Statistical analysis of compressible turbulent shear flows with special emphasis on turbulence modeling

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Compressible turbulent shear flows are analyzed using a two-scale direct-interaction (propagator-renormalization) approximation with the density, velocity, and internal energy as the primitive variables. This method elucidates the effects of fluid compressibility on various important correlation functions that appear in the equations for the mean field. Specifically, density fluctuations are confirmed to be closely linked with compressibility effects. These results show that the current one-point turbulence model based on mass-weighted averaging, which is widely used in aeronautical studies, obscures many important features of fluid compressibility. On this basis, modeling based on the combined use of ensemble averaging and primitive variables is recommended as the one-point turbulence modeling applicable to the study of aerospace and astrophysical phenomena.

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I. INTRODUCTION

High-speed flows in aerospace phenomena, which often accompany large changes of fluid density, are of very high Reynolds numbers and are beyond the scope of the direct numerical simulation of the primitive fluid-dynamical equations in the foreseeable future. Such a difficulty is more prominent in astrophysical phenomena associated with huge spatial scales and large velocity. As a result, some kind of turbulence model is indispensable for supplementing the process of energy transfer to small-scale motions that result in the loss of kinetic energy as heat generation. The first step in the compressible turbulence modeling is the straightforward extension of the one-point incompressible turbulence models to the compressible ones with the change of the mean density incorporated. This step has been made usually under the mass-weighted averaging [1]. The averaging keeps the resulting system of equations very similar to the incompressible counterpart and brings the least alteration of the current computational codes for compressible turbulence.

The reverse side of the mass-weighted-mean models is that the effects of density change are apt to be obscured. As a result, the success of the mass-weighted-mean models hinges on how properly the effects of compressibility can be incorporated into the mean correlation functions in the mean-field equations whose forms are very similar to the solenoidal counterparts. A typical instance of the attempts of incorporating compressibility effects into the solenoidal two-equation turbulence models of the k-ϵ type is the modeling of the effect of dilatational energy dissipation. Most simply, the increase in the energy dissipation due to the dilatation is related to the turbulent Mach number and the dissipation rate related to the velocity strain [2,3]. Under this modeling, the solenoidal k-ϵ models based on the eddy-viscosity approximation are extended to the compressible ones by interpreting k as the sum of the solenoidal and compressible turbulent kinetic energy and ϵ as the dissipation rate due to the velocity strain motion. The extended model has been confirmed to work well in unseparated flows along a plate for Mach numbers as large as 10 [4]. The correction to the density-change effects, however, has not yet succeeded in coping with the high-compressibility effects that are encountered in shock-wave–turbulence interactions. This fact signifies that the incorporation of compressibility effects through the mean density is rather weak.

Another shortcoming of the modeling based on the mass-weighted averaging is that the results of the turbulence theories based on two-point closure or spectral methods, which are always founded on the simple ensemble averaging, cannot be used apart from the analogical use. In the context of the solenoidal turbulence modeling, some two-point closure theories have contributed to the theoretical examination of the models constructed in a phenomenological manner and their improvements [5–12]. For compressible turbulence, the author [13] previously proposed an attempt of reconciling a two-point closure theory with the mass-weighted-mean modeling. In it, a two-scale direct-interaction (propagator-renormalization) approximation (TSDIA) is combined with the density-weighted velocity and internal energy in place of the velocity and internal energy themselves. One of the major results is the suggestion about the pressure-dilatation correlation function, which is a representative quantity associated with fluid compressibility. Namely, the function cannot be expressed without the information about density fluctuations. This result inevitably leads to a three-equation model with the density-variance equation added. Such a three-equation model has been further developed and the importance of density-fluctuation effects has been confirmed in the application to compressible boundary layer and mixing layer flows [14].

The previous TSDIA analysis of compressible turbulent shear flows, however, suffers from the shortcomings closely related with the density-weighted velocity, although it, as well as the density-weighted internal energy, is instrumental to preserving the properties intrinsic to the mass-weighted averaging. Its typical shortcoming is the difficulty in the satisfaction of the Galilean-transformation rule. It comes from the fact that the velocity fluctuation is Galilean invariant, whereas the counterpart of the mass-weighted velocity is not so. The renormalization methods of perturbational expansions like the TSDIA often lead to the breakage of the Galilean-transformation rule. As a result, those results are incomplete in the use as a guiding principle of compressible turbulence modeling. To overcome this difficulty is urgent, considering the importance of the construction of a turbulence model that can cope with high-compressibility effects. This point is one of the major motivations of this work.

Another motivation of this study is associated with the difficulty with the mass-weighted averaging that leads to the mixture of the mass-weighted-mean and simple ensemble-mean quantities. Under the averaging, the pure ensemble-mean quantities also appear, except the advection terms in the momentum and internal energy equations to which the density always attaches. A typical instance of such a mixture comes from the viscosity term in the velocity equation, and it becomes necessary to represent the ensemble mean of the velocity in terms of the mass-weighted-mean quantities. As a result, the ensemble mean of the velocity fluctuation around the mass-weighted-mean value is required. At present, we do not have any theoretical guiding principle for estimating it. The usual ensemble-mean modeling is free from such a difficulty. This fact signifies that the mass-weighted-mean modeling is not so simple, although it is widely used. Considering the above circumstances, the ensemble-mean compressible turbulence modeling deserves serious consideration. To this end, it is necessary to construct a two-point closure theory for compressible turbulent shear flows and get some useful information about their mathematical properties.

In this paper, we shall apply the TSDIA based on the original primitive variables, that is, the density, velocity, and internal energy to clarify the mathematical structures of various important correlation functions in compressible turbulent shear flows. This paper is organized as follows. The fundamental equations are given in Sec. II and the comparison between the ensemble and mass-weighted averagings is made in Sec. III. The major steps of the TSDIA and the final results are given in Sec. IV. In Sec. V, the results obtained are discussed in detail from the viewpoint of application to the ensemble-mean turbulence modeling. The supplementary explanation about the details of the TSDIA calculation is given in Appendix A and the prototype of a compressible turbulence model based on the ensemble-mean quantities is proposed and discussed in Appendix B.

II. FUNDAMENTAL EQUATIONS

The motion of a viscous compressible fluid is governed by the conservation laws for mass, momentum, and internal energy:

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0 , \tag{1}$$

$$\frac{\partial}{\partial t}\rho u_i+\frac{\partial}{\partial x_j}\rho u_j u_i=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\mu s_{ji} , \tag{2}$$

$$\frac{\partial}{\partial t}\rho e+\nabla\cdot(\rho e\mathbf{u})=\nabla\cdot(\lambda\nabla\theta)-p\nabla\cdot\mathbf{u}+\phi . \tag{3}$$

Here ρ is the fluid density, $\mathbf{u}$ is the velocity, p is the pressure, e is the internal energy, θ is the temperature, μ is the viscosity, and λ is the heat conductivity. The deviatoric part of the velocity strain s_{ij} is given by

$$s_{ij}=\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}-\tfrac{2}{3}\nabla\cdot\mathbf{u}\delta_{ij} , \tag{4}$$

and the dissipation function ϕ is defined as

$$\phi=\mu s_{ij}\frac{\partial u_j}{\partial x_i} . \tag{5}$$

As the thermodynamic relation, we assume that the gas is a perfect gas; namely, we have

$$p=R\rho\theta=(\gamma-1)\rho e , \tag{6}$$

where

$$e=C_v(\theta)\theta , \tag{7}$$

$$\gamma=C_p(\theta)/C_v(\theta) \tag{8}$$

(C_v and C_p are the specific heats at constant volume and pressure, respectively).

III. ENSEMBLE AND MASS-WEIGHTED AVERAGINGS

We firstly give the ensemble-mean and mass-weighted-mean forms of the fundamental equations (1)–(3). Afterwards, we discuss the major features of the two.

A. Ensemble averaging

The ensemble averaging of a quantity f is written as

$$f=F+f', \quad F=\langle f\rangle , \tag{9a}$$

$$f'=f-F , \tag{9b}$$

where

$$f=(\rho,\mathbf{u},p,e,\theta,\phi) , \tag{10a}$$

$$F=(\rho_M,\mathbf{U},P,E,\Theta,\Phi) , \tag{10b}$$

$$f'=(\rho',\mathbf{u}',p',e',\theta',\phi') \tag{10c}$$

(subscript M denotes mean).

On taking the ensemble average of Eqs. (1)–(3), we have

$$\frac{\partial \rho_M}{\partial t}+\nabla\cdot(\rho_M \mathbf{U}+\langle \rho'\mathbf{u}'\rangle)=0 , \tag{11}$$

$$\frac{\partial}{\partial t}(\rho_M U_i+\langle \rho' u_i'\rangle)+\frac{\partial}{\partial x_j}(\rho_M U_j U_i+\rho_M\langle u_j' u_i'\rangle+U_j\langle \rho' u_i'\rangle+U_i\langle \rho' u_j'\rangle)=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\mu_M S_{ji} , \tag{12}$$

$$\frac{\partial}{\partial t}(\rho_M E+\langle \rho' e'\rangle)+\nabla\cdot(\rho_M \mathbf{U}E+\rho_M\langle \mathbf{u}'e'\rangle+\mathbf{U}\langle \rho' e'\rangle+E\langle \rho'\mathbf{u}'\rangle)=\nabla\cdot(\lambda_M\nabla\Theta)-(P\nabla\cdot\mathbf{U}+\langle p'\nabla\cdot\mathbf{u}'\rangle)+\Phi , \tag{13}$$

where μ_M and λ_M are the mean of μ and λ, respectively, and the fluctuations of μ and λ have been neglected. Moreover, S_{ij}, P, and p' are given by

$$S_{ij}=\langle s_{ij}\rangle, \quad s_{ij}=S_{ij}+s_{ij}' , \tag{14}$$

$$P=(\gamma-1)(\rho_M E+\langle \rho' e'\rangle) , \tag{15}$$

$$p'=(\gamma-1)(\rho_M e'+\rho' E+\rho' e'-\langle \rho' e'\rangle) \simeq(\gamma-1)(\rho_M e'+\rho' E) . \tag{16}$$

In Eqs. (12) and (13), we have dropped $\langle \rho' u_j' u_i'\rangle$ and $\langle \rho' u' e'\rangle$, compared with $U_j\langle \rho' u_i'\rangle$ and $\mathbf{U}\langle \rho' e'\rangle$. The inclusion of the former two does not bring any essential difficulty, but the latter two directly dependent on the mean flow are considered more important. The mean dissipation function Φ is

$$\Phi=\mu_M S_{ij}\frac{\partial U_j}{\partial x_i}+\rho_M\epsilon , \tag{17}$$

where ϵ is the contribution of the fluctuation $\mathbf{u}'$ to the dissipation of the kinetic energy, which is

$$\epsilon=\nu_M\left\langle s_{ij}'\frac{\partial u_j'}{\partial x_i}\right\rangle \tag{18}$$

with

$$\nu_M=\mu_M/\rho_M . \tag{19}$$

In the case that $\mathbf{u}'$ can be regarded as nearly isotropic, Eq. (18) is approximated as

$$\epsilon=\nu_M\left\langle \left[\partial u_j'/\partial x_i\right]^2+\tfrac{1}{3}(\nabla\cdot\mathbf{u}')^2\right\rangle , \tag{20}$$

where the second part represents the dissipation effect due to the dilatation.

