Stochastic resonance in a nonlinear system driven by an aperiodic force

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A bistable system is simulated by an electric circuit. The system is driven by an informational aperiodic signal and a noise force. Under the stochastic resonance condition, the portion of information received by the bistable system can be greatly enhanced.

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I. INTRODUCTION

In the past decade, the study of stochastic resonance (SR) of bistable systems has become an active field in the investigation of the effects played by noise and nonlinearity, and especially by the cooperative action of both [1–7]. However, in all previous investigations, a periodic signal, or specifically, a sinusoid like input, has been applied. The application of the purely sinusoidal signal is rather limited. In many cases of practical interest the input signal is modulated by certain pulses which contain a certain amount of information. In contrast, the information contained in a purely sinusoidal signal is zero. Therefore it is extremely interesting to extend the study of stochastic resonance to a general signal that is more complicated than a sinusoidal signal.

To our knowledge, there has not been a theory to deal with the SR problem with consideration of an aperiodic force. However, in Refs. [4] and [5], it is argued that the effect of stochastic resonance is not very sensitive to the frequency of the input signal. Thus we may expect that the phenomenon of SR may be observed also for an aperiodic signal, of course, with a certain modified definition of SR. In this paper we present the result of an analog simulation of this problem. An electric circuit is used to simulate the nonlinear system

$$\dot{X}=ax-bx^3+\mathcal{E}(t)+Q(t)\ , \tag{1.1}$$

where the noise is Gaussian and white

$$\langle Q(t)\rangle=0\ ,\qquad \langle Q(t)Q(t')\rangle=2D\delta(t-t')\ . \tag{1.2}$$

The main step of our simulation advanced from all previous works is that we use a word consisting of 100 pulses as our input signal $\mathcal{E}(t)$. Each pulse takes a value

$$\mathcal{E}(t)=U \quad \text{or} \quad \mathcal{E}(t)=-U \tag{1.3}$$

and persists for time τ_0. The voltage U can be directly read from the signal generator. The words are designed by a computer which can arbitrarily modify τ_0 and determine the value of each pulse to produce any word needed. In Fig. 6(a) the first 50 pulses in one of our words are shown. The word does not have any characteristic frequency. It is obvious that each word contains information of

$$I=\log_2(2^{100})=100\ . \tag{1.4}$$

In Sec. II we describe how the information is received. A theoretical study verifies the correctness of our measurement. In Sec. III we input $\mathcal{E}(t)+Q(t)$ into our SR device, and measure the percentage of information that can be received from the output. We find that the portion of the information obtained from the noise output can be greatly enhanced by the bistable system, Eq. (1.1), under the proper condition of stochastic resonance.

II. THE INFORMATION RECEIVED FROM THE INPUT

First we should emphasize that the noise is, actually, colored. A true white noise must have infinite power, and then cannot be practically produced. Moreover, it is impossible to directly modulate a true white noise by finite pulses [5]. Nevertheless, the correlation time of noise τ_c, which is less than 10 μs in our case, is much smaller than τ_0 which is of order 10 ms, and then this correlation time can be approximately identical to zero. On the other hand, the Gaussian distribution of the noise has finite width. This noise can be modulated by a finite signal. In Fig. 1 we plot the profile of the probability distribution of noise voltage $H(t)$ (dots) when the effective noise strength H read from our noise generator is 1.2 V. Throughout the paper we use the effective noise strength H instead of the coefficient D in Eq. (1.2). It is obvious that

$$H\propto\sqrt{D/\tau_c}\ . \tag{2.1}$$

The probability distribution for a colored noise is

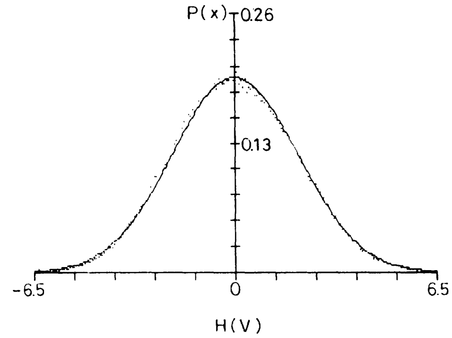

FIG. 1. The probability distribution of the input noise $\Gamma(t)$. $H=1.2$ V. The dots represent the experimentally measured data. The solid line corresponds to the Gaussian distribution (2.2) the height h of which is given by the dots.

