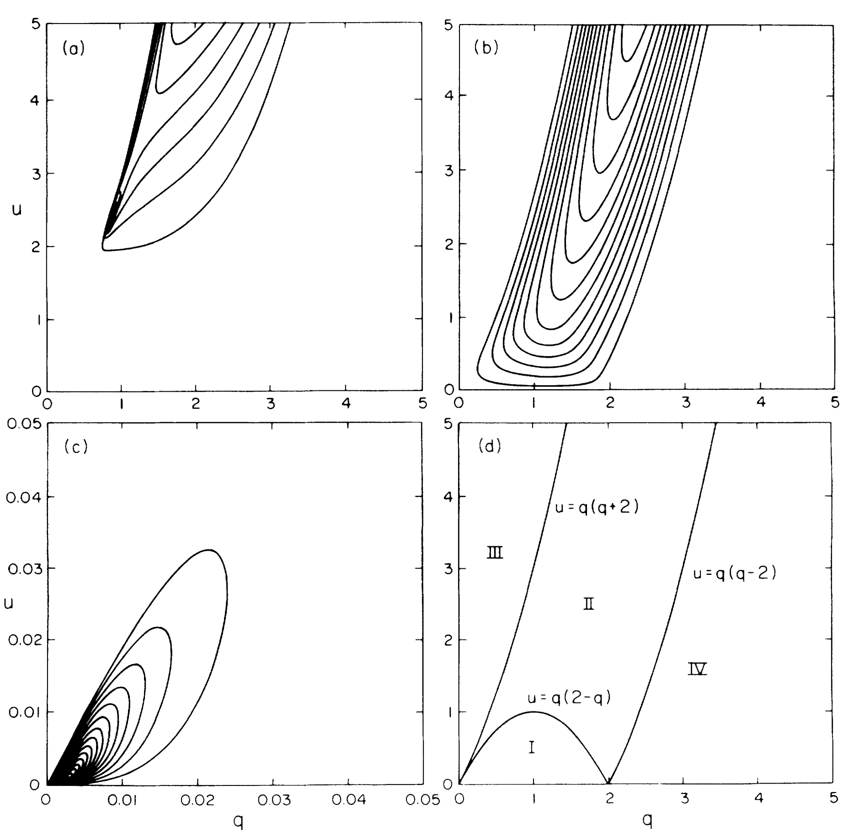

FIG. 11. Fluctuation power spectra of various quantities in a degenerate plasma. Plasmon energy divided by Fermi energy is 1.49. (a) Power spectrum of parallel current fluctuations $\langle j_\parallel^2 \rangle_{\mathbf{k}\omega}(\hbar/E_F^2)(k\hbar/p_F)^2/2\pi^2$. Contours run from 0 to about 0.03. The contours are highest at the top of the graph, and at the "island" near $(q,u)=(0.75,2.3)$. The outside contour is close to zero. (b) Power spectrum of transverse current fluctuations $\langle j_\perp^2 \rangle_{\mathbf{k}\omega}(\hbar/E_F^2)(k\hbar/p_F)^2/2\pi^2$. Contours run from 0 to 0.024. The contours are highest at the top of the graph. The outside contour is close to zero. (c) Power spectrum of magnetic-field fluctuations $(\langle B^2 \rangle_{\mathbf{k}\omega} k^2/8\pi)(\hbar/2m_e c)^2/2\pi^2$. The contour interval is 1×10^{-7}. Contours diverge at the origin. The outside contour is close to zero. (d) Finite fluctuations of nonpropagating electrostatic and electromagnetic modes can occur only in regions I and II for a fully degenerate plasma.

