Exact effective-stress rules in rock mechanics

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The standard paradigm for analysis of rock deformation arises from postulating the existence of "an equivalent homogeneous porous rock." However, data on the pore-pressure dependence of fluid permeability for some rocks cannot be explained using any equivalent homogeneous porous medium. In contrast, a positive result shows that deformation measurements on both high-porosity sandstones and low-porosity granites can be explained adequately in terms of an equivalent two-constituent model of porous rocks, for which exact results have recently been discovered.

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In poroelasticity [1], variations in confining pressure and pore pressure both induce volume strains, but the signs of these strains are opposite and the magnitude differ. Some linear combination of these stresses will therefore produce no measurable strain even though the pressures themselves are changing. This fact leads naturally to the concept of effective stress [2].

Virtually all previous theoretical analyses of effective-stress relations for rocks [2,3] have used the same restrictive assumption used by Gassmann [4], postulating a microscopically homogeneous solid frame. Since natural rocks are normally quite heterogeneous and therefore obviously do not satisfy the homogeneity condition, the validity of such analyses rests on an implicit assumption that an "equivalent homogeneous rock" can be found and that the analysis of this fictitious homogeneous rock will satisfactorily explain all available data. However, we give a rigorous demonstration that some effective stress data on fluid transport through porous rocks cannot be explained in terms of any equivalent homogeneous rock. This counterexample to a common practice in rock mechanics shows clearly that more sophisticated methods are required to explain the behavior of porous media.

Rather than being model-based, our analysis relies on scaling rules that porous media must obey. For example, an insulating porous rock saturated with a conducting brine solution is known to have the conductivity $g=g_f/F$, where g_f is the conductivity of the brine and F is called the formation factor. Neglecting some small internal surface conduction effects, the formation factor is a bulk property depending only on the twisted shape of the internal pore space of the rock. Furthermore, F is a scale-invariant property of the rock; if the rock and its pore space could be uniformly expanded or contracted everywhere, then neither the porosity ϕ nor the formation factor F would change. Although, in general, a change of confining pressure δp_c and fluid pressure δp_f in a rock does not produce uniform swelling or shrinking, there is one set of circumstances where this happens: Consider a microhomogeneous rock (one containing a single type of solid grain) with no change in differential pressure $\delta p_d \equiv \delta p_c - \delta p_f$. Then, it is well known [2] that this idealized rock undergoes a uniform expansion or contraction (implying constant ϕ and F), and therefore the formation factor of such a rock can only be a function of the change in differential pressure. Corresponding arguments for the bulk and shear moduli (both of which are also scale-invariant properties) show that they must also be functions of the differential pressure, assuming only that the material bulk and shear constants K_m and μ_m for the grains do not change significantly as a function of the ambient pressure.

In terms of fluid pressure δp_f and differential pressure δp_d variations, the isotropic-stress–volume-strain relations are

$$-\frac{\delta V}{V}=\frac{\delta p_d}{K}+\frac{\delta p_f}{K_s} \tag{1}$$

for the total volume strain and

$$-\frac{\delta V_\phi}{V_\phi}=\frac{\delta p_d}{K_p}+\frac{\delta p_f}{K_\phi} \tag{2}$$

for the pore volume strain. The definitions of the various moduli may be easily inferred from the statements of these relations. The bulk modulus K is known as the "jacketed" modulus, K_s is the "unjacketed" modulus, and reciprocity [5,6] shows that $K_p=\phi K/\alpha$, where

$$\alpha \equiv 1-\frac{K}{K_s} . \tag{3}$$

The remaining modulus K_ϕ is independent of the others in general. Thus, the total volume and pore volume strains depend only on the porosity and the three constants: K, K_s, and K_ϕ.

The effective stress principle for total volume V follows immediately from the general stress-strain relation (1), giving

$$-\frac{\delta V}{V}=\frac{1}{K}(\delta p_c-\alpha\delta p_f) , \tag{4}$$

where the coefficient α was defined in (3). This coefficient is often measured [3,7]. See Table I. The usual range of values for α is $\phi \le \alpha \le 1$.

Considering variations in porosity $\phi = V_\phi/V$, we find

TABLE I. Values of ϕ, K_s, K, and α for two different confining pressures applied to various rocks measured by Coyner [7]. The value of the effective-stress coefficient θ for expansion and contraction is computed from the given values of K_s and K.