$$P(y)=\left[\frac{\tau_c}{2\pi D}\right]^{1/2}\exp\left[-\frac{\tau_c}{2D}y^2\right]$$
$$=h\exp[-\pi h^2y^2]\ , \qquad (2.2)$$

where the height of the probability peak is

$$h=\left[\frac{\tau_c}{2\pi D}\right]^{1/2}\propto\frac{1}{H} \qquad (2.3)$$

In Fig. 1 the solid line shows a Gaussian distribution, the height of which is given by the peak of the dots. The theoretical result is identical to the experimental one. All the measurements in our set verify (2.2) and (2.3) perfectly.

Let us now design the way to reconstruct a word from a noise signal. We measure the voltage of the noise signal once each τ_1. Thus $m=\tau_0/\tau_1$ measurements are performed in the duration of each pulse. The m data are simply summed. The summation is regarded as the value of the received pulse. The sign of the received pulse is compared with that of the corresponding input pulse. We regard the input information correctly received if both signs are identical, or lost otherwise. For a given combination of U and H, which can be directly read from the signal generator and the noise generator, respectively, we measure the percentage of which information is received.

Before we go to the main point, receiving information by using the SR devices, let us first analyze the percentage of information one can receive from the input $\mathcal{E}(t)+\Gamma(t)$ for given U and H. In this case a theoretical analysis is available.

We assume

$$\tau_c \ll \tau_1 \ll \tau_0\ . \qquad (2.4)$$

(These conditions can be easily satisfied in our simulation since $\tau_0>10$ ms, $\tau_c<0.01$ ms.) Then each measurement can be regarded as independent of the others. By one measurement the probability of correctly receiving a pulse can be computed as

$$P_0=\tfrac{1}{2}+\int_0^U h\exp[-\pi h^2y^2]dy\ . \qquad (2.5)$$

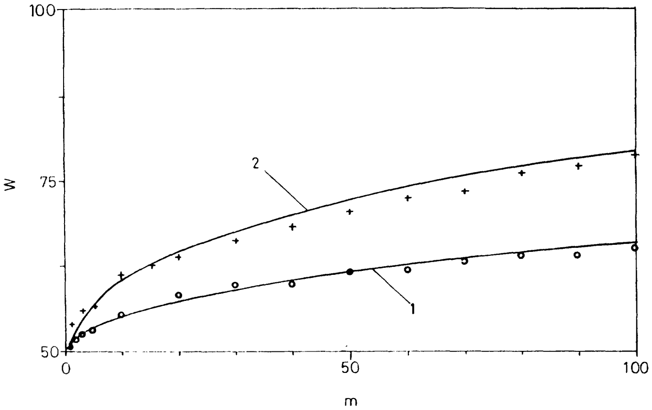

FIG. 2. W plotted vs m. $H=1.2$ V. $U=0.08$ V (curve 1) and 0.16 V (curve 2), respectively. The circles and crosses represent the experimental data and the solid lines show the theoretical result, Eq. (2.6). Both theoretical and experimental results coincide to each other.

According to the well-known law of large numbers, the probability in which one can correctly receive the information by m measurements is

$$W=\tfrac{1}{2}+\sqrt{m}\,h\int_0^U\exp[-m\pi h^2x^2]dx\ . \qquad (2.6)$$

For small U and large H we have

$$P_0\ll 1\ . \qquad (2.7)$$

Therefore Eq. (2.6) can be replaced by

$$W=\tfrac{1}{2}+\sqrt{m}\,P_0=\tfrac{1}{2}+\sqrt{m}\,hU\ . \qquad (2.8)$$

In Fig. 2 we plot W vs m by fixing $H=1.2$ V. The solid lines represent the theoretical results. The quantity h in (2.6) is taken from the experimental result in Fig. 1. The circles and crosses represent the experimental results. Each experimental plot is determined by the average of 40 experiments. Curves 1 and 2 correspond to $U=0.08$ and 0.16 V, respectively. The agreement between the theoretical and the experimental results is perfect. [Actually, Eq. (2.8) is already a very good approximation in the case of Fig. 2.]