$$\epsilon_l(\omega,\mathbf{k})=1-\frac{\omega_{pe}^2}{\omega^2}-\frac{4\pi e^2}{\omega^2 m^2}\frac{1}{V}\sum_{\mathbf{p}} n_{\mathbf{p}}\left[p^2-\frac{(\mathbf{k}\cdot\mathbf{p})^2}{k^2}\right]\left[\frac{1}{\omega-E_{\mathbf{p}-\mathbf{k}}+E_{\mathbf{p}}+io}-\frac{1}{\omega-E_{\mathbf{p}}-E_{\mathbf{p}-\mathbf{k}}+io}\right], \quad (149)$$

where V is the volume of the gas and $n_{\mathbf{p}}$ is the Fermionic occupation number of the state $\mathbf{p}$:

$$n_{\mathbf{p}}=\left[e^{[(E_{\mathbf{p}}-\mu)/T]}+1\right]^{-1}. \quad (150)$$

If the separation between the momenta of existing states Δp is much smaller than the range of momenta of the occupied states p, the summation can be approximated by an integral. The number io is a small, positive imaginary number which indicates the path of the integral in the complex p plane.

Since the electron wave functions are plane waves, the energy of state $\mathbf{p}$, E_p, is $p^2/2m_e$. Assuming a large number of electrons, with two electrons per $\mathbf{p}$ state, as allowed by the Pauli exclusion principle, we can approximate the summation here by an integration. If the boundary conditions on the gas are periodic, we find

$$\epsilon_t'(\omega,\mathbf{k})=1-\frac{3}{8}\frac{\omega_{pe}^2}{E_F^2}\frac{1}{u^2}\left\{1+\frac{q^2}{4}+\frac{3}{4}\frac{u^2}{q^2}+\frac{1}{2q}\left[1-\frac{1}{4}\left[q+\frac{u}{q}\right]^2\right]^2\ln\left|\frac{q(q+2)+u}{q(q-2)+u}\right|\right.$$
$$\left.-\frac{1}{2q}\left[1-\frac{1}{4}\left[q-\frac{u}{q}\right]^2\right]^2\ln\left|\frac{q(q+2)-u}{q(q-2)-u}\right|\right\},$$

$$\epsilon_t''=\begin{cases}-\dfrac{3\pi}{16}\dfrac{\omega_{pe}^2}{E_F^2}\dfrac{1}{uq}\left[2-\dfrac{1}{4q^2}\right](u+q^2)^2-\dfrac{1}{4q^2}(-u+q^2)^2, & 0<u<q(2-q)\\ -\dfrac{3\pi}{16}\dfrac{\omega_{pe}^2}{E_F^2}\dfrac{1}{u^2q}\left[1-\dfrac{1}{4q^2}(-u+q^2)^2\right]^2, & q|q-2|<u<q(q+2)\\ 0, \quad 0<u<q(q-2),\ u>q(q+2)\,. \end{cases} \quad (151)$$

The transverse current fluctuation spectrum is given by

$$\langle\delta j_\perp^2\rangle_{\mathbf{k}\omega}=\frac{1}{2\pi}\frac{\omega^2}{e^{\omega/T}-1}\left[1-\frac{c^2k^2}{\omega^2}\right]^2\frac{\epsilon_t''}{\left|\epsilon_t-\dfrac{c^2k^2}{\omega^2}\right|^2}. \quad (152)$$

This function, weighted with $4\pi k^2$, is plotted in contour form in Fig. 11(c). In analogy with the longitudinal case, $\mathrm{Im}[1/(\epsilon_t-c^2k^2/\omega^2)]$ is a nonzero δ function along the dispersion relation curve of the propagating electromagnetic wave. The dispersion relation is given, approximately, by

$$\omega^2=\omega_{pe}^2+c^2k^2+\tfrac{1}{5}v_F^2k^2,\quad c^2k^2\ll\omega_{pe}^2$$
$$\omega^2=c^2k^2+\omega_{pe}^2\left[1+\frac{1}{5}\frac{v_F^2}{c^2}\right],\quad c^2k^2\gg\omega_{pe}^2\,. \quad (153)$$