	$p_c=0$	$p_c=10$ MPa			$p_c=25$ MPa			
Rock sample	ϕ (%)	K_s (GPa)	K (GPa)	α	K_s (GPa)	K (GPa)	α	θ
Weber sandstone	9.5	37.0	4.0	0.89	38.0	10.0	0.74	0.995
Navajo sandstone	11.8	34.0	13.0	0.62	34.5	16.5	0.52	0.974
Berea sandstone	17.8	39.0	6.0	0.85	39.0	10.0	0.74	1.000
Bedford limestone	11.9	66.0	23.0	0.65	66.0	27.0	0.59	1.000
Barre granite	0.7	54.5	13.5	0.75	55.5	21.5	0.61	0.988
Westerly granite (red)	0.8	53.0	24.0	0.55	54.0	34.0	0.37	0.971
Chelmsford granite	1.1	54.5	8.0	0.85	55.5	17.0	0.69	0.995

$$-\frac{\delta\phi}{\phi}=\left[\frac{\alpha-\phi}{\phi K}\right](\delta p_c-\chi\delta p_f)\ , \tag{5}$$

where the coefficient is given by

$$\chi\equiv 1-\frac{1/K_\phi-1/K_s}{1/K_p-1/K}\ . \tag{6}$$

Experiments consistently show that an increase in confining pressure results in a decrease in porosity, so empirically the leading coefficient in (5) is positive and $\phi\le\alpha$.

Note that when only one solid constituent is present, $K_s=K_\phi=K_m$, so $\chi\equiv 1$; thus, in the Gassmann limit, the effective pressure for porosity is just the differential pressure p_d, as expected. Fatt [8] has shown that $\chi\simeq 0.85$ in some sandstones. Since $\chi\equiv 1$ is required for all homogeneous solid frames, this result is one clue that no equivalent homogeneous rock or set of rocks can be used to explain all available effective stress data.

In contrast to formation factor F and porosity ϕ, fluid permeability for porous media is not a scale-invariant material property: Darcy's constant k has the dimensions of (length)^2, so a uniform swelling or shrinking of the isotropic porous medium changes the value of the permeability proportional to $V^{2/3}$. The dependence of the permeability on geometry may therefore be expressed in general as

$$k=\text{const}\times HV^{2/3}\ , \tag{7}$$

where H depends only on the relative positioning of the grains and is therefore rigorously scale invariant. Like F, the factor H will generally be a complicated function of the confining and fluid pressures with no combination leaving it invariant. However, also like F in the Gassmann limit, when the pore space swells or shrinks at the same rate as the grains, H is rigorously seen to be a function only of the differential pressure. Well-known arguments for the formation factor [9,10] show that $\delta F/F\simeq m\,\delta\phi/\phi$, where $m\simeq 2$ is Archie's cementation exponent. In analogy with these arguments, suppose that

$$\frac{\delta H}{H}\simeq n\frac{\delta\phi}{\phi}=-n\left[\frac{\alpha-\phi}{\phi K}\right]\delta p_d\ . \tag{8}$$

The constant n may be related approximately to m through a Kozeny-Carman relation [11,12] of the form $k\simeq\phi^2/2s^2F$, where s is a measure of the specific surface area (for an equivalent smooth-walled pore), so $s^{-2}=\text{const}\times V^{2/3}$. Thus, $H\simeq\phi_2/F=\phi^{2+m}$ and $n\simeq 2+m\simeq 4$, which is in reasonable agreement with experiment [13].

To find the effective stress for the permeability *in the Gassmann limit,* we combine (4), (7), and (8). Then we find

$$\frac{\delta k}{k}=\frac{\delta H}{H}+\frac{2}{3}\frac{\delta V}{V}=-\left[n\left[\frac{\alpha-\phi}{\phi K}\right]+\frac{2}{3K}\right](\delta p_c-\kappa\delta p_f)\ , \tag{9}$$

where the effective-stress coefficient for permeability is

$$\kappa=1-\frac{2\phi(1-\alpha)}{3n(\alpha-\phi)+2\phi}\le 1\ . \tag{10}$$

The inequality follows from the facts that $\phi\le\alpha\le 1$ for the homogeneous frame and that the denominator is always positive as long as $\phi>0$.

The bound (10) is important for an analysis of rocks. If we suppose that any porous rock can be well approximated by an equivalent homogeneous rock, then (10) makes a definite prediction that the effective stress coefficient κ must be less than unity for microhomogeneous porous materials. Considering Table II, this prediction is verified by the data on two corundum (Al_2O_3) samples with no

TABLE II. Sample properties from Zoback and Byerlee [14] and Nur *et al.* [15].