From Eqs. (11)–(13), the mathematical properties of the following correlation functions need to be elucidated:

$$\langle \rho'\mathbf{u}'\rangle, \quad \langle u_i' u_j'\rangle, \quad \langle \rho' e'\rangle, \quad \langle e'\mathbf{u}'\rangle, \quad \langle p'\nabla\cdot\mathbf{u}'\rangle . \tag{21}$$

The quantities $\langle \rho'\mathbf{u}'\rangle$, $\langle u_i' u_j'\rangle$, and $\langle e'\mathbf{u}'\rangle$ represent the transport rates of mass, momentum, and internal energy, respectively. The pressure-dilatation correlation function $\langle p'\nabla\cdot\mathbf{u}'\rangle$ plays an important role in close connection with the enhancement of the conversion of kinetic energy to heat by fluid compressibility. This point will later become clear.

Next we consider the equations for the fluctuations $(\rho,\mathbf{u}',e')$. To this end, we rewrite the left-hand sides or the advection terms of Eqs. (2) and (3) as

$$\frac{\partial}{\partial t}\rho u_i+\frac{\partial}{\partial x_j}\rho u_j u_i=\rho\left[\frac{\partial u_i}{\partial t}+(\mathbf{u}\cdot\nabla)u_i\right] , \tag{22a}$$

$$\frac{\partial}{\partial t}\rho e+\nabla\cdot(\rho\mathbf{u}e)=\rho\left[\frac{\partial e}{\partial t}+(\mathbf{u}\cdot\nabla)e\right] , \tag{22b}$$

using Eq. (1). Under Eq. (22), the $\rho\mathbf{u}$ and ρe equations change into the time evolution equations for $\mathbf{u}$ and e. This rewriting not only considerably reduces the mathematical burden in the TSDIA analysis of the fluctuating field, but it is also instrumental to keeping the results from breaking the Galilean-transformation rule, unlike the previous treatment [13], as will be later discussed. Therefore Eq. (22) is not trivial in constructing a compressible turbulence theory.

The fluctuating field $(\rho',\mathbf{u}',e')$ obeys

$$\frac{D\rho'}{Dt}+\nabla\cdot(\rho'\mathbf{u}'-\langle \rho'\mathbf{u}'\rangle)+\rho_M\nabla\cdot\mathbf{u}'=-(\mathbf{u}'\cdot\nabla)\rho_M-\rho'\nabla\cdot\mathbf{U} , \tag{23}$$

$$\frac{Du_i'}{Dt}+(\mathbf{u}'\cdot\nabla)u_i'-\langle(\mathbf{u}'\cdot\nabla)u_i'\rangle-\frac{\partial}{\partial x_j}\nu_M s_{ji}'+\frac{1}{\rho_M}\frac{\partial p'}{\partial x_i}+\frac{\rho'}{\rho_M}\frac{Du_i'}{Dt}-\left\langle\frac{\rho'}{\rho_M}\frac{Du_i'}{Dt}\right\rangle$$
$$=-(\mathbf{u}'\cdot\nabla)U_i-\frac{\rho'}{\rho_M}\frac{DU_i}{Dt}-\frac{1}{\rho_M}(\rho' u_j'-\langle \rho' u_j'\rangle)\frac{\partial U_i}{\partial x_j} , \tag{24}$$

$$\frac{De'}{Dt}+(\mathbf{u}'\cdot\nabla)e'-\langle(\mathbf{u}'\cdot\nabla)e'\rangle-\nabla\cdot(\kappa_M\nabla e')+(P/\rho_M)\nabla\cdot\mathbf{u}'+\frac{\rho'}{\rho_M}\frac{De'}{Dt}-\left\langle\frac{\rho'}{\rho_M}\frac{De'}{Dt}\right\rangle+(p'/\rho_M)\nabla\cdot\mathbf{u}'-\langle(p'/\rho_M)\nabla\cdot\mathbf{u}'\rangle$$
$$=-(\mathbf{u}'\cdot\nabla)E-\frac{\rho'}{\rho_M}\frac{DE}{Dt}-(p'/\rho_M)\nabla\cdot\mathbf{U}-(1/\rho_M)(\rho'\mathbf{u}'-\langle \rho'\mathbf{u}'\rangle)\cdot\nabla E , \tag{25}$$

where the quantities of third order in ρ', $\mathbf{u}'$, and e' have been dropped, as in Eqs. (12) and (13), and

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+(\mathbf{U}\cdot\nabla)\ , \tag{26}$$

$$\kappa_M=\lambda_M/(C_v\rho_M)\ . \tag{27}$$

In what follows, we shall show that the spectra of the density and velocity variances defined by

$$\langle\rho'^2\rangle/2=K_d\ , \tag{28}$$

$$\langle\mathbf{u}'^2\rangle/2=K\ , \tag{29}$$

play a key role in various correlation functions. They obey

$$\frac{DK_d}{Dt}=-\langle\rho'\mathbf{u}'\rangle\cdot\nabla\rho_M-K_d\nabla\cdot\mathbf{U}+\rho_M\langle\mathbf{u}'\cdot\nabla\rho'\rangle -\rho_M\nabla\cdot\langle\rho'\mathbf{u}'\rangle-\langle\rho'\nabla\cdot(\rho'\mathbf{u}')\rangle\ , \tag{30}$$

$$\frac{DK}{Dt}=-\langle u_i'u_j'\rangle\frac{\partial U_j}{\partial x_i}-(1/\rho_M)\langle\rho'\mathbf{u}'\rangle\cdot\frac{D\mathbf{U}}{Dt} -[\epsilon-(1/\rho_M)\langle p'\nabla\cdot\mathbf{u}'\rangle]+\mathcal{D}\ , \tag{31}$$

where $\mathcal{D}$ denotes the diffusionlike terms which are defined by

$$\mathcal{D}=-(1/\rho_M)\nabla\cdot\langle p'\mathbf{u}'\rangle-\nabla\cdot\langle(\mathbf{u}'^2/2)\mathbf{u}'\rangle+\langle(\mathbf{u}'^2/2)\nabla\cdot\mathbf{u}'\rangle -(1/\rho_M)\langle\rho'u_i'u_j'\rangle\frac{\partial U_j}{\partial x_i}-(1/\rho_M)\left\langle\rho'u'\frac{Du'}{Dt}\right\rangle +\frac{\partial}{\partial x_j}\nu_M\langle s_{ji}'u_i'\rangle\ . \tag{32}$$

B. Mass-weighted averaging

The mass-weighted averaging of a quantity f is defined by

$$\hat{f}\equiv\{f\}=\langle\rho f\rangle/\rho_M\ , \tag{33a}$$

$$f''=f-\hat{f}\ . \tag{33b}$$

Taking the ensemble mean of Eqs. (1)–(3) and using Eq. (33), we have

$$\frac{\partial\rho_M}{\partial t}+\nabla\cdot(\rho_M\hat{\mathbf{u}})=0\ , \tag{34}$$

$$\frac{\partial}{\partial t}\rho_M\hat{u}_i+\frac{\partial}{\partial x_j}\rho_M\hat{u}_j\hat{u}_i=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}(-\rho_M\{u_j''u_i''\}) +\frac{\partial}{\partial x_j}(\mu_M S_{ji})\ , \tag{35}$$

$$\frac{\partial}{\partial t}\rho_M\hat{e}+\nabla\cdot(\rho_M\hat{\mathbf{u}}\hat{e})=\nabla\cdot(\lambda_M\nabla\Theta)+\nabla\cdot(-\rho_M\{\mathbf{u}''e''\}) -\langle p\nabla\cdot\mathbf{u}\rangle+\Phi\ . \tag{36}$$

Here the ensemble-mean dissipation function Φ has already been given by Eq. (17). On the right-hand side of Eq. (35), we should note that P and S_{ij} are the ensemble-mean quantities, whose appearance illustrates the mixture of ensemble-mean and mass-weighted-mean quantities under the mass-weighted averaging. The pressure-dilatation correlation function $\langle p\nabla\cdot\mathbf{u}\rangle$ is written as

$$\langle p\nabla\cdot\mathbf{u}\rangle=(\gamma-1)\rho_M[\hat{e}\{\nabla\cdot\mathbf{u}\}+\{e''(\nabla\cdot\mathbf{u})''\}]\ , \tag{37}$$

using Eq. (6). Here we should note that

$$\{\nabla\cdot\mathbf{u}\}-\nabla\cdot\hat{\mathbf{u}}=-\langle(\mathbf{u}''\cdot\nabla)\rho\rangle/\rho_M \simeq-[(\langle\mathbf{u}''\rangle\cdot\nabla)\rho_M]/\rho_M\ ; \tag{38}$$

namely, we have $\{\nabla\cdot\mathbf{u}\}\neq\nabla\cdot\hat{\mathbf{u}}$. This point will later be referred to.

Entirely similarly, we can construct the equation for the mass-weighted-mean turbulent energy K_M defined by

$$K_M=\{\mathbf{u}''^2\}/2\ . \tag{39}$$

The detailed are not given here since we adopt the ensemble-averaging procedure in the later theoretical treatment (see Ref. [4] for the details).

C. Comparison between the ensemble and mass-weighted averagings

In order to capture the difference between the ensemble and mass-weighted averagings, we consider two kinds of mean-field equations (11)–(13) and (34)–(36). A prominent feature of the mass-weighted averaging is the compactness of the left-hand sides of Eqs. (35) and (36). Namely, the Reynolds stress and the turbulent internal-energy flux are given by

$$-\rho_M\{u_i''u_j''\}\ , \tag{40}$$

$$\rho_M\{u''e''\}\ , \tag{41}$$

respectively. Their counterparts under the ensemble averaging are

$$-(\rho_M\langle u_i'u_j'\rangle+U_j\langle\rho u_i'\rangle+U_i\langle\rho u_j'\rangle)\ , \tag{42}$$

$$\rho_M\langle\mathbf{u}'e'\rangle+\mathbf{U}\langle\rho'e'\rangle+E\langle\rho'\mathbf{u}'\rangle\ . \tag{43}$$

The reverse side of the compactness under the mass-weighted averaging results in the complexity of all the terms except the advection terms. Its typical instance is the pressure-dilatation effect $\langle p\nabla\cdot\mathbf{u}\rangle$, which is given by Eq. (37). In order to express the mass-weighted mean of $\nabla\cdot\mathbf{u}$ or $\{\nabla\cdot\mathbf{u}\}$ in terms of the mass-weighted-mean quantities like $\hat{\mathbf{u}}$, we need Eq. (38), which includes the ensemble mean of the velocity fluctuation around the mass-weighted-mean value. From the relation that

$$\langle\mathbf{u}''\rangle=-\langle\rho'\mathbf{u}''\rangle/\rho_M\ , \tag{44}$$

we have to construct a model for the turbulent mass flux $\langle\rho'\mathbf{u}''\rangle$, which is similar to $\langle\rho'\mathbf{u}'\rangle$ in Eq. (42).