We find the frequency spectrum of the transverse current

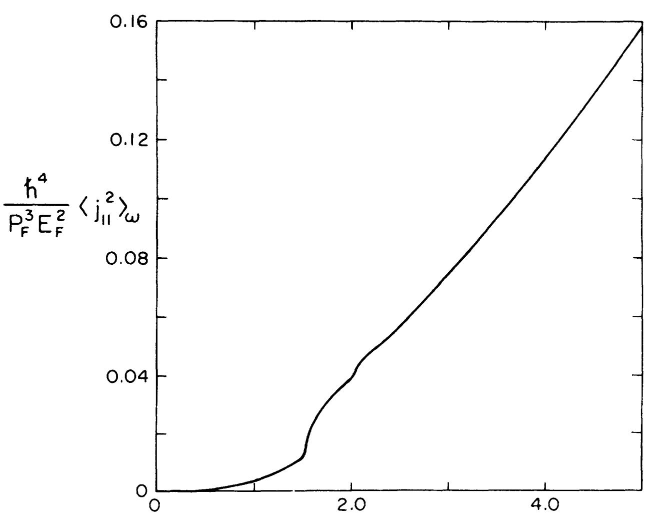

FIG. 12. Frequency power spectrum of fluctuations in longitudinal current in degenerate electron plasma. The plasmon energy divided by the Fermi energy is 1.49.

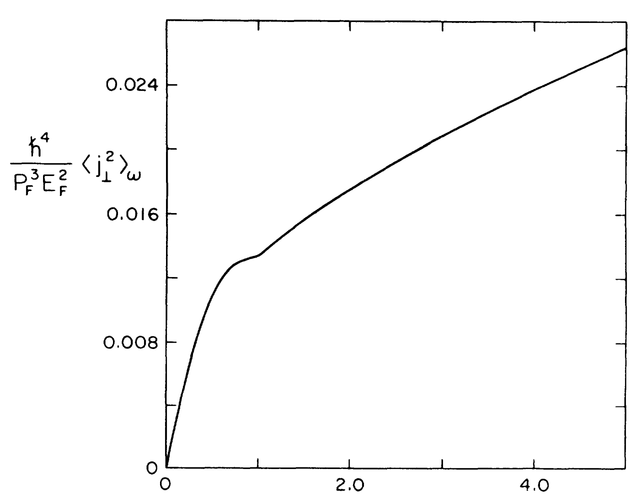

FIG. 13. Frequency power spectrum of fluctuations in transverse current in degenerate electron plasma. The plasmon energy divided by the Fermi energy is 1.49.

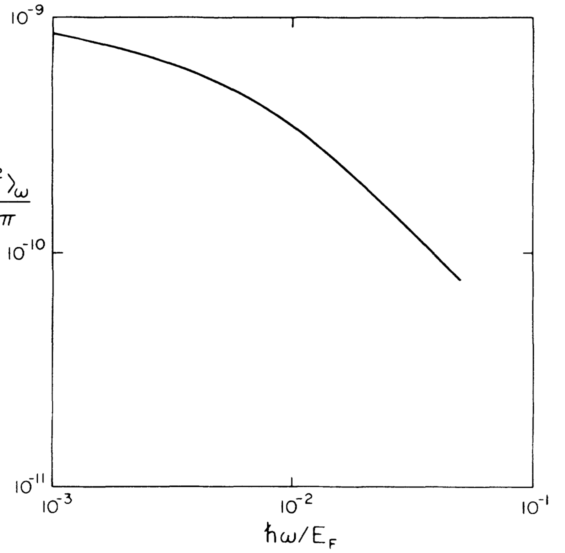

FIG. 14. Frequency power spectrum of fluctuations in magnetic field in degenerate electron plasma. The plasmon energy divided by the Fermi energy is 1.49.