It is clear from (2.5) and (2.6) that W can be, in principle, arbitrarily close to 100% as m is sufficiently large. However, two facts definitely restrict W. First, m cannot be infinitely large due to the limitation of the devices. For instance, the maximal m allowed by our present device is $m\approx 100$. For small P_0 a very large m is needed for W to reach 95%. For instance, as $P_0=5\%$ (in our experiment P_0 is around 1%) m should be larger than 1000 for $W>95\%$; that is too far beyond the capacity of our devices. Second, even if we have a capable computer and an analog-to-digital converter (ADC), the W is still limited by the conditions (2.4). If $m=\tau_0/\tau_1\approx\tau_0/\tau_c$ (i.e., if the time interval between two adjacent measurements is about or smaller than the correlation time of noise) various measurements can no longer be regarded as independent of each other. Then, Eq. (2.6) is no longer valid. In this case the actual W is considerably smaller than that given by Eq. (2.6). This point can be clearly seen in the next section.

III. INFORMATION RECEPTION BY UTILIZING THE SR DEVICE

For nonlinear system (1.1) a systematical theory for computing W is not available as of yet. A study of (1.1) with a small pulse amplitude can be seen in Ref. [8]. In this paper we provide experimental results from which one can see that under certain optimal modulation conditions the information received by our SR device can be much more than that given by Eq. (2.6) and Fig. 2.

The experimental block is the same as that in Ref. [9]. Similar devices have been used by various groups to study the problem of SR [10–13]. However, the signal input in the present case is an informational aperiodic word rather than a sinusoidal function. The word consists of 100 pulses of which the value is either U or $-U$ determined arbitrarily by a computer. In our simulation we adjust the experiment parameters such that $a=b=1$. The output of the SR device, $X(t)$, is received by the computer via an ADC. The pulses in $X(t)$ are reconstructed and the information contained in $X(t)$ is received according to the rule described in Sec. II.

In Fig. 3 we plot $I_{\rm NL}$ and $I_{\rm L}$ against H by fixing the input pulse amplitude to $U=0.16$ V and taking $\tau_0=24$ ms and $m=20$. The information received from the input $\mathcal{E}(t)+\Gamma(t)$ is denoted as $I_{\rm L}$ (circles), that from the output of the bistable system is denoted as $I_{\rm NL}$ (crosses) (the subscripts L and NL stand for linear and nonlinear). The behavior of the crosses ($I_{\rm NL}$) is extremely interesting. There are four characteristic stages in changing H. As H is very small ($H<0.35$ V) the switchings between the two basins happen rather seldom, and then the system practically stays in a single basin. In this first stage the behavior of $I_{\rm NL}$ and $I_{\rm L}$ is very similar to each other. (Note, in this stage the sign of the pulse reconstructed from the output of the bistable system is defined such that the stable solutions of the bistable system at $U=H=0$ is regarded to be zero.) When H increases (the second stage, $0.35\ {\rm V}<H<0.7$ V), random switchings between the two basins take place more frequently. $I_{\rm NL}$ decreases sharply. In this stage, $I_{\rm NL}$ decreases even lower than $I_{\rm L}$ ($I_{\rm NL}<I_{\rm L}$). Further increasing H (the third stage, $0.7\ {\rm V}<H<1.5$ V) may lead to a dramatic increase of $I_{\rm NL}$ while the quantity $I_{\rm L}$ monotonically decreases. There is a $I_{\rm NL}$ plateau on the cross line. On this plateau, the information received by our nonlinear device is practically 100%. In the same region $I_{\rm L}$ is lower than 65%, i.e., almost 70% of the input information has been lost for $I_{\rm L}$. (We regard 100% of the information lost if $I_{\rm L}=50\%$.) In the fourth stage ($H>1.6$ V), increasing H may slowly reduce $I_{\rm NL}$.