In the usual modeling, $\langle\rho'\mathbf{u}''\rangle$ is written as

$$\langle\rho'\mathbf{u}''\rangle=-\nu_d\nabla\rho_M\ , \tag{45}$$

with resort to the concept of gradient diffusion, where ν_d

is a dimensional coefficient. The counterpart under the ensemble averaging or $\langle \rho' \mathbf{u}' \rangle$, which appears in the mean density equation (11) as well as Eqs. (42) and (43), is modeled as

$$\langle \rho' \mathbf{u}' \rangle = -\nu'_d \nabla \rho_M \tag{46}$$

in correspondence to Eq. (45). On substituting Eq. (46) into Eq. (11), the resulting equation is very similar to the mean equation for the scalarlike temperature. Here ν'_d is the so-called turbulent diffusivity and expresses a kind of scalar cascade process. Such a process is natural for the internal energy or temperature since the equation originally includes the diffusion term given by the first on the right-hand side of Eq. (3). On the other hand, the original ρ equation has no counterpart. Therefore the relevance of the modeling (45) and (46) is doubtful. Eventually, the modeling of the turbulent mass flux is indispensable for both the mass-weighted and ensemble averagings. What is a difficulty in the mass-weighted-mean modeling is that the current turbulence theories cannot become a guiding principle for the modeling since they are always founded on the ensemble averaging.

Another feature of the reverse side of the compactness of the advection terms under the mass-weighted averaging is linked with the modeling of the Reynolds stress (40) and the turbulent internal-energy flux (41). For instance, Eq. (40) is usually modeled as

$$-\rho_M \{u''_i u''_j\} = -\tfrac{2}{3}\rho_M K_M \delta_{ij} + \rho_M \nu_e \{s_{ij}\} , \tag{47}$$

using the eddy-viscosity concept. As a result, the direct effects of density change appear only through ρ_M. On the other hand, the counterpart (42) is directly dependent on the density fluctuation through $\langle \rho' \mathbf{u}' \rangle$ in addition to the mean density ρ_M. This fact signifies that the usual ensemble-mean models can more easily incorporate the effects of density change, specifically those of density fluctuation, compared with the mass-weighted-mean models.

The modeling of $\langle \rho' \mathbf{u}' \rangle$ also becomes critical in the study of the low-Reynolds-number effects near a solid wall. The viscosity or μ_M-related term in Eq. (35) under the mass-weighted averaging is also written using the ensemble-mean velocity $\mathbf{U}$ through S_{ij} [Eq. (14)]. This term is important in the construction of a model possessing the correct asymptotic near-wall behavior, which leads to accurate results, as was stated in Sec. I [4]. Under the mass-weighted averaging, it is indispensable to express $\mathbf{U}$ using the mass-weighted-mean velocity $\hat{\mathbf{u}}$. These two velocities are related as

$$\mathbf{U} = \hat{\mathbf{u}} - \langle \rho' \mathbf{u}' \rangle / \rho_M . \tag{48}$$

Therefore the modeling of $\langle \rho' \mathbf{u}' \rangle$ is also necessary in the context of the viscosity effects.

Finally, let us simply see the structures of the equations for the density variance and the turbulent kinetic energy. Under the ensemble averaging, the K equation (31) reduces to

$$\frac{DK}{Dt} = -\langle u'_i u'_j \rangle \frac{\partial U_j}{\partial x_i} - \epsilon + \nabla \cdot [-\langle (p'/\rho + \mathbf{u}'^2/2) \mathbf{u}' \rangle + \nu \nabla K] \tag{49}$$

in the solenoidal limit ($\nabla \cdot \mathbf{u}' = 0$), where $\nu = \mu/\rho$. The role of each term on the right-hand side of Eq. (49) is clear and these terms are called the production, dissipation, and diffusion rates, respectively. The clarity of the physical meaning of each term comes from the fact that the mean total kinetic energy $\int \langle \mathbf{u}^2/2 \rangle dV$ is conserved in the absence of ν. In the case of compressible flows, its counterpart in the absence of ν and λ is

$$\int \langle \rho(\mathbf{u}^2/2 + e) \rangle dV . \tag{50}$$

The kinetic part $\int \langle \rho \mathbf{u}^2/2 \rangle dV$ leads to

$$\int (\rho \mathbf{U}^2/2 + \rho_M K + \mathbf{U} \cdot \langle \rho' \mathbf{u}' \rangle + \langle \rho' \mathbf{u}'^2/2 \rangle) dV . \tag{51}$$

In this work, the last term will be neglected, compared with the third term.

The K equation (31) is similar to Eq. (49). The first term in Eq. (31) corresponds to the first term of Eq. (49). The role of each term, however, is not so simple as in Eq. (49) since K is part of the turbulent kinetic energy, as can be seen from Eq. (51). The pressure-dilatation effect ($\langle p' \nabla \cdot \mathbf{u}' \rangle$), which is combined with ϵ in Eq. (31), leads to the enhancement of kinetic-energy dissipation in the case that it is negative. The role of the effect will be clear in the later analysis. In the diffusionlike term (32), the first term is the counterpart of the p'-related part in Eq. (49). In the solenoidal limit, $\langle (p'/\rho) \mathbf{u}' \rangle$ is combined with $\langle (\mathbf{u}'^2/2) \mathbf{u}' \rangle$ to be modeled as

$$\langle (p'/\rho + \mathbf{u}'^2/2) \mathbf{u}' \rangle = -\nu_K \nabla K , \tag{52}$$

where ν_K is the diffusivity of K. As a result, the feature of p' is obscured, but its effect is generally considered small compared with the triple-velocity correlation (this point is also confirmed using the TSDIA analysis [15]). In compressible turbulence, however, p' is given by Eq. (16) and is directly linked with the mean field E and ρ_M. In a highly compressed region near a shock wave where the temperature rises, the first term in Eq. (32) possibly becomes more important than the second term or the triple-velocity correlation. In such a case, the relative importance of the two terms reverses, compared with the solenoidal case. Not enough attention, however, has been paid to this point in the current turbulence modeling.

In the K_d equation (30), the first and second terms on the right-hand side bear a role of the K_d production rate. The first term expresses the contribution from $\nabla \rho_M$, just as in the effect of $\nabla \Theta$ for the $\langle \theta'^2 \rangle$ production in the case of temperature diffusion. The second term also expresses a kind of K_d production effect since $\nabla \cdot \mathbf{U}$ is large and negative in a highly compressed region just in front of a shock wave. The situation with large $|\nabla \cdot \mathbf{U}|$ is much different from the case subject to homogeneous shear, where $\nabla \cdot \mathbf{U} = 0$. This point will be discussed in Appendix B. The properties of the remaining terms in Eq. (30), specifically those of the third and fifth terms, are not clear. This point is closely related to the fact that $\int \langle \rho^2 \rangle dV$ is not a conserved quantity. The two-point closure analysis of these density-related quantities is indispensable for the theoretically sound turbulence modeling of those terms.

IV. TSDIA ANALYSIS OF COMPRESSIBLE TURBULENCE

Theoretical studies of compressible turbulence are few. One of them is the study by Hartke, Canuto, and Alonso [16], who used the direct-interaction approximation (DIA) [17] to investigate the homogeneous turbulence subject to a uniform temperature gradient. Another is the author's application [13] of the TSDIA to turbulent shear flows, which aims at incorporating the effect of density fluctuations into the turbulence modeling. In the latter, ρ, $\mathbf{j}$ $(=\rho\mathbf{u})$ and w $(=\rho e)$ were adopted as the fundamental variables. This choice seems very natural since the fundamental conservation laws are written for these variables. We have encountered, however, a difficulty with the satisfaction of the Galilean-transformation rule. Namely, when we move from one coordinate system A to another one B with the relative constant velocity $\mathbf{V}$, we have the relation

$$\mathbf{j}_B=\mathbf{j}_A+\rho\mathbf{V} . \tag{53}$$

In the previous formalism using $\mathbf{j}$, the correlation function $\langle j'_i j'_j\rangle/\rho_M$ bears a role of the Reynolds stress in the mean momentum equation for $\langle j\rangle$, which should obey the transformation rule

$$\langle j'_{Bi}j'_{Bj}\rangle=\langle j'_{Ai}j'_{Aj}\rangle+V_i\langle\rho' j'_{Aj}\rangle+V_j\langle\rho' j'_{Ai}\rangle+\langle\rho'^2\rangle V_iV_j . \tag{54}$$

The theoretical methods like the DIA are founded on some chains of approximations like the renormalization of a perturbational solution [17]. On using $\mathbf{j}$, whose fluctuating part $\mathbf{j}'$ is not Galilean invariant unlike $\mathbf{u}'$, it is specifically difficult to construct a formalism that exactly satisfies Eq. (54). A method of reducing the above difficulty is the direct use of ensemble-mean quantities. Namely, the use of the nonconservational form (24) and (25) for the $\mathbf{u}'$ and $\mathbf{e}'$ equations is helpful in retaining the Galilean invariance under the DIA since $\mathbf{u}'$ and e' are combined with the Galilean-invariant operator D/Dt. In what follows, we shall proceed to the TSDIA analysis along this line. Here we shall give only the main procedures and results.

A. Key mathematical procedures

In the TSDIA, we first introduce two time and space variables

$$\xi\ (=\mathbf{x}),\quad \mathbf{X}\ (=\delta\mathbf{x}),\quad s\ (=t),\quad T\ (=\delta t) , \tag{55}$$

using a small-scale parameter δ. Namely, ξ and s are the fast variables expressing the rapid variation of the fluctuating field, whereas the slow variables $\mathbf{X}$ and T describe the slow variation of the mean field. In terms of (55), a quantity f [Eq. (9)] is written as

$$f=F(\mathbf{X};T)+f'(\xi,\mathbf{X};s,T) . \tag{56}$$

Here the slow variables $\mathbf{X}$ and T are also important in connecting f' with F, since turbulence fluctuations are maintained by the gradient of the mean field like the mean velocity shear.