fluctuations $\langle \delta j_\perp^2 \rangle_\omega$ by integrating over d^3k and dividing by $(2\pi)^3$. Our numerical result is shown in Fig. 13. The plasmon energy divided by the Fermi energy is, again, 1.49. $\langle \delta j_\perp^2 \rangle_\omega$ does not exhibit a power-law behavior at low frequencies. There is a kink in the spectrum at $\omega = E_F/\hbar$. This is due to the difference in behavior of ϵ_t'' below and above the curve $u = q(2-q)$. Above the kink, the spectrum rises approximately as $\omega^{0.45}$. The contribution to the current fluctuation spectrum given by the propagating electromagnetic wave is about 10^{-5} times that of the spectrum contributed by the zero-sound noise; therefore, it is not distinguishable in our figure.

The magnetic-field fluctuation spectrum is also studied. Maxwell's equations yield

$$\frac{\langle B^2 \rangle_{\mathbf{k}\omega}}{8\pi} = \frac{2\pi}{c^2} \frac{k^2}{\left[\frac{\omega^2}{c^2} - k^2 \right]^2} \langle \delta j_\perp^2 \rangle_{\mathbf{k}\omega} . \tag{154}$$

This function is shown in contour form, with geometrical weighting of $4\pi k^2$, in Fig. 11(d). We obtain the frequency spectrum of the magnetic-field fluctuations $\langle B^2 \rangle_{\mathbf{k}\omega}/8\pi$ by integrating over $\mathbf{k}$. Our numerical result is shown in Fig. 14. Here we have a quantity whose spectrum clearly diverges at low frequencies. Note that for $\omega < 10^{-2}(E_F/\hbar)$, $\langle B^2 \rangle_\omega / 8\pi$ falls off as $\omega^{-1/3}$, whereas for $\omega > 10^{-2}(E_F/\hbar)$, it falls off as ω^{-1}.

X. CONSEQUENCES AND APPLICATIONS

The present theory indicates that the amount of low-frequency magnetic fluctuations in a plasma, Eq. (50), is proportional to the temperature T and density of the plasma to the three-halves power $n^{3/2}$. Thus, the higher the density and/or temperature, the greater these fluctuations are. More significant examples may be found, therefore, in high-n and/or -T plasmas. Three concrete cases are discussed here.

A. Electron density fluctuations in gaseous plasmas

In addition to calculating the magnetic-field spectrum, we have elsewhere [22] calculated the longitudinal ion-density fluctuation spectrum arising from ion acoustic waves in a fluid plasma. We find a fluctuation spectrum given by

$$\langle \delta n_i^2 \rangle_{\mathbf{k}\omega} = \frac{\hbar\omega}{e^{\hbar\omega/T} - 1} \frac{\eta \omega_{pi}^2}{2\pi e^2} \left[1 + \frac{k_D^2}{k^2} \right]^2 \times \frac{k^2}{\left[\omega^2 \left[1 + \frac{k_D^2}{k^2} \right] - \omega_{pi}^2 \right]^2 + \eta^2 \omega^2 \left[1 + \frac{k_D^2}{k^2} \right]^2} ,$$

where k_D is the Debye wave number. This spectrum is plotted as a function of frequency at $k = 0.1 k_D$, for a hydrogen plasma of temperature $T = 100$ eV and density $n = 10^{16}$ cm^{-3} in Fig. 15(a). The spectrum peaks around the ion acoustic frequency of the given wave number. It should be noted that in an ion plasma, mass density is nearly proportional to ion density; therefore, the ion-