Porous sample	ϕ (%)	Clay content (%)	k_0 (mdarcy)	κ
Al_2O_3(A)	26	0	817	0.43
Al_2O_3(B)	29	0	50	0.86
St. Peter	20	0.5	944	1.2
Brownstone	21	4.5	624	2.4
Berea 500	21	4.3	470	3.2
Massilon	24	6	995	3.5
Berea	19	8	42	4.0
Bandera	16	20.0	0.4	7.1

clay content. However, (10) is in direct conflict with all the other experimental results in Table II, showing that the effective stress coefficient κ for fluid permeability can be significantly greater than unity for a variety of rocks containing multiple constituents [7,14,15]. Thus, it is impossible to explain this aspect of the behavior of these clay-bearing porous rocks under stress in terms of an equivalent homogeneous frame. This result does not imply that it is never appropriate to use "an equivalent homogeneous frame" postulate when analyzing rock data, but it does show that circumstances can arise in very inhomogeneous rocks that invalidate such a postulate.

This negative result provides a strong motivation to pursue a more rigorous analysis of porous media containing at least two constituents. Suppose the solid frame is composed of two distinct porous constituents (say, type 1 and type 2), each of which obeys a volume stress-strain relation analogous to (1) so that, microscopically, we have $-\delta V^{(i)}/V^{(i)}=\delta p_d^{(i)}/K^{(i)}+\delta p_f^{(i)}/K_m^{(i)}$, for $i=1,2$. For two constituents, Berryman and Milton [6] have shown that there exists a ratio of the macroscopic pressure increments $\delta p_c/\delta p_f\equiv\theta$ such that the relative change in the volumes of each constituent (and therefore of the composite) is the same. Thus, the composite porous medium undergoes a uniform swelling or shrinking so the shapes and relative positions of all the porous constituents remain fixed while the overall size increases or decreases. Furthermore, the microscopic pressure changes equal the macroscopic ones, so $\delta p_d=\delta p_d^{(1)}=\delta p_d^{(2)}$ and $\delta p_f=\delta p_f^{(1)}=\delta p_f^{(2)}$. The resulting formula for θ is

$$\frac{1-\theta}{K}-\frac{1}{K_s}=\frac{\alpha^{(1)}-\alpha^{(2)}}{K^{(1)}-K^{(2)}} . \tag{11}$$

Both this result and another one for α [6] show how these coefficients depend on the various constants of the composite's constituents.

The analysis just presented also demonstrates the existence of a new effective-stress principle. For the two constituent medium, relative positions of the porous constituents remain unchanged if the quantity $\delta p_c-\theta\delta p_f=\text{const}$, so changes in geometry depend only on changes in this new effective stress. In the Gassmann limit, $\theta=\chi=1$.

Supposing the jacketed modulus K could be varied without changing the properties of the two constituents (for example, by somehow changing only the volume fractions), then (11) implies

$$\theta=1-\frac{\partial(1/K_s)}{\partial(1/K)} . \tag{12}$$

This rule can be used to compute θ from experimental data on K and K_s as a function of confining pressure [7] to the extent that both properties are in fact changing due to variations in the volume fractions of the constituents. For this approach to be valid, it is necessary (but not sufficient) to find that the value of θ computed this way remains constant over some finite range of variation δK, or equivalently that $1/K_s$ is a linear function of $1/K$. It is normally observed that $\partial K_s/\partial K\geq 0$ and, therefore, that $\partial(1/K_s)/\partial(1/K)\geq 0$.

To check whether these ideas agree with experiment, we have replotted some of Coyner's [7] data on K_s and K (see Fig. 1) for various rocks to illustrate the linear dependence of $1/K_s$ on $1/K$ for confining pressures less than 30 MPa. We have purposely excluded data on all the materials for higher pressures since these rocks are not expected to satisfy the simple linear model presented here at the higher pressures. To validate the equivalent homogeneous rock paradigm, these curves should all be constant. To validate the "two-constituent porous medium" paradigm, they only need to be linear or nearly linear over a small range of pressures. Although some of the curves are indeed constant (Berea sandstone, Bedford limestone), all the curves are observed to be nearly linear over this range of pressures. Table I summarizes the results for θ.