Second, we express f' in the Fourier-representation form of ξ as

$$f'(\xi,\mathbf{X};s,T)=\int f'(\mathbf{k},\mathbf{X};s,T)\exp[-i\mathbf{k}\cdot(\xi-\mathbf{U}s)]d\mathbf{k} . \tag{57}$$

This procedure is equivalent to the viewpoint that the fluctuating motion consists of a lot of small eddies in the frame moving with the mean velocity $\mathbf{U}$. Applying Eqs. (55)–(57) to the equations for the fluctuating field (23)–(25), we have

$$\frac{\partial}{\partial s}\rho'(\mathbf{k};s)-ik_i\int\int\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\,u'_i(\mathbf{p};s)\rho'(\mathbf{q};s)-ik_i\rho_M u'_i(\mathbf{k};s)=\delta\left[-u'_i(\mathbf{k};s)\frac{\partial\rho_M}{\partial X_i}-\rho'(\mathbf{k};s)\frac{\partial U_i}{\partial X_i}-\frac{D}{DT_I}\rho'(\mathbf{k};s)+\cdots\right] , \tag{58}$$

$$\frac{\partial}{\partial s}u'_i(\mathbf{k};s)-i\int\int\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\,M_{ijm}(\mathbf{p},\mathbf{q})u'_j(\mathbf{p};s)u'_m(\mathbf{q};s)+\nu_M k^2u'_i(\mathbf{k};s)+\nu'_M k_ik_ju'_j(\mathbf{k};s)-ik_i\frac{1}{\rho_M}p'(\mathbf{k};s)+\mathcal{R}=\delta\left[-u'_j(\mathbf{k};s)\frac{\partial U_j}{\partial X_i}-\frac{1}{\rho_M}\rho'(\mathbf{k};s)\frac{DU_i}{DT}-\frac{D}{DT_I}u'_i(\mathbf{k};s)-\frac{1}{\rho_M}\frac{\partial}{\partial X_{Ii}}p'(\mathbf{k};s)+\mathcal{R}\right] , \tag{59}$$

$$\frac{\partial}{\partial s}e'(\mathbf{k};s)-i\int\int\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})q_i d\mathbf{p}\,d\mathbf{q}\,u'_i(\mathbf{p};s)e'(\mathbf{q};s)+\kappa_M k^2e'(\mathbf{k};s)-ik_i\frac{P}{\rho_M}u'_i(\mathbf{k};s)+\mathcal{R}=\delta\left[-u'_i(\mathbf{k};s)\frac{\partial E}{\partial X_i}-\frac{1}{\rho_M}\rho'(\mathbf{k};s)\frac{DE}{DT}-\frac{1}{\rho_M}p'(\mathbf{k};s)\frac{\partial U_i}{\partial X_i}-\frac{D}{DT_I}e'(\mathbf{k};s)-\frac{P}{\rho_M}\frac{\partial}{\partial X_{Ii}}u'_i(\mathbf{k};s)+\mathcal{R}\right] , \tag{60}$$

where $\nu'_M = \nu_M/3$ and

$$M_{ijk}(\mathbf{p},\mathbf{q}) = (q_j\delta_{ik} + p_k\delta_{ij})/2 , \quad (61)$$

$$\left[\frac{D}{DT_I}, \frac{\partial}{\partial X_{Ii}}\right] = \exp(-i\mathbf{k}\cdot\mathbf{U}s)\left[\frac{D}{DT}, \frac{\partial}{\partial X_i}\right]\exp(i\mathbf{k}\cdot\mathbf{U}s) , \quad (62)$$

with $D/DT = \partial/\partial T + U_i(\partial/\partial X_i)$ and $\mathcal{R}$ denotes the terms nonlinear in the fluctuating field of $O(\delta^0)$ and $O(\delta)$ (their details are omitted here). Here and hereafter, the dependence of f' on the slow variables $\mathbf{X}$ and T is not written explicitly except when it is necessary for avoiding confusion.

Third, we expand the fluctuating field f' using a scale parameter δ as

$$f'(\mathbf{k};s) = \sum_{n=0} \delta^n f'_n(\mathbf{k};s) . \quad (63)$$

On applying Eq. (63) to Eqs. (58)–(60), the direct effects of inhomogeneity given rise to by the mean-field gradients like $\nabla\mathbf{U}$, ∇E, etc., appear in the equations for f'_n ($n \geq 1$), which depend on both f'_0 and the gradients of ρ_M, $\mathbf{U}$, and E. The f'_n equations can be integrated formally by introducing the Green's functions for Eqs. (58)–(60). They satisfy

$$\frac{\partial}{\partial s}G'_d(\mathbf{k};s,s') - ik_i\int\int\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\,u'_{Bi}(\mathbf{p};s)G'_d(\mathbf{q};s,s') = \delta(s-s') , \quad (64)$$

$$\frac{\partial}{\partial s}G'_{ij}(\mathbf{k};s,s') - 2i\int\int\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\,M_{ikm}(\mathbf{p},\mathbf{q})u'_{Bk}(\mathbf{p};s)G'_{mj}(\mathbf{q};s,s') + \nu_M k^2 G'_{ij}(\mathbf{k};s,s') + \nu'_M k_i k_m G'_{mj}(\mathbf{k};s,s') = \delta_{ij}\delta(s-s') , \quad (65)$$

$$\frac{\partial}{\partial s}G'_e(\mathbf{k};s,s') - i\int\int\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\,q_i u'_{Bi}(\mathbf{p};s)G'_e(\mathbf{q};s,s') + \kappa_M k^2 G'_e(\mathbf{k};s,s') = \delta(s-s') , \quad (66)$$

respectively. Here $\mathbf{u}'_B$ is the basic velocity field defined by Eq. (A1) in Appendix A and represents part of $\mathbf{u}'_0$ whose interactions with ρ'_0 and e'_0 are smallest. In Eqs. (64)–(66), it is more accurate to write, for instance, $G'_d(\mathbf{k},\mathbf{k}';s,s')$ and $\delta(\mathbf{k}-\mathbf{k}')\delta(s-s')$ in place of $G'_d(\mathbf{k};s,s')$ and $\delta(s-s')$, respectively. Within the present framework based on the DIA, we can put

$$G'_d(\mathbf{k},\mathbf{k}';s,s') = \delta(\mathbf{k}-\mathbf{k}')G'_d(\mathbf{k};s,s') , \quad (67)$$

which leads to Eq. (64). As a result, we reach the same conclusion.

The solution f'_n ($n \geq 1$) can be formally integrated using the Green's functions and are written in terms of f'_0. We apply the DIA renormalization procedure to calculate various important correlation functions. They are expressed in terms of the statistics of the $O(\delta^0)$ field f'_0. We write these statistics as

$$\langle\rho'_0(\mathbf{k},\mathbf{X};s,T)\rho'_0(-\mathbf{k},\mathbf{X};s',T)\rangle/\delta(0) = Q_d(k,\mathbf{X};s,s',T) , \quad (68)$$

$$\langle u'_{0i}(\mathbf{k},\mathbf{X};s,T)u'_{0j}(-\mathbf{k},\mathbf{X};s',T)\rangle/\delta(0) = Q_{ij}(\mathbf{k},\mathbf{X};s,s',T) \quad (69a)$$

$$= D_{ij}(\mathbf{k})Q_s(k,\mathbf{X};s,s',T) + \Pi_{ij}(\mathbf{k})Q_c(k,\mathbf{X};s,s',T) , \quad (69b)$$

$$\langle e'_0(\mathbf{k},\mathbf{X};s,T)e'_0(-\mathbf{k},\mathbf{X};s',T)\rangle/\delta(0) = Q_e(k,\mathbf{X};s,s',T) , \quad (70)$$

$$\langle G'_d(\mathbf{k},\mathbf{X};s,s',T)\rangle = G_d(k,\mathbf{X};s,s',T) , \quad (71)$$

$$\langle G'_{ij}(\mathbf{k},\mathbf{X};s,s',T)\rangle = G_{ij}(\mathbf{k},\mathbf{X};s,s',T) \quad (72a)$$

$$= D_{ij}(\mathbf{k})G_s(k,\mathbf{X};s,s',T) + \Pi_{ij}(\mathbf{k})G_c(k,\mathbf{X};s,s',T) , \quad (72b)$$

$$\langle G'_e(\mathbf{k},\mathbf{X};s,s',T)\rangle = G_e(k,\mathbf{X};s,s',T) , \quad (73)$$

where subscripts S and C denote solenoidal and compressible parts, respectively, and

$$D_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2, \quad \Pi_{ij}(\mathbf{k}) = k_i k_j/k^2 . \quad (74)$$

In Eqs. (68)–(73), we have assumed the isotropy of the $O(\delta^0)$ field. This approximation is plausible since the field is not directly dependent on the mean-field gradients leading to anisotropy, except the implicit dependence through the slow variables $\mathbf{X}$ and T. The density variance Q_d is related to the compressible part of the velocity variance Q_c as

$$Q_d(k,\mathbf{X};s,s',T) = k^2\rho_M^2\int^s ds_1\int^{s'} ds_2 Q_c(k,\mathbf{X};s_1,s_2,T) . \quad (75)$$

Here and hereafter, the lower limit of the time integral is the infinite past or $-\infty$. The fact that the density variance is linked with the variance of the compressible part of the velocity fluctuation will become important in incorporating the latter effects into the one-point turbulence modeling.

B. TSDIA results

In order to show the results compactly, we introduce the following abbreviation form for time and wave-number integrals:

$$I_0\{A\}=\int A(k,x;s,s,t)d\mathbf{k} \ , \tag{76a}$$

$$I_n\{A,B\}=\int k^{2n}d\mathbf{k}\int^{s}ds_1\,A(k,x;s,s_1,t)B(k,x;s,s_1,t) \ , \tag{76b}$$

where the lower limit of time integral is the infinite past or $-\infty$, as has already been noted.

Using Eq. (76), we finally have

$$\langle\rho'\mathbf{u}'\rangle=-\tfrac{1}{3}(2I_0\{G_d,Q_s\}+I_0\{G_d,Q_c\})\nabla\rho_M-\tfrac{1}{3}(2I_0\{G_s,Q_d\}+I_0\{G_c,Q_d\})(\gamma-1)(1/\rho_M)\nabla E$$
$$-\tfrac{1}{3}(2I_0\{G_s,\nabla Q_d\}+I_0\{G_c,\nabla Q_d\})(\gamma-1)(1/\rho_M)E-\tfrac{1}{3}(2I_0\{G_s,Q_d\}+I_0\{G_c,Q_d\})(1/\rho_M)\frac{D\mathbf{U}}{Dt} \ , \tag{77}$$

$$\langle u_i'u_j'\rangle=\tfrac{2}{3}K\delta_{ij}-\tfrac{1}{15}(7I_0\{G_s,Q_s\}+3I_0\{G_s,Q_c\}+3I_0\{G_c,Q_s\}+2I_0\{G_c,Q_c\})S_{ij} \ , \tag{78}$$

$$\langle\rho'e'\rangle=-I_0\{G_e,Q_d\}(1/\rho_M)\left[(\gamma-1)E\nabla\cdot\mathbf{U}+\frac{DE}{DT}\right] , \tag{79}$$

$$\langle e'\mathbf{u}'\rangle=-\tfrac{1}{3}(2I_0\{G_e,Q_s\}+I_0\{G_e,Q_c\})\nabla E-\tfrac{1}{6}(2I_0\{G_e,\nabla Q_s\}+I_0\{G_e,\nabla Q_c\})(\gamma-1)E$$
$$-\tfrac{1}{3}(2I_0\{G_s,Q_e\}+I_0\{G_c,Q_e\})(\gamma-1)(1/\rho_M)\nabla\rho_M-\tfrac{1}{6}(2I_0\{G_s,\nabla Q_e\}+I_0\{G_c,\nabla Q_e\})(\gamma-1)E \ , \tag{80}$$

$$\langle p'\nabla\cdot\mathbf{u}'\rangle=-I_1\{G_s,Q_c\}(\gamma-1)\rho_M E-I_1\{G_e,Q_c\}(\gamma-1)^2\rho_M+I_1\{G_c,Q_e\}(\gamma-1)^2\rho_M \ , \tag{81}$$

$$\langle p'\mathbf{u}'\rangle=(\gamma-1)(\langle\rho'\mathbf{u}'\rangle E+\langle e'\mathbf{u}'\rangle\rho_M) \ , \tag{82}$$

$$\langle\mathbf{u}'\cdot\nabla\rho'\rangle=-I_1\{G_c,Q_d\}(\gamma-1)E/\rho_M+I_1\{G_d,Q_c\}\rho_M \ . \tag{83}$$