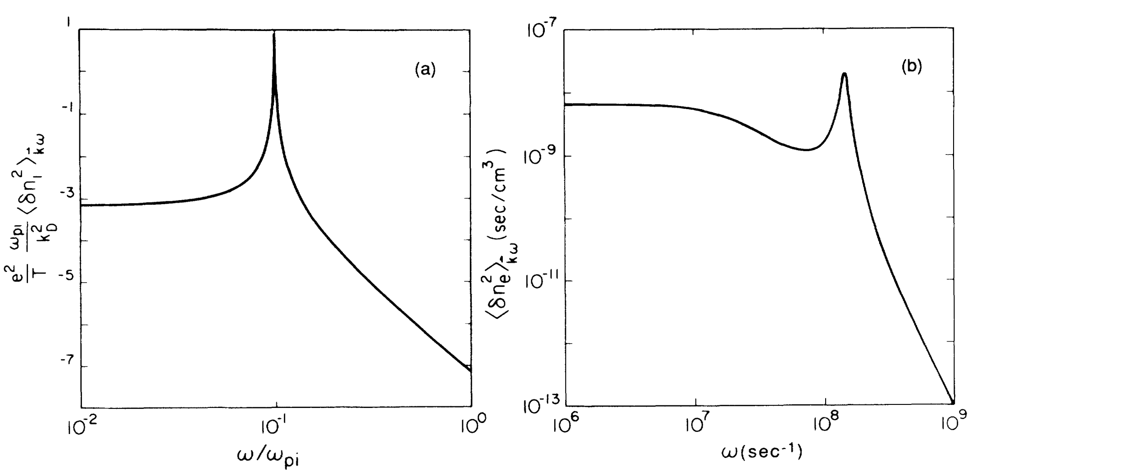

FIG. 15. Density fluctuation power spectra: (a) Ion fluctuation power spectrum $\langle \delta n_i \rangle_{\mathbf{k}\omega}$ as a function of frequency at $k = 0.1 k_D$. (b) Electron-density fluctuation power spectrum after Zhang and DeSilva [23]. $\langle \delta n_e^2 \rangle_{\mathbf{k}\omega}$ as a function of ω at $k = 415$ cm^{-1}.

density spectrum automatically gives the mass-density spectrum as well.

Zhang and DeSilva [23] have included more elaborate dissipation effects by way of the Braginskii transport equations (a set of two-fluid equations accounting for interspecies and intraspecies collisions, electron and ion thermal conductivity, electron and ion viscosity, and longitudinal electric fields). They have calculated and measured the low-frequency electron-density fluctuation spectrum in an Ar plasma. Figure 15(b) shows the result for $\langle \delta n_e^2 \rangle_{\mathbf{k}\omega}$ as a function of frequency at $k=415$ cm^{-1}, based on their theoretical treatment. The spectrum shown has been generated using the transport coefficients for an Ar plasma that Zhang and DeSilva derived from their experimental studies. In addition to the ion acoustic peak, there is a strong peak at $\omega=0$. Its existence was confirmed experimentally by Zhang and DeSilva. Their work shows that in a plasma where thermal conductivity and viscosity are important, fluctuations can be sustained in particle density as well as in magnetic field. In addition to Zhang and DeSilva's work, Stenzel's work on magnetic fluctuations [24] may have bearing on the present theory. He has experimentally found low-frequency magnetic spectra in a nonmagnetized plasma which may, upon further analysis, prove to be consistent with the results presented here.

It may be of interest to measure particle transport in a plasma sustaining such magnetic fluctuations. Trace particles may be followed in an experiment. Some theoretical treatment in such a direction has been laid out recently [25].

B. Cosmological consequences

We have discovered in the previous sections that electromagnetic waves in a plasma fall into two categories: those with large wavelengths ($k \lesssim \omega_{pe}/c$) and nearly zero frequency ($\omega \ll \omega_{pe}$) and those with small wavelengths ($k \gg \omega_{pe}/c$) and frequency greater than ω_{pe}. Those modes with $k > \omega_p/c$ are not significantly modified by the presence of the plasma ("hard photon"), while those with $k < \omega_p/c$ are significantly modified ("soft or plastic photon") [9,10]. It is those plastic photons or their magnetic fields that we are interested in, as they can have more magnetic fields in nature and may have left possible structural imprints on the primordial plasma. The strength of magnetic fluctuations $\langle B^2 \rangle_\lambda / 8\pi$, whose wavelengths are longer than λ, is given by