Now consider a special case called the "clayey-sandstone model." One of the constituents of this model has no porosity, so $K^{(2)}=K_m^{(2)}$ and $\alpha^{(2)}=\phi^{(2)}=0$. The other constituent has a very soft frame, so $K^{(1)}\to 0$ and $\alpha^{(1)}\to 1$. Substituting these limits into (11), we find that $\alpha\simeq 1-K/K_m^{(2)}$ and $\theta\to\alpha^{(1)}\to 1$. So $K_s\simeq K_m^{(2)}$ and, using the exact results [6], we find

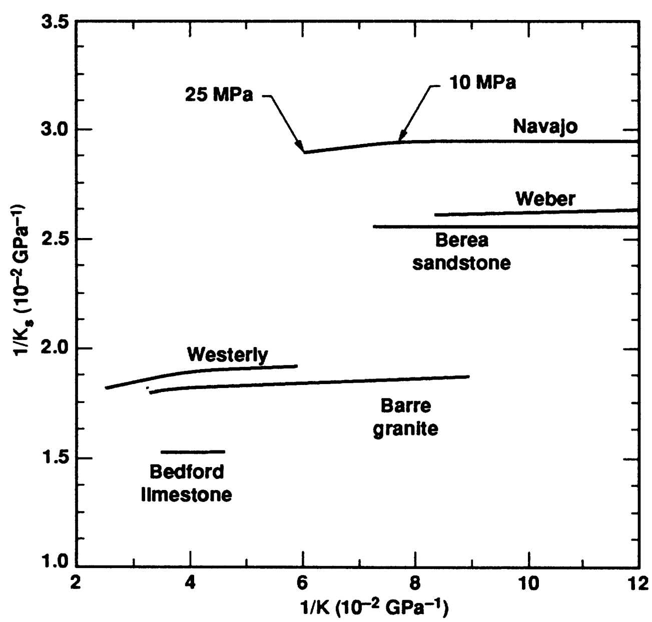

FIG. 1. The unjacketed compressibility $1/K_s$ as a function of the jacketed compressibility $1/K$ for six of Coyner's [7] suite of seven rocks. (Chelmsford granite is not shown since its curve is close to that for Barre granite.) Pressure variation is illustrated on the curve for Navajo sandstone, showing that the high end corresponds to lower confining pressure (10 MPa) and the low end to the higher pressures (25 MPa). In fact, Coyner's measurements continue to 100 MPa and almost all the curves begin to deviate from linearity for the higher pressures, but this behavior is beyond the scope of the present study and therefore is not shown.

$$\frac{1}{K_\phi} \simeq \frac{1}{K_m^{(1)}} + \frac{1}{\phi^{(1)}} \left[\frac{1}{K_m^{(2)}} - \frac{1}{K_m^{(1)}} \right] . \tag{13}$$

Defining $v_A = V^{(1)}/V$ (the volume fraction occupied by clay and voids), for this model the variation of v_A is given approximately by

$$\frac{\delta v_A}{v_A} \simeq -\frac{1}{K^{(1)}}(\delta p_c - \theta \delta p_f) . \tag{14}$$

Then, we find that

$$\chi \simeq 1 + \left[\frac{v_A - \phi}{\alpha - \phi} \right] \left[\frac{1}{K_m^{(1)}} - \frac{1}{K_m^{(2)}} \right] K . \tag{15}$$

The porosity is given by $\phi = v_A \phi^{(1)}$ initially. We use these results when we need to evaluate formulas for the effective-stress coefficient of fluid permeability.

It is shown elsewhere [16] that $k \simeq k_1 / F_A$, where k_1 is the intrinsic permeability of the clay-void assemblage and F_A is the formation factor associated with the void space surrounding the sand grains (i.e., with the clay absent). Supposing that, as in (7), $k_1 \simeq \text{const} \times \phi_1^{n_1} V_1^{2/3}$, where $n_1 \simeq 2 + m_1 \simeq 4$ and using $V_1 = v_A V$ together with Archie's law for $F_A = v_A^{-m_A}$, the effective-stress formula for the permeability bound in the clayey-sandstone model is given by

$$\begin{aligned} \frac{\delta k}{k} &= n_1 \frac{\delta \phi}{\phi} + \frac{2}{3} \frac{\delta V}{V} - q \frac{\delta v_A}{v_A} \\ &= -\left[n_1 \left[\frac{\alpha - \phi}{\phi K} \right] + \frac{2}{3K} - \frac{q}{K^{(1)}} \right] (\delta p_c - \kappa \delta p_f) , \end{aligned} \tag{16}$$

where $\phi = \phi_1 v_A$, $q = n_1 - m_A - \frac{2}{3}$, and

$$\kappa = \alpha + \frac{3n_1(\alpha - \phi)(\chi - \alpha) - 3q\phi K(\theta - \alpha)/K^{(1)}}{3n_1(\alpha - \phi) + 2\phi - 3q\phi K/K^{(1)}} . \tag{17}$$

Rigorous bounds on κ are difficult to obtain because the terms in the denominator of (17) do not have the same sign.