In Eq. (78), K $(=\langle\mathbf{u}'^2/2\rangle)$ is the energy of the velocity fluctuation given by

$$K=I_0\{Q_s\}+I_0\{Q_c\}/2-I_0\{G_s,DQ_s/Dt\}-\tfrac{1}{2}I_0\{G_c,DQ_c/Dt\}-\tfrac{1}{3}(2I_0\{G_s,Q_s\}+\tfrac{1}{2}I_0\{G_c,Q_c\})\nabla\cdot\mathbf{U} \ . \tag{84}$$

Finally, we summarize the relationship between the above correlation functions and the equations for ρ_M, $\mathbf{U}$, E, K, and K_d, which is given by the following:

(a) ρ_M equation [Eq. (11)]: $\langle\rho'\mathbf{u}'\rangle$; (85a)

(b) U equation [Eq. (12)]: $\langle u_i'u_j'\rangle$, $\langle\rho'\mathbf{u}'\rangle$, $\langle\rho'e'\rangle$ through P [Eq. (15)] ; (85b)

(c) E equation [Eq. (13)]: $\langle e'\mathbf{u}'\rangle$, $\langle\rho'e'\rangle$, $\langle\rho'\mathbf{u}'\rangle$, $\langle p'\nabla\cdot\mathbf{u}'\rangle$; (85c)

(d) K_d equation [Eq. (30)]: $\langle\rho'\mathbf{u}'\rangle$, $\langle\mathbf{u}'\cdot\nabla\rho'\rangle$, (85d)

(e) K equation [Eq. (31)]: $\langle u_i'u_j'\rangle$, $\langle\rho'u'\rangle$, $\langle p'\nabla\cdot\mathbf{u}'\rangle$, $\langle p'\mathbf{u}'\rangle$ through $\mathcal{D}$ [Eq. (32)] . (85e)

Here we should note the importance of the mass flux $\langle\rho'\mathbf{u}'\rangle$ since it enters all of the transport equations. This fact signifies that $\langle\rho'\mathbf{u}'\rangle$ is most closely connected with compressibility effects. Therefore its proper modeling is indispensable for the construction of a turbulence model that can cope with strong compressibility effects encountered in shock-wave–turbulence interactions.

V. DISCUSSIONS AND SUGGESTIONS TO TURBULENCE MODELING

A. Two-point closure modeling

In the analysis of Sec. IV, we have reached the mathematical expressions for the correlation functions that appear in the equations for the man field $(\rho_M,\mathbf{U},E)$. These correlation functions are related to the mean field and the two-time statistics of the $O(\delta^0)$ fluctuating field. The $O(\delta^0)$ field does not directly depend on the mean-field gradient, as can be easily seen from the left-hand sides of Eqs. (58)–(60), and obeys the same system as for homogeneous compressible turbulence, except for the implicit dependence on the slow variables. The most orthodox theoretical approach to turbulent shear flows is to construct the equations for the $O(\delta^0)$ statistics using the DIA or other two-point closure formalisms and connect these quantities with the mean-field equations that contain them through the correlation functions Eqs. (77)–(83). Here the $O(\delta^0)$ statistics are sustained by the mean-field gradients through the K_d and K equations, etc. As a result, these statistics are determined, in principle, in a way consistent with the mean field. Such a procedure in the solenoidal case is given in detail in Ref. [18].

In the context of two-point closures, the eddy viscosity ν_e, which is the coefficient of S_{ij} in $\langle u_i' u_j' \rangle$ [Eq. (78)], is written as

$$\nu_e = \int \nu_e(k) d\mathbf{k} \ , \tag{86}$$

where the eddy-viscosity spectrum $\nu_e(k)$ is

$$\begin{aligned} \nu_e(k) = \tfrac{1}{15} \int^s ds_1 [& 7G_s(k;s,s_1)Q_s(k;s,s_1) \\ & + 3G_s(k;s,s_1)Q_c(k;s,s_1) \\ & + 3G_c(k;s,s_1)Q_s(k;s,s_1) \\ & + 2G_c(k;s,s_1)Q_c(k;s,s_1)] \ . \end{aligned} \tag{87}$$

Equation (87) shows that the eddy-viscosity spectrum in compressible turbulence consists of four kinds of effects. Namely, they are the combinations of the solenoidal and compressible time scales (the Green's functions) and turbulent intensities. As a result, the eddy viscosity in compressible turbulence may behave differently, compared with the solenoidal case.

It is surely academically interesting to pursue the above line of two-point closure theories, but it is entirely not feasible in the study of real-world phenomena appearing in the aerospace and astrophysical fields. For instance, the DIA system of equations derived by Hartke, Canuto, and Alonso [16] for homogeneous turbulence, which corresponds to the $O(\delta^0)$ field, is suggestive in the qualitative investigation of compressibility effects, but the quantitative results have not been abstracted at least at the present since the system is very complicated.

Inhomogeneity of turbulence, which is the cause of the complexity not comparable to the homogeneous counterpart, is an essential factor in aerospace and astrophysical phenomena. Therefore it is much more fruitful to make full use of the results from two-point closure theories and proceed to the one-point turbulence modeling that possesses a theoretically sound basis. Specifically, the construction of such a model that can cope with strong compressibility effects in a region with a shock wave is urgent in the aeronautical field in close relation to the design of high-speed aircrafts. In reality, a lot of compressible turbulence models are being proposed from the conventional methods mainly using dimensional analysis. The present results are expected to give a starting point in the sense that the relationship of the correlation functions with the mean field and the turbulence quantities characterizing compressibility effects has been obtained.

B. Importance of the mass flux $\langle \rho \mathbf{u}' \rangle$

First, we look at $\langle u_i' u_j' \rangle$ [Eq. (78)], which is the counterpart of the Reynolds stress in the solenoidal case. As has already been stated below Eq. (87), the way to model the eddy viscosity ν_e depends on whether we distinguish between the solenoidal and compressible parts of velocity intensities. In the current one-point turbulence modeling, we do not distinguish between them and adopt their sum or K [Eq. (29)] as an indicator of the strength of turbulent velocity intensities since the separate treatment needs at least two more turbulence transport equations. The increase in the number of equations not only leads to an increase in computational burden, but it also gives rise to difficulties with boundary conditions (this point can be easily understood considering the upstream condition on the dissipation rate of the compressible turbulent energy). From the standpoint of not distinguishing between the solenoidal and compressible turbulence intensities, the simplest model for ν_e is

$$\nu_e = C_t K^2 / \epsilon \ , \tag{88}$$

using K and its corresponding dissipation rate ϵ [Eq. (18)], where C_t is a model constant. This modeling of ν_e results in covering the explicit effects of compressibility and therefore they need to be incorporated through other correlation functions.

The above point linked with the eddy-viscosity approximation is crucial in the mass-weighted averaging. In the averaging, the Reynolds stress [Eq. (47)] results in not including the density effect under the eddy-viscosity approximation of type (88), except through the mean density ρ_M. On the other hand, the counterpart (42) under the ensemble averaging can be directly dependent on the density fluctuations through $\langle \rho' \mathbf{u}' \rangle$. Therefore compressibility effects can be incorporated under the use of the eddy-viscosity approximation based on Eq. (88) for $\langle u_i' u_j' \rangle$ if $\langle \rho' \mathbf{u}' \rangle$ is properly modeled. Entirely the same situation holds for the mean internal-energy equations (13) and (36) [see Eqs. (41) and (43)]. In this sense, the ensemble-mean modeling is more appropriate for the efficient description of compressibility effects than the mass-weighted-mean one.

The absence of the ρ'-related terms in the mass-weighted-mean Reynolds stress does not always reduce the burden of turbulence modeling. In the averaging, the effect very similar to $\langle \rho' \mathbf{u}' \rangle$ has to be modeled in connection with the pressure-dilatation effect in the internal-energy equation (36) as well as the molecular-viscosity effect near a solid wall. This point has already been referred to in Sec. III C [see Eqs. (37) and (38)]. In this context, the mass-weighted averaging does not always lead to the essential simplification of compressible turbulence modeling.

Second, let us examine the property of the mass flux $\langle \rho' \mathbf{u}' \rangle$. The elucidation of this quantity is a major concern in the present work since it typically expresses the effects of compressibility on turbulence. It is simply expressed by writing Eq. (46) using the concept of gradient diffusion [4,14]. As has already been noted below Eq. (46), this approximation based on the eddy density diffusivity inevitably results in the enhanced diffusion of the density that is very similar to that of the temperature and energy. The ρ equation (1) originally contains no density-diffusion term, entirely different from the λ-related term in the e equation (3). Therefore the familiar eddy-diffusivity approximation for $\langle \rho' \mathbf{u}' \rangle$ is doubtful.

The present TSDIA result (77) clearly shows that $\langle \rho' \mathbf{u}' \rangle$ consists of several kinds of effects. Namely, the first term in Eq. (77) corresponds to the familiar eddy-diffusivity representation. Of the remaining effects, we pay special attention to the second term dependent on ∇E. In the case of increasing E or the temperature, ρ often decreases. In such a case, the $\nabla \rho_M$ and ∇E effects in Eq. (79) have the opposite contributions and tend to cancel each other. In other words, the existence of the $\nabla \rho_M$-related term does not always signify the cascade of ρ. This point is consistent with the fact that the mass conservation law (1) does not possess the diffusion term. Therefore some proper effects like the second ∇E-related term in Eq. (77) should be taken into account in the modeling of $\langle \rho' \mathbf{u}' \rangle$ so as not to destroy the fundamental mathematical property of the mass conservation law. In the connection with the effect of $\langle \rho' \mathbf{u}' \rangle$, we should refer to the previous work [13] based on the variable $\mathbf{j}$ ($=\rho \mathbf{u}$). In it, the Reynolds stress in the $\langle \mathbf{j} \rangle$ equation is given by $\langle j_i' j_j' \rangle / \rho_M$. The TSDIA analysis shows that the quantity includes the U-dependent term in addition to the eddy-viscosity effect. The origin of the term also depending on the density-variance spectrum is linked with the ρ'-related term in Eq. (42).