$$\langle B^2 \rangle_\lambda / 8\pi = (T/2)(4\pi/3)\lambda^{-3} .$$

For $\lambda_p = 2\pi c/\omega_p$,

$$(\langle B^2 \rangle_{\lambda_p})^{1/2} = 1.4\times 10^{-12}[n/(10^4\ \mathrm{cm}^3)]^{3/4}[T/(10^4\ \mathrm{K})]^{1/2}\ \mathrm{G} . \tag{155}$$

The electron magnetic energy $\langle B^2 \rangle_\omega^{\mathrm{bb}}$ contained in the blackbody radiation is proportional to ω^2 and $\langle B^2 \rangle_\omega^{\mathrm{bb}} \propto T^3 \propto a^{-3}$, where a is the scale factor of the expanding Universe. On the other hand, the zero-frequency magnetic fluctuation energy

$$\langle B^2 \rangle_{\omega \to 0} \propto T\omega_p^3/\mu \propto a^{-4} ,$$

where μ is the kinetic viscosity of the plasma. Thus the ratio of the zero-frequency fluctuations to the blackbody energy is proportional to a^{-1}. If we assume here that the level of magnetic fields is determined by Eq. (50) at each instance of time after ω integration, the plasma β scales as

$$\beta = nT/(\langle B^2 \rangle^0/8\pi) \propto n(c/\omega_p)^3 \propto a^{3/2} .$$

(This is based on the instantaneous adjustment of the magnetic fields to the level of thermal energy of the Universe.) This implies that the earlier the epoch of the Universe (small a), the greater is the relative importance of magnetic fluctuations with respect to the particle pressure. In fact, when $t=1$ s after the Big Bang, the amount of magnetic fluctuations is so large that β is nearly of order unity.

The significance of the presence of static (or nearly zero-frequency) magnetic fields in the cosmological plasma may be appreciated in the following. Two main scenarios [26] have been considered for primordial fluctuations: adiabatic fluctuations and isothermal fluctuations. The adiabatic (or isentropic) fluctuations are like those accompanied by ordinary sound waves. In such fluctuations the density of matter [electrons, positrons, and protons (and helium ions) for the case of the early radiation epoch] is accompanied by that of photons. Therefore, after electrons and positrons annihilate around $t=1$ s, or after electrons and ions recombine around $t=10^{13}$ s, the imprint of matter fluctuations would remain in photon fluctuations as a fossil of the primordial plasma structure. Thus the background microwave spectra would show a certain fluctuation or an anisotropy or inhomogeneity on top of the blackbody spectra. This would be a contradiction to the latest observations by the cosmic-background explorer (COBE) earth satellite, etc. [27,28]. On the other hand, imagine that, as we have shown, there exist static magnetic fields in the primordial plasma. Charged particles in the early radiation epoch ($t \lesssim 1$ s) or in the late radiation epoch ($t \lesssim 10^{13}$ s) readily respond to these magnetic fields. Where stronger magnetic fields exist, there will be less matter of charged particles, be it electron-positron-proton or electron-proton plasma. The matter will be distributed in such a way as to maintain the total matter pressure and magnetic pressure constant in space. Now, in addition, photons are present. Photons do couple strongly with charged particles, but not as strongly as static magnetic fields do with charged particles. Furthermore, photons are less strongly coupled with magnetic fields. This should leave a landscape of fluctuations in such a way that the sum of the magnetic and charged particle pressure is constant in space, while the photon pressure remains nearly constant in space. Such fluctuations are similar to the second category of isothermal fluctuations [26] (but they can be isentropic at the same time), as they are nearly frequencyless. This problem is closely related to the question of structure formation apparently without the so-called Sachs-Wolfe effect [29], which is discussed in detail in Ref. [30].