In general, the total volume effective-stress coefficient satisfies $\alpha < 1$, but the porosity coefficient χ can have values either less than or greater than unity. Also, θ is restricted by the empirical inequalities $\alpha \le \theta \le 1$. Thus, we find that the expression (17) can take a wide variety of values because of the variability of χ and $K^{(1)}$. For the clayey-sandstone model, taking the limit $K^{(1)} \to 0$ we find that $\kappa \to \theta \to 1$. However, if $K^{(1)}/K \ll 1$ but remains finite, then we can get a magnification effect due to some cancellation in the denominator of (17). In the case of most interest, the result for κ is given approximately by

$$\kappa \simeq \alpha + M(\chi - \theta) , \tag{18}$$

where the magnification factor $M \simeq 40$ if $n_1 = 4$, $\alpha = 0.85$, $v_A = 0.25$, $\phi = 0.2$, $q = \frac{4}{3}$, and $K/K^{(1)} = 10$. Evaluating (17) assuming $\chi \simeq 1.1$, the result for the effective-stress coefficient is $\kappa \simeq 5$. This estimate agrees reasonably well with the experimental result for the Berea sandstone considered by Zoback and Byerlee [14] and Coyner [7]. Thus, if the effective grain modulus of the pore-filling material $K_m^{(1)}$ is sufficiently smaller than that of the sand grains $K_m^{(2)}$, we can easily find that both $\chi > 1$ and $\kappa > 1$.

The theory shows that it is possible for the effective-stress coefficient κ to be greater than unity as observed by Zoback and Byerlee [14], Nur *et al.* [15], and Coyner [7]. To obtain better quantitative agreement between theory and experiment, we need to know values of constants usually not measured, such as K_ϕ or χ.

Having demonstrated the necessity of using a two-constituent paradigm for clay-rich sandstones, the next question is how generally applicable this approach may be. The presence of cracks in rock implies the presence of at least two types of constituents: (1) the intact microhomogeneous material and (2) regions of otherwise intact matrix material altered by the presence of cracks. The deformation properties of these two types of constituents may be quite different, since intact material generally will have a bulk modulus that is essentially independent of pressure for a very wide range of pressures, whereas the bulk modulus of the cracked material is much smaller. If—during the fluid-saturation process—some fraction of the cracks remains unsaturated while the remaining cracks become fully saturated, then a two-constituent porous medium paradigm will be appropriate. Table I and Fig. 1 show that, even for quite homogeneous granites like Westerly, the effective stress coefficient θ deviates measurably from unity—suggesting that the two-constituent model could be used in place of the demonstrably inadequate equivalent homogeneous rock approach.

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[1] M. A. Biot, J. Appl. Phys. **12**, 155 (1941).
[2] M. M. Carroll, in *Theoretical and Applied Mechanics*, Proceedings of the 15th International Congress of Theoretical and Applied Mechanics, Toronto, 1980, edited by F. P. J. Rimrott and B. Tabarrok (North-Holland, Amsterdam, 1980), pp. 251–262.
[3] A. Nur and J. D. Byerlee, J. Geophys. Res. **76**, 6414 (1971).
[4] F. Gassmann, Verh. Naturforsch. Ges. Zürich **96**, 1 (1951).
[5] R. J. S. Brown and J. Korringa, Geophys. **40**, 608 (1975).
[6] J. G. Berryman and G. W. Milton, Geophys. **56**, 1950

(1991).
[7] K. B. Coyner, Ph.D. thesis, Massachusetts Institute of Technology, 1984.
[8] I. Fatt, Soc. Pet. Eng. Trans. AIME **213**, 362 (1958).
[9] G. E. Archie, Soc. Pet. Eng. Trans. AIME **146**, 54 (1942).
[10] P. N. Sen, C. Scala, and M. H. Cohen, Geophys. **46**, 781 (1981).
[11] M. S. Paterson, Mech. Mater. **2**, 345 (1983).
[12] J. B. Walsh and W. F. Brace, J. Geophys. Res. **89**, 9425 (1984).
[13] P. M. Adler, C. G. Jacquin, and J. A. Quiblier, Int. J. Multiphase Flow **16**, 691 (1990).
[14] M. D. Zoback and J. D. Byerlee, Bull. Am. Assoc. Pet. Geol. **59**, 154 (1975).
[15] A. M. Nur, J. D. Walls, K. Winkler, and J. DeVilbiss, Soc. Pet. Eng. J. **20**, 450 (1980).
[16] J. G. Berryman, J. Geophys. Res. (to be published).