The origin of the ∇E-related term in $\langle \rho' \mathbf{u}' \rangle$ [Eq. (77)] is clear. On considering the generation process of $\langle \rho' \mathbf{u}' \rangle$, the first term on the right-hand side of Eq. (23) generates the $\nabla \rho_M$-related effect in Eq. (77). On the other hand, the fifth term on the left-hand side of Eq. (24) or $(1/\rho_M)\nabla p'$ produces the ∇E-related term, as can be easily understood from Eq. (16) for p' [note that the E-related term in Eq. (16) includes ρ', which leads to the dependence of $\langle \rho' \mathbf{u}' \rangle$ on the density-variance spectrum]. In other words, the $\nabla \rho_M$ effect in $\langle \rho' \mathbf{u}' \rangle$ is the contribution from the ρ' equation, whereas the ∇E effect comes from the $\mathbf{u}'$ equation.

The mass flux $\langle \rho' \mathbf{u}' \rangle$ also becomes important near a solid wall, in close relation to the low-Reynolds-number effects under the mass-weighted averaging, as was noted near Eq. (48). Therefore the above point also deserves serious consideration under the averaging.

C. Effects of pressure

Let us discuss the pressure-dilatation correlation function $\langle p' \nabla \cdot \mathbf{u}' \rangle$. In general, the pressure decreases in the expansion phase and increases in the contraction phase. This fact suggests that $\langle p' \nabla \cdot \mathbf{u}' \rangle$ is negative. In the present result (81), the leading term supports the suggestion. Concerning the sign of $\langle p' \nabla \cdot \mathbf{u}' \rangle$, some interesting works based on the direct numerical simulation have recently been done for decaying isotropic turbulence and homogeneous shear turbulence [19–21]. These works show that $\langle p' \nabla \cdot \mathbf{u}' \rangle$ is predominantly positive in decaying isotropic turbulence, whereas it is predominantly negative in homogeneous shear turbulence. Specifically, Sarkar [21] examined the generation mechanism of the quantity to elucidate the difference between the two cases. In this sense, we should note that the present theoretical result partially accounts for the effect of the pressure-dilatation correlation.

In the case of negative $\langle p' \nabla \cdot \mathbf{u}' \rangle$, the quantity gives rise to the increase in the kinetic-energy dissipation, as can be seen from the combination with ϵ in the kinetic-energy equation (31) and the counterpart under the mass-weighted averaging. It has been confirmed that compressibility effects lead to the suppression of turbulence. Therefore $\langle p' \nabla \cdot \mathbf{u}' \rangle$ is a promising candidate for explaining such a suppression mechanism. This point has already been pointed out in the author's previous work [13], where the result similar to Eq. (81) was obtained. As the first term of Eq. (81) shows, $\langle p' \nabla \cdot \mathbf{u}' \rangle$ is closely connected with the compressible part of the velocity fluctuation, which is related to the density fluctuation, as in Eq. (75). Therefore it is relevant to relate the first term to the density fluctuation so long as we do not distinguish between the solenoidal and compressible parts of the velocity fluctuation. The importance of the role borne by the density fluctuation can also be seen from the Q_c- and Q_d-related effects in the other correlation functions, for instance, the first term in $\langle \rho' e' \rangle$ [Eq. (79)]. In order to incorporate the density-fluctuation effects into the one-point compressible turbulence modeling, we need the equation governing the density variance K_d.

Another important effect of the pressure comes from the first pressure diffusion term $\langle p' \mathbf{u}' \rangle$ in $\mathcal{D}$ [Eq. (32)]. As is seen from Eq. (52), no special attention has been paid so far to $\langle p' \mathbf{u}' \rangle$ in the solenoidal case. This point makes a sharp contrast with the situation concerning the presure-strain correlation $\langle p' s_{ij}' \rangle$. In fact, some experimental supports have been given to the approximation of neglecting $\langle p' \mathbf{u}' \rangle$, compared with the triple-velocity correlation function $\langle (\mathbf{u}'^2/2)\mathbf{u}' \rangle$. In compressible turbulence, this situation may change drastically at high Mach numbers since p' can contain the contribution through the thermodynamic relation (6). Specifically, $\langle p' \mathbf{u}' \rangle$ is related to the mass flux $\langle \rho' \mathbf{u}' \rangle$ and E, as is seen from the first part of Eq. (82). In a highly compressed region where the density change and the temperature rise are large, $\langle p' \mathbf{u}' \rangle$ is considered to be more important than $\langle (\mathbf{u}'^2/2)\mathbf{u}' \rangle$ (note that the intensity of velocity fluctuation is suppressed in such a region). Therefore the validity of the straightforward use of the solenoidal model (52) in compressible turbulence is doubtful.

D. One-point modeling

Let us refer to the construction of a one-point model based on the ensemble averaging. Such a model is indispensable for the aerospace and astrophysical research where the Reynolds number of flows is extremely high. A critical point in the modeling is the choice of the one-point quantities properly characterizing compressible turbulence. The present TSDIA results (77)–(83) show that the quantities indispensable for the modeling are

$$K_d, \quad K, \quad \epsilon \tag{89}$$

since Q_s and Q_c give the major part of K and Q_d is the counterpart of K_d. Here K_d is a quantity necessary for expressing the degree of compressibility effects, whereas K and ϵ are needed for expressing the intensity and time scale of turbulence. It is more relevant to include K_e ($=\langle e'^2 \rangle$) in Eq. (89) since some of the correlation functions (77)–(83) are also dependent on the internal-energy variance Q_e [for instance, see the third and fourth terms in Eq. (80)]. Most of those correlation functions, however, are linked with the density and velocity variances. Therefore the one-point modeling based on Eq. (89) is considered to satisfy the least requirement from the TSDIA of compressible turbulence. A proposal for the model, making full use of the present theoretical results, is given in Appendix B. One remarkable point in this model is that we have the nondimensional parameter χ [Eq. (B13)] as the indicator of the importance of compressibility effects, which is written in terms of ρ_M, E, K, and K_d.

VI. CONCLUSION

We performed a two-scale DIA analysis of compressible turbulent shear flows to gain some information that is useful in the construction of an ensemble-mean one-point turbulence model. Such a model is indispensable for the study of aeronautical and astrophysical flows at extremely high Reynolds numbers. One of the major achievements of this work is to clarify the cause of the paradoxical points related to the eddy-diffusivity expression for the turbulent mass flux whose original conservation law does not possess the molecular diffusion effect. Another is the estimate of the pressure-dilatation and pressure-velocity correlation functions in the transport equation for the turbulent kinetic energy. Specifically, the importance of the latter has been missing in the current turbulence modeling since the counterpart in the solenoidal case is not so important as the triple-velocity correlation function.

Through the present investigation of various correlation functions appearing in the mean equations, the importance of the effects of density fluctuations has been elucidated. On the basis of this result, a three-equation model with a transport equation for the density variance added to a current model of the two-equation type was proposed for aerospace applications.

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APPENDIX A: SUPPLEMENTARY EXPLANATION OF TSDIA

In the DIA renormalization procedures, the fundamental equations for the fluctuations (58)–(60) are solved using a perturbation expansion method. There the $O(\delta^0)$ equations, which are given by neglecting the δ-related terms on the right-hand sides of Eqs. (58)–(60), still couple with one another. In order to solve such a system systematically, we introduce the concept of the basic field $(\mathbf{u}'_B, e'_B)$, obeying

$$\frac{\partial}{\partial s}u'_{Bi}(\mathbf{k};s)-i\int\int\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\,M_{ijm}(\mathbf{p},\mathbf{q})u'_{Bj}(\mathbf{p};s)u'_{Bm}(\mathbf{q};s)+\nu_M k^2 u'_{Bi}(\mathbf{k};s)+\nu'_M k_i k_j u'_{Bj}(\mathbf{k};s)=0 \ , \qquad \text{(A1)}$$

$$\frac{\partial}{\partial s}e'_B(\mathbf{k};s)-i\int\int\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}\,d\mathbf{q}\,q_i u'_{Bi}(\mathbf{p};s)e'_B(\mathbf{q};s)+\kappa_M k^2 e'_B(\mathbf{k};s)=0 \ . \qquad \text{(A2)}$$

Here we should note that Eqs. (A1) and (A2) do not couple with each other directly.

Using Eqs. (A1) and (A2), the $O(\delta^0)$ equations are integrated formally as

$$u'_{0i}(\mathbf{k};s)=u'_{Bi}(\mathbf{k};s)+\int^s ds_1 G'_{ij}(\mathbf{k};s,s_1) \times[i(1/\rho_M)k_j p'_0(\mathbf{k};s_1)+\mathcal{R}] \ , \qquad \text{(A3)}$$

$$e'_0(\mathbf{k};s)=e'_B(\mathbf{k};s)+\int^s ds_1 G'_e(\mathbf{k};s,s_1) \times[i(P/\rho_M)k_i u'_{0i}(\mathbf{k};s_1)+\mathcal{R}] \qquad \text{(A4)}$$

around the basic field $(\mathbf{u}'_B, e'_B)$, where G' and G'_e are the Green's functions for Eqs. (A1) and (A2), which obey Eqs. (65) and (66), respectively. On the other hand, ρ'_0 is related to $\mathbf{u}'_0$ as

$$\rho'_0(\mathbf{k};s)=i\rho_M k_i\int^s ds_1 G'_d(\mathbf{k};s,s_1)u'_{0i}(\mathbf{k};s_1) \ , \qquad \text{(A5)}$$

with G'_d obeying Eq. (64).

The $O(\delta)$ field $(\rho'_1,\mathbf{u}'_1,e'_1)$ satisfies Eqs. (58)–(60) with $(\rho',\mathbf{u}',e')$ on the right-hand sides replaced by $(\rho'_0,\mathbf{u}'_0,e'_0)$. They can be integrated entirely similarly using the Green's functions.

Using these solutions of $O(\delta^0)$ and $O(\delta)$, we can calculate various correlation functions with resort to the DIA renormalization method. The resulting expressions are written in terms of the statistics of the basic field, which are obtained by replacing $(\rho'_0,\mathbf{u}'_0,e'_0)$ in Eqs. (68)–(70) with $(\rho'_B,\mathbf{u}'_B,e'_B)$. Our final aim is to write various important correlation functions using the statistics of the $O(\delta^0)$ field. To this end, we first express Q_{ij} [Eq. (69)], etc., in terms of the counterparts of the basic field Q_{Bij}. Thereafter, we revert such expressions by iteration [17] to get, for instance,

$$Q_{Bs}(k;s,s')=Q_s(k;s,s')+\text{higher-order terms} \ , \qquad \text{(A6)}$$

$$Q_{Bc}(k;s,s')=Q_c(k;s,s')+(\gamma-1)Ek^2\Big[\int^s dw\int^w dw_1 G_c(k;s,w)G_d(k;w,w')Q_c(k;s',w') + \int^{s'} dw\int^w dw_1 G_c(k;s',w)G_d(k;w,w')Q_c(k;s,w')\Big] \ , \qquad \text{(A7)}$$

where the higher-order terms depend on the terms that are of fourth order in the Green's functions and the velocity or internal-energy variances.