C. Anomalous spin relaxation in condensed matter

Another example of high-n "plasma" is electrons or other matter in a condensed state. When one tries to cool a metal below tens of mK by the standard nuclear adiabatic demagnetization cryostat technique, the spins of metallic electrons are manipulated from the external magnets. The standard Korringa theory [31] predicts that the spin equilibration time τ is inversely proportional to the temperature T of electrons. However, experiments [32] usually show an anomalous decrease in the product $T\tau$. A similar phenomenon was first observed in the spin-equilibration time anomaly in liquid He^3 in the superfluid phase by Avenel *et al.* [33]. Although this anomaly is not well understood at present, it is typically explained by resorting to impurity scattering. We suggest that it may be possible to explain the phenomenon of anomalously rapid relaxation as due to the magnetic fluctuations spontaneously created in the condensed matter due to the present mechanism as discussed in Sec. IV and the interactions between these fields and particle spins.

In addition to these three examples, the present theory may be found useful to tackle tough problems that have resisted full resolution to date, such as the $1/f$ noise [34] and the fluctuations in a (stable) nonuniform plasma (e.g., a certain type of stellarator plasma).

ACKNOWLEDGMENTS

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APPENDIX A: EVALUATING THE INTEGRAL OF EQ. (46)

The integral we need to evaluate is

$$\frac{2\hbar\omega}{e^{\hbar\omega/T}-1}\eta\omega_p^2c^2\int_0^{k_{\rm cut}}\frac{dk}{(2\pi)^3}4\pi\frac{k^4}{Ac^4k^4+Bc^2k^2+C}\ , \tag{A1}$$

where

$$A=\omega^2+\eta^2\ ,$$

$$B=2\omega^2(\omega_p^2-\omega^2-\eta^2)\ ,$$

$$C=[(\omega^2-\omega_p^2)^2+\eta^2\omega^2]\omega^2\ .$$

Normalizing all frequencies by ω_{pe} gives

$$\frac{2\hbar\omega'}{e^{(\hbar\omega_{pe}/T)\omega'}-1}\frac{\omega_p{'}^2\eta'}{2\pi^2}\left[\frac{\omega_{pe}}{c}\right]^3\times\int_0^{x_{\rm cut}}dx\frac{x^4}{A'x^4+B'x^2+C'}\ , \tag{A2}$$

where all primed quantities have been made dimensionless by division by the appropriate power of ω_{pe} (e.g., $\eta'=\eta/\omega_{pe}$), and $x=ck/\omega_{pe}$.

The first step to handling this integral is to rewrite it as

$$\frac{1}{A}\int_0^{x_c}dx-\frac{1}{A}\int_0^{x_c}dx\frac{Bx^2/A+C/A}{x^4+Bx^2/A+C/A}\ .$$

The first integral gives

$$\frac{x_c}{A}=\frac{x_c}{\omega'^2+\eta'^2}\ .$$

To evaluate the remaining integral, we find the (often complex) roots of the integrand's denominator:

$$x^2=\frac{-B\pm(B^2-4AC)^{1/2}}{2A}=r_\pm\ .$$

For the case $r_+\neq r_-$,

$$\begin{aligned}-\frac{1}{A}\int_0^{x_c}dx\frac{Bx^2/A+C/A}{x^4+Bx^2/A+C/A}&=-\frac{1}{A^2}\int_0^{x_c}dx\frac{Bx^2+C}{(x^2-r_+)(x^2-r_-)}\\&=-\frac{1}{A^2}\int_0^{x_c}dx\left[\frac{Bx^2+C}{(r_+-r_-)(x^2-r_+)}+\frac{Bx^2+C}{(r_--r_+)(x^2-r_-)}\right]\ .\end{aligned} \tag{A3}$$