The existence of the third stage, or say, the existence of the high plateau on the cross curve, is particularly interesting. In this region the high level of $I_{\rm NL}$ is due to the optimal switchings which are the key point of stochastic resonance [3,4,10–13]. Therefore, under the condition of SR the information obtained from the noise output of the bistable system can be enormously enhanced.

In Fig. 4 we plot the same curves as in Fig. 3 by reducing the input signal strength to $U=0.08$ V. For such a small signal, the $I_{\rm L}$ curve goes down quickly to the 50% horizontal axis, i.e., the input information can hardly be taken back under such bad conditions for the signal-to-noise ratio. However, there is still a high plateau on the $I_{\rm NL}$ curve (though the width of the plateau is narrower and the height of the plateau is lower than that in Fig. 3) on which $I_{\rm NL}$ can be above 90%.

Figure 5 is useful for understanding the rule played by the correlation time of a colored noise. In Fig. 5 we do the same thing as in Fig. 4 except that the number m is increased to 80 (i.e., we measure 80 times during a period of each pulse $\tau_0=24$ ms). A comparison of Figs. 3–5 is interesting. For $I_{\rm L}$ the circle curves in Figs. 3 and 5 are almost the same, and in both curves $I_{\rm L}$'s are about two times higher than $I_{\rm L}$ in Fig. 4 at each H. This experimental result is identical to the theoretical analysis of Eq. (2.8). The law of large numbers works well since the correlation time of $Q(t)$ is extremely small, and the condition $\tau_c \ll \tau_1=\tau_0/m$ is satisfied for both $m=20$ and 80.

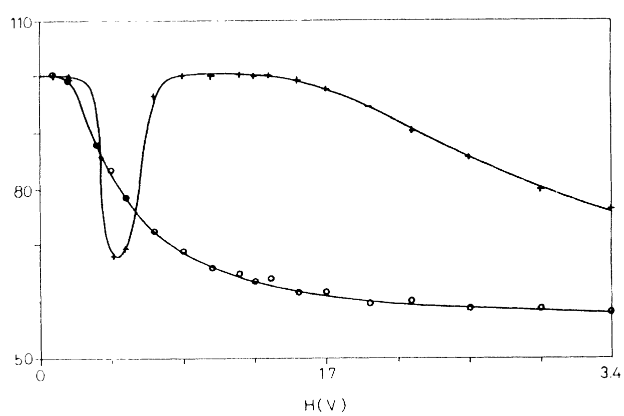

FIG. 3. The portions of information received shown vs the input noise. $U=0.16$ V, $\tau_0=24$ ms, $m=20$. $I_{\rm NL}(+)$ represents result from the output of the bistable system [$a=b=1$ in Eq. (1.1)]. $I_{\rm L}$ ($\bigcirc$) represents the result directly from the mixture of the input word and noise.

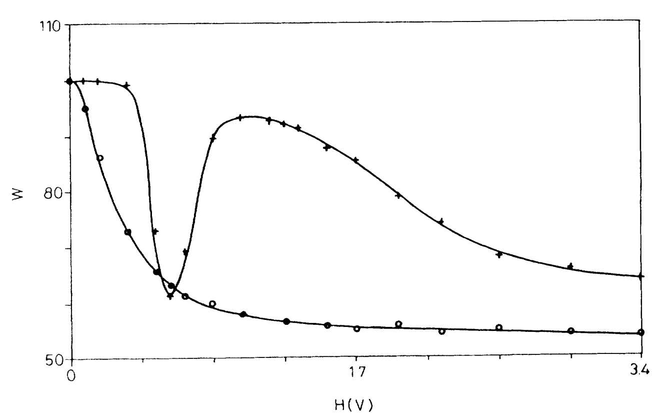

FIG. 4. The same as in Fig. 3 with U reduced to $U=0.08$ V.