We substitute Eqs. (A6), (A7), etc. into the correlation functions expressed using the statistics of the basic field Q_B, etc., and obtain the final results. Within the framework of the DIA, only the leading terms in Eqs. (A6) and (A7), etc., are usually retained to lead to Eqs. (70)–(84).

APPENDIX B: THREE-EQUATION MODEL

1. Modeling

Let us propose a turbulence model based on the present TSDIA results. The aim of this modeling is to construct a model that is as simple as possible and retain some essential properties of compressible turbulence. The simplicity is also important from the viewpoint of applicability to real-world interesting phenomena in the aerospace and astrophysical fields. The starting point of modeling is the choice of three fundamental one-point quantities K_d, K, and ϵ [Eq. (89)]. Here K and ϵ are indispensable for constructing the dimensions of length and time, whereas K_d is also needed for incorporating the density change in a highly compressed region since various density-related correlation functions are linked with its spectrum. In the following modeling, only the most important aspect of the TSDIA results will be taken into account.

The mean-field equations are

$$\frac{\partial \rho_M}{\partial t}+\nabla\cdot(\rho_M \mathbf{U})=\nabla\cdot(-\langle \rho'\mathbf{u}'\rangle) , \tag{B1}$$

$$\frac{\partial}{\partial t}\rho_M U_i+\frac{\partial}{\partial x_j}(\rho_M U_j U_i) = -\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}(-\rho_M\langle u_j' u_i'\rangle - U_i\langle \rho' u_j'\rangle - U_j\langle \rho' u_i'\rangle)+\frac{\partial}{\partial x_j}\mu_M S_{ji} , \tag{B2}$$

$$\frac{\partial}{\partial t}\rho_M E+\nabla\cdot(\rho_M \mathbf{U}E) =\nabla\cdot(\lambda_M\nabla\Theta)+\nabla\cdot(-\rho_M\langle e'\mathbf{u}'\rangle-\mathbf{U}\langle\rho' e'\rangle - E\langle \rho'\mathbf{u}'\rangle) -(P\nabla\cdot\mathbf{U}+\langle p'\nabla\cdot\mathbf{u}'\rangle)+\Phi , \tag{B3}$$

where

$$P=(\gamma-1)(\rho_M E+\langle\rho' e'\rangle)\simeq(\gamma-1)\rho_M E , \tag{B4}$$

$$\Theta=E/C_v , \tag{B5}$$

$$\Phi=\lambda_M S_{ij}\frac{\partial U_j}{\partial x_i}+\rho_M\epsilon \tag{B6}$$

[S_{ij} is defined by Eqs. (4) and (14)]. On the left-hand sides of Eqs. (B2) and (B3), $(\partial/\partial t)\langle\rho'\mathbf{u}'\rangle$ and $(\partial/\partial t)\langle\rho' e'\rangle$ have been neglected since the ensemble-mean model aims at mainly treating the stationary properties of turbulence. The terms $\partial\rho_M/\partial t$, etc., are retained since such a system is usually solved by a time-marching method.

The correlation functions in Eqs. (B1)–(B3) are modeled as

$$\langle\rho' u'\rangle=-(\mu_e/\sigma_1)\nabla(\ln\rho_M)-(\mu_e/\sigma_2)\chi\nabla(\ln E) , \tag{B7}$$

$$\langle u_i' u_j'\rangle=\tfrac{2}{3}K\delta_{ij}-\nu_e S_{ij} , \tag{B8}$$

$$\langle\rho' e'\rangle=-(\mu_e/\sigma_3)\chi\nabla\cdot\mathbf{U} , \tag{B9}$$

$$\langle e'\mathbf{u}'\rangle=-(\nu_e/\sigma_4)\nabla E , \tag{B10}$$

$$\langle p'\nabla\cdot\mathbf{u}'\rangle=-(\mu_e/\sigma_5)\chi(\epsilon/K)^2 , \tag{B11}$$

with $\mu_e=\rho_M\nu_e$, where

$$\nu_e=C_1 K^2/\epsilon , \tag{B12}$$

$$\chi=(\gamma-1)(E/K)/(\rho_M^2/K_d) , \tag{B13}$$

and σ_n ($n=1-5$) and C_1 are positive model constants.

The modeling of Eqs. (B7)–(B11) using the one-point quantities (89) is rather trivial by dimensional analysis once the dependence of these correlation functions on the mean field and the spectra of (89) has been clarified on the basis of the TSDIA, as in Eqs. (77)–(83). Here we should note that Q_d gives the leading contribution to K_d, whereas the sum of Q_s and Q_c is the counterpart of K since they are the contributions of the $O(\delta^0)$ field. In the modeling of $\langle\rho'\mathbf{u}'\rangle$ [Eq. (77)], we have retained the first two terms [the importance of the second term in Eq. (B7) for eliminating the effect of virtual density cascade was discussed in Sec. V B and will be further discussed below]. In Eq. (80) for $\langle e'\mathbf{u}'\rangle$, we modeled only the first term in accordance with the eddy-viscosity representation for $\langle u_i' u_j'\rangle$ [Eq. (78)]. In Eq. (81) for $\langle p'\nabla\cdot\mathbf{u}'\rangle$, the first two terms are essentially the same, where the third term was neglected since $\langle e'^2\rangle$ is not adopted as the fundamental turbulence quantities in this modeling.

The remarkable feature in Eqs. (B7)–(B11) is that the effects of compressibility are tightly linked with the nondimensional parameter χ [Eq. (B13)], which depends on ρ_M, E, K, and K_d as well as the ratio of specific heat γ. In Eq. (B13), E/K is the inverse of the magnitude of the turbulent kinetic energy relative to the mean internal energy, whereas ρ_M^2/K_d is the inverse of the magnitude of the density variance normalized using the squared mean density. At very low Mach numbers regarded as incompressible flows, ρ_M^2/K_d becomes infinitely large, whereas E/K can remain finite. As a result, χ vanishes. This fact signifies that χ is a proper indicator of the strength of compressibility effect. At higher Mach numbers where ρ_M^2/K_d is finite, the increasing importance of the χ effects is natural from the viewpoint of modeling $\langle\rho'\mathbf{u}'\rangle$ [Eq. (B7)]. On neglecting the second χ-related term, Eq. (B7) leads to the enhanced diffusion effect of ρ_M, but the original mass conservation law (1) has no diffusion term, as was discussed in Sec. III C. If the χ-related term becomes important with the increasing Mach number or χ, the relative importance of the ρ_M-related term decreases in Eq. (B7) and the problem of the virtual enhanced diffusion effect can be resolved.

The transport equations for the turbulence quantities K_d, K, and ϵ are given by

$$\frac{DK_d}{Dt} = -\langle \rho' \mathbf{u}' \rangle \cdot \nabla \rho_M - K_d \nabla \cdot \mathbf{U} - C_2 \rho_M^2 (\epsilon/K) \chi + \rho_M \nabla \cdot (-\langle \rho' \mathbf{u}' \rangle) , \quad (B14)$$

$$\frac{DK}{Dt} = -\langle u_i' u_j' \rangle \partial U_j / \partial x_i - (\epsilon - \langle p' \nabla \cdot \mathbf{u}' \rangle / \rho_M) + (1/\rho_M) \nabla \cdot (-\langle p' \mathbf{u}' \rangle) + \nabla \cdot [(\nu_e / \sigma_K) \nabla K] , \quad (B15)$$

$$\frac{D\epsilon}{Dt} = -C_3 (\epsilon/K) \langle u_i' u_j' \rangle \frac{\partial U_j}{\partial x_i} - C_4 (\epsilon/K) [\epsilon - \langle p' \nabla \cdot \mathbf{u}' \rangle / \rho_M + (1/\rho_M) \nabla \cdot (-\langle p' \mathbf{u}' \rangle)] + \nabla \cdot [(\nu_e / \sigma_D) \nabla \epsilon] , \quad (B16)$$

where

$$\langle p' \mathbf{u}' \rangle = (\gamma - 1)(\langle \rho' \mathbf{u}' \rangle E + \langle e' \mathbf{u}' \rangle \rho_M) , \quad (B17)$$

and $C_2 - C_4$, σ_K, and σ_D are model constants. On the left-hand sides of Eqs. (B14)–(B16), for instance, DK_d/Dt can be rewritten as

$$\frac{DK_d}{Dt} = \frac{1}{\rho_M} \left[\frac{\partial}{\partial t} \rho_M K_d + \nabla \cdot (\rho_M \mathbf{U} K_d) + K_d \nabla \cdot \langle \rho' \mathbf{u}' \rangle \right] \quad (B18)$$

with the aid of Eq. (B1).

In the K_d equation (30), the last term that is of third order in ρ' and $\mathbf{u}'$ was neglected, compared with the second-order terms. The term to be newly modeled in the equation is the third term $\rho_M \langle \mathbf{u}' \cdot \nabla \rho' \rangle$. The TSDIA result for it is given by Eq. (83), consisting of two terms. Of the two, we have retained the first E-related effect since such an effect can become important in a region near a shock wave with large E or high temperature. In the K equation (31), the first two terms in $\mathcal{D}$ [Eq. (32)] have been retained. In the solenoidal case, they are combined to be modeled as $\nabla \cdot [(\nu_e/\sigma_K) \nabla K]$, as was referred to in Sec. III C. In compressible turbulence, however, $\langle p' \mathbf{u}' \rangle$ is directly dependent on $\langle \rho' \mathbf{u}' \rangle$, E, etc., as is seen from Eq. (B17), which are expected to become important in a highly compressed region. Therefore this pressure-transport effect should be separately taken into account, as in Eq. (B15). The second $D\mathbf{U}/Dt$-related term in the K equation (31) was dropped since the stationary properties of turbulence is our major concern. The theoretical derivation of the ϵ model equation is difficult in general. The author has proposed a systematic method [9,10] for recovering most of the major effects in the current model ϵ equation by using the TSDIA results. Equation (B16) has been derived following the method (see also Appendix C) of Ref. [13].

In this model, we have 11 model constants, which are C_n $(n = 1-4)$, σ_n $(n = 1-5)$, σ_K, and σ_D. In the solenoidal limit, this model should reduce to the familiar two-equation model for the K-ϵ type. This constraint fixes six model constants as

$$C_1 = 0.09, C_3 = 1.43, \quad C_4 = 1.9 , \quad \sigma_4 = 0.7, \quad \sigma_K = 1 , \quad \sigma_D = 1.4 . \quad (B19)$$

The reasonableless of some of these constants has been confirmed using the TSDIA [7,13]. As a result, five constants $\sigma_1 - \sigma_3$, σ_5, and C_2 are to be determined using direct numerical simulation data or through the application of the model.