The full integral in (164) becomes

$$\frac{x_c}{A}-\frac{1}{A^2}(C+Br_+)\frac{1}{\sqrt{-r_+}}\frac{1}{(r_+-r_-)}\tan^{-1}\frac{x_c}{\sqrt{-r_+}}-\frac{1}{A^2}(C+Br_-)\frac{1}{\sqrt{-r_-}}\frac{1}{(r_--r_+)}\tan^{-1}\frac{x_c}{\sqrt{-r_-}}\ . \tag{A4}$$

It also shows the Lorentzian behavior of $\langle B^2\rangle_\omega/8\pi$ near $\omega=0$. Notice that when ω becomes small, B and C both vanish. Remembering that $A=\omega'^2+\eta'^2$ and multiplying by the leading factor we left behind in Eq. (A2), we find

$$\frac{\langle B^2\rangle_\omega}{8\pi}=\frac{2\hbar\omega'}{e^{(\hbar\omega_{pe}/T)\omega'}-1}\frac{\omega_p'^2}{2\pi^2}\left[\frac{\omega_{pe}}{c}\right]^3\frac{\eta'}{\omega'^2+\eta'^2}x_c\ . \tag{A5}$$

Notice that if $\eta\rightarrow0$, this expression does *not* vanish. Rather, it becomes a Dirac δ function.

For the exceptional case, $r_+=r_-=r$, we write the integral as

$$\frac{x_c}{A} - \frac{1}{A^2}\int_0^{x_c} dx \frac{Bx^2+C}{(x^2-r)^2} .$$

The integrals left to do are

$$-\frac{C}{A^2}\int_0^{x_c} dx = \frac{C}{2A^2 r}\frac{x_c}{x_c^2-r} + \frac{C}{4A^2 r\sqrt{r}}\ln\left|\frac{\sqrt{r}-x_c}{\sqrt{r}+x_c}\right| , \tag{A6}$$

$$-\frac{B}{A^2}\int_0^{x_c} dx = \frac{B}{2A^2}\frac{x_c}{x_c^2-r} + \frac{B}{4A^2\sqrt{r}}\ln\left|\frac{\sqrt{r}-x_c}{\sqrt{r}+x_c}\right| .$$

When $r_+ = r_-$, it is true that $B^2=4AC$ and $r=-B/2A$. The full integral is, then,

$$\frac{x_c}{A} + \left|\frac{B}{4A^2}\right|\frac{x_c}{x_c^2-r} - \frac{3B}{8A^2\sqrt{r}}\ln\left|\frac{\sqrt{r}-x_c}{\sqrt{r}+x_c}\right| . \tag{A7}$$

APPENDIX B: EVALUATING THE INTEGRAL OF EQ. (124)

Here we need to do an integral of the form

$$\int_0^{x_c} dx \frac{x^5}{x^6+px^4+qx^2+r} . \tag{B1}$$

We first make a change of variable:

$$u=x^2, \quad dx=du/(2u^{1/2}) ,$$

which leads to

$$\int_0^{u_c} \frac{du}{2}\frac{u^2}{u^3+pu^2+qu+r} . \tag{B2}$$

We can find the (usually complex) roots a, b, and c of the integrand's denominator. Since the only negative term in the denominator is vanishingly small in our region of interest, the roots will all be distinct. So we rewrite the integral as

$$\int_0^{u_c} \frac{du}{2}\frac{u^2}{(u-a)(u-b)(u-c)} . \tag{B3}$$

The integrand can be rewritten

$$\frac{a^2}{(a-b)(a-c)(u-a)} + \frac{b^2}{(b-c)(b-a)(u-b)} + \frac{c^2}{(c-a)(c-b)(u-c)} . \tag{B4}$$

Each of these terms integrates to give a natural logarithm. The integral is given by

$$\frac{1}{2}\left[\frac{a^2}{(a-b)(a-c)}\ln|u-a|_0^{x_c^2} + (a\to b\to c\to a) + (a\to b\to c\to a)\right] . \tag{B5}$$

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