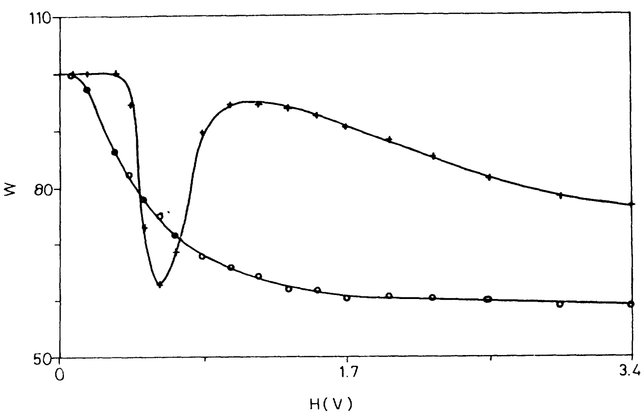

FIG. 5. The same as in Fig. 3 with m increased to $m=80$.

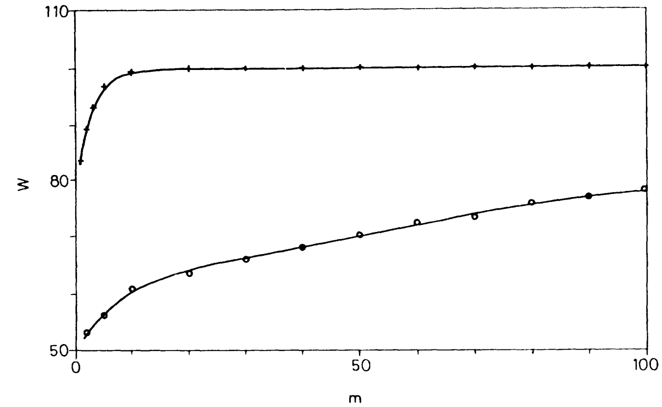

FIG. 7. I_{NL} (crosses) and I_L (circles) plotted vs m. $H=1.2$ V, $U=0.16$ V, $\tau_0=24$ ms. I_{NL} is much larger than I_L.

The situation for I_{NL} is completely different though in each figure I_{NL}'s are considerably higher than I_L's. It is obvious by comparing Figs. 4 and 5 that increasing m does not much change I_{NL} for small H while it does apparently increase I_{NL} for large H. The reason is simple. For small H the noise correlation time of the output $X(t)$ is large (which is of the order of the mean first passage time of the bistable system), and then different measurements cannot be regarded as independent of each other. Thus W is not sensitive to the number of the measurement times m. Increasing H can effectively reduce the noise correlation time of the output $X(t)$. Various measurements can be more and more independent of each other for larger and larger H. For sufficiently large H increasing m can definitely enhance I_{NL} according to the argument in Eqs. (2.6) and (2.8).

To get a clear impression how the nonlinear device can effectively increase the portion of the information received we present Fig. 6, where we fix $U=0.16$ V,

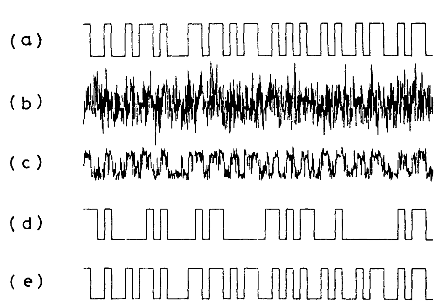

FIG. 6. (a) The first 50 pulses of an input word consisting of 100 pulses. (b) The mixture of the input signal and noise. $U=0.16$ V, $H=1.2$ V. The signal is almost completely destroyed by the noise. (c) The output from our nonlinear device with U and H being the same as in (a). The trace of the input word [see (a)] can be seen from the modulation of the noise output. (d) The first 50 pulses of the word reconstructed from (b). Many wrong pulses appear in comparison with (a). (e) The first 50 pulses of the word reconstructed from (c). The word of (a) is completely recovered.

$H=1.2$ V. Figure 6(a) shows the first 50 pulses of an input word. Figures 6(b) and 6(c) provide the time series of $\mathcal{E}(t)+\Gamma(t)$ and $X(t)$, respectively. It is obvious that without the nonlinear device the information contained in the signal is completely destroyed by the strong noise while with the nonlinear device the trace of the input word can still be seen from the modulation of the output. Figures 6(d) and 6(e) show the first 50 pulses reconstructed from Figs. 6(b) and 6(c), respectively, in the manner described in Sec. II. In Fig. 6(d) about one-third of the pulses have the wrong sign. On the contrary, in Fig. 6(e) the entire input word is practically restored after filtering the noise.

For a thorough comparison of I_{NL} and I_L we give the experimental results, Figs. 7 and 8, where I_{NL} and I_L are plotted versus m which varies from 1 to 100. Note, for our device, $m=100$ is the maximal number of the measurings during the period of one pulse τ_0. In Figs. 7 and 8 we fix $H=1.2$ V, $\tau_0=24$ ms, and take $U=0.16$ and 0.08 V, respectively. We find that for these small modulations I_{NL} is greatly larger than I_L. In Fig. 8 we

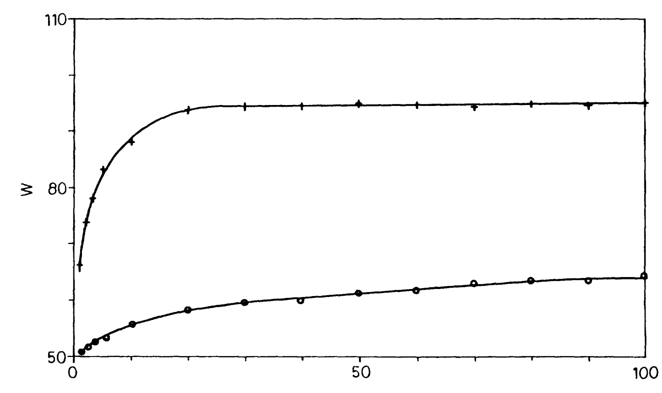

FIG. 8. The same as in Fig. 7 with the pulse amplitude reduced to $U=0.08$ V. The I_{NL} is insensitive to increasing m as $m>40$.

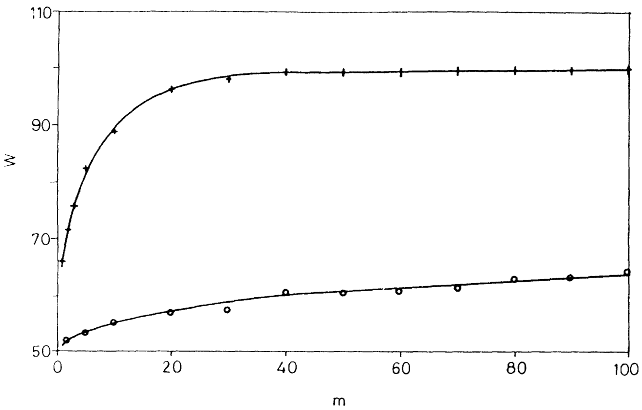

FIG. 9. The same as in Fig. 7 with τ_0 increased to $\tau_0=130$ ms. The curve of I_L is practically the same as that in Fig. 8. Now I_{NL} can be enhanced close to 100% (99.7%) by increasing m while in Fig. 8 the maximal I_{NL} is 94.5%.

notice again that after $m>40$ increasing m no longer increases I_{NL} due to the constraint of the finite noise correlation time of $X(t)$.

In Fig. 9 we take the same parameters as in Fig. 8 with τ_0 replaced by 130 ms. For such a large τ_0, $\tau_1=\tau_0/m$ becomes relatively larger than that of the noise correlation time of $X(t)$. Therefore, I_{NL} can be effectively enhanced close to 100% (99.7%) by increasing m. It is interesting to see that I_L is not modified by increasing τ_0 since the condition $\tau_c \ll \tau_0/m$ is fulfilled for both $\tau_0=24$ ms and