The present model given by the system of equations (B1)–(B3) and (B14)–(B16) seems to include many correlation functions, compared with the counterpart under the mass-weighted averaging. However, this is not the case. The number of different correlation functions is five, including Eqs. (B8) and (B10). Modeling the counterparts of $\langle \rho' \mathbf{u}' \rangle$, $\langle u_i' u_j' \rangle$, $\langle e' \mathbf{u}' \rangle$, and $\langle p' \nabla \cdot \mathbf{u}' \rangle$ is also indispensable under the mass-weighted-mean modeling, as has already been noted.

2. Qualitative assessment

One prominent feature of this model is the role borne by the density variance K_d. In a highly compressed region near a shock wave where ρ_M and E, as well as their space derivatives, become large, K_d is generated by the first two terms in Eq. (B14) or the $\nabla \rho_M$- and ∇E-related terms (note that $\nabla \cdot \mathbf{U}$ is negative in a compressed region). The effects of K_d thus generated enter the other equations through the K_d- or χ-related terms in the correlation functions. Specifically, negative $\langle p' \nabla \cdot u' \rangle$ [Eq. (B11)] leads to the enhancement of energy dissipation in the K equation (B15). This effect is considered to be closely linked with the mechanism of turbulence suppression due to compressibility effects in turbulent shear flows, as was discussed in the previous works [19,21].

In the context of the pressure-dilatation correlation function $\langle p' \nabla \cdot \mathbf{u}' \rangle$, we should refer to the relationship of the present model with homogeneous shear turbulence that has already been investigated by Sarkar, Erlebacher, and Hussaini [19] and Blaisdell, Reynolds, and Mansour [20] using the direct numerical simulation (DNS). In the flow situation, we have no mean dilatation or $\nabla \cdot \mathbf{U} = 0$; namely, the mean velocity field is solenoidal. Of the two K_d-generation mechanisms that are connected with $\nabla \rho_M$ and $\nabla \cdot \mathbf{U}$ in Eq. (B14), respectively, the latter is lost. As a result, the present model leads to vanishing of $\langle p' \nabla \cdot \mathbf{u}' \rangle$ since we usually have $\nabla \rho_M = 0$ under $\nabla \cdot \mathbf{U} = 0$. The DNS of homogeneous shear turbulence [19,20] shows that $\langle p' \nabla \cdot \mathbf{u}' \rangle$ is still nonzero, specifically, negative. Therefore the present model lacks the K_d-generation mechanism under no mean dilatation or $\nabla \cdot \mathbf{U} = 0$. Sarkar [21] used the DNS data to propose a model for $\langle p' \nabla \cdot \mathbf{u}' \rangle$ that can deal with homogeneous turbulence for Mach numbers as large as about 0.6. As will be discussed below, the present three-equation model possesses an interesting property in the case of $\nabla \cdot \mathbf{U} \neq 0$ as in a highly-compressed region near a shock wave. Therefore Eq. (B11) for $\langle p' \nabla \cdot \mathbf{u}' \rangle$ is expected to bear a role of supplementing Sarkar's model at high Mach numbers leading to large $|\nabla \cdot \mathbf{U}|$.

We have already mentioned that our main interest lies in the study of high compressibility effects that are encountered in shock-wave–turbulence interactions. In such a case, we have $\nabla \cdot \mathbf{U} \neq 0$ as well as $\nabla \rho_M \neq 0$. In order to see the potential usefulness of the present model, let us perform the qualitative assessment of the model in a flow situation retaining the feature of a shock-wave region. As such a typical example, we consider the case where $\mathbf{U}[=(U,0,0)]$, ρ_M, and E change rapidly in the streamwise or $\mathbf{x}$ direction. Their profiles are written as

$$U=U_F-(U_F-U_B)H_a(x) , \quad \text{(B20a)}$$

$$\rho_M=\rho_F+(\rho_B-\rho_F)H_a(x) , \quad \text{(B20b)}$$

$$E=E_F+(E_B-E_F)H_a(x) , \quad \text{(B20c)}$$

with

$$U_F>U_B, \quad \rho_F<\rho_B, \quad E_F<E_B , \quad \text{(21)}$$

where subscripts F and B denote front and back, respectively, and

$$H_a(x)=\tan^{-1}(x/a) \quad \text{(B22)}$$

(a is a positive constant representing the width of the region with steeply varying flow structures). From Eq. (B22), we have

$$H_a'(x)\equiv\frac{d}{dx}H_a(x)=\frac{a}{x^2+a^2} , \quad \text{(B23)}$$

which leads to

$$H_a'(x)\rightarrow\delta(x) \quad \text{(B24)}$$

in the limit of vanishing a [$\delta(x)$ is the Dirac δ function]. Namely, $H_a(x)$ tends to the step function $H(x)$ in the limit and the flow structure described by Eq. (B20) retains the feature of a shock-wave region (the center of the region is located at $x=0$). The regions for $x<0$ and $x>0$ correspond to the ones in front of and behind a shock wave, respectively. This type of flow structure is familiar in considering Rankine-Hugoniot's relations among shock-related quantities.

Let us investigate the roles of various terms in the present model in the case of small but finite a. As the representative example, we consider the mean velocity equation (B2). First, we look at the contribution of the eddy-viscosity approximation [Eq. (B8)] to $-\rho_M\langle u_i'u_j'\rangle$ in Eq. (B2), which is given by

$$\left[\frac{d}{dx}(-\rho_M\langle u'^2\rangle)\right]\Big/\tfrac{4}{3}\nu_e\simeq\rho_M U''+\rho_M' U' , \quad \text{(B25)}$$

where $U'=dU/dx$, u' is the streamwise velocity fluctuation, and the spatial change of ν_e has been neglected for the simplicity of discussion. From Eqs. (B20) and (B23), we have

$$U'=-(U_F-U_B)[a/(x^2+a^2)] , \quad \text{(B26a)}$$

$$U''=2(U_F-U_B)[ax/(x^2+a^2)^2] , \quad \text{(B26b)}$$

$$\rho_M'=(\rho_B-\rho_F)[a/(x^2+a^2)] , \quad \text{(B26c)}$$

$$\rho_M''=-2(\rho_B-\rho_F)[ax/(x^2+a^2)^2] , \quad \text{(B26d)}$$

$$E'=(E_B-E_F)[a/(x^2+a^2)] , \quad \text{(B26e)}$$

$$E''=-2(E_B-E_F)[ax/(x^2+a^2)^2] . \quad \text{(B26f)}$$

Using Eqs. (B21) and (B26), the first part in Eq. (B25) is negative for $x<0$ and positive for $x>0$. On the other hand, the second term in Eq. (B25) is negative for both $x<0$ and $x>0$.

Next we consider the contributions of $-U_i\langle\rho' u_j'\rangle$ and $-U_j\langle\rho' u_i'\rangle$ to show the importance of $\langle\rho'\mathbf{u}'\rangle$. Both the terms lead to

$$-\frac{d}{dx}U\langle\rho' u'\rangle\simeq U[(\nu_e/\sigma_1)\rho_M''+(\mu_e/\sigma_1)(\chi/E)E''] + U'[(\nu_e/\sigma_1)\rho_M'+(\mu_e/\sigma_1)(\chi/E)E'] . \quad \text{(B27)}$$

From Eq. (B26), the first or U-related part including the second-order derivatives is positive for $x<0$ and negative for $x>0$, whereas the second part is negative for both $x<0$ and $x>0$.

From the above consideration, we have found the following properties of Eqs. (B25) and (B27): (a) the first parts dependent on second-order derivatives have the opposite signs in each of $x<0$ and $x>0$, and (b) the second parts dependent on first-order derivatives have the same signs in each of $x<0$ and $x>0$.

A prominent feature of the eddy-viscosity approximation to $-\rho_M\langle u_i'u_j'\rangle$ lies in the diffusive property characterized by second-order spatial derivatives, which leads to the destruction of steeply-varying structures like a shock-wave region and tends to smooth out such structures. This point is considered to be a cause for the insufficiency of the eddy-viscosity approximation in the study of shock-wave–turbulence interactions. The property (a) shows that the diffusive property of the eddy-viscosity approximation is weakened by the effect given rise to by $\langle\rho'\mathbf{u}'\rangle$. Therefore the incorporation of the effect into compressible turbulence modeling is considered to be instrumental to the proper description of shock-wave structures in turbulent flows. The similar argument can be done for the E equation (B3). At this time, $\langle\rho' e'\rangle$ bears the role similar to $\langle\rho'\mathbf{u}'\rangle$ in the $\mathbf{U}$ equation (B2).

Finally, this model does not include the so-called low-Reynolds-number effects. It has already been confirmed that the correct asymptotic near-wall behaviors of the model are very important for obtaining accurate results in both solenoidal and compressible turbulence [4,22,23]. The study of those effects is beyond the scope of this paper, but some of the findings about the near-wall behaviors of models obtained in Ref. [4] are applicable to this model.

[1] M. W. Rubesin (unpublished).
[2] O. Zeman, Phys. Fluids A **2**, 178 (1990).
[3] S. Sarkar, G. Erlebacher, M. Y. Hussaini, and H. O. Kreiss, J. Fluid Mech. **227**, 473 (1991).
[4] H. S. Zhang, R. W. C. So, C. G. Speziale, and Y. G. Lai, (unpublished).
[5] J. Weinstock, J. Fluid Mech. **105**, 369 (1981).
[6] J. Weinstock, J. Fluid Mech. **116**, 1 (1982).

[7] A. Yoshizawa, Phys. Fluids **27**, 1377 (1984).
[8] V. Yakhot and S. A. Orszag, J. Sci. Comput. **1**, 3 (1986).
[9] A. Yoshizawa, Phys. Fluids **30**, 628 (1987).
[10] A. Yoshizawa, J. Fluid Mech. **195**, 541 (1989).
[11] R. Rubinstein and J. M. Barton, Phys. Fluids A **2**, 1472 (1990).
[12] R. Rubinstein and J. M. Barton, Phys. Fluids A **3**, 415 (1991).
[13] A. Yoshizawa, Phys. Fluids A **2**, 838 (1990).
[14] D. Taulbee and J. Van Osdal (unpublished).
[15] A. Yoshizawa, J. Phys. Soc. Jpn. **51**, 2326 (1982).
[16] G. J. Hartke, V. M. Canuto, and C. T. Alonso, Phys. Fluids **31**, 1034 (1986).
[17] R. H. Kraichnan, J. Fluid Mech. **83**, 349 (1977).
[18] A. Yoshizawa, Phys. Fluids **28**, 59 (1985).
[19] S. Sarkar, G. Erlebacher, and M. Y. Hussaini, Theor. Comput. Fluid Dyn. **2**, 291 (1991).
[20] G. A. Blaisdell, W. C. Reynolds, and N. N. Mansour, in *Proceedings of the Eighth Symposium on Turbulent Shear Flows* (International Association for Hydraulic Research, Munich, 1991), Sec. 1-1.
[21] S. Sarkar (unpublished).
[22] Y. Nagano and H. Hishida, J. Fluids Eng. **109**, 156 (1987).
[23] H. K. Myong and N. Kasagi, JSME Int. J. II **33**, 63 (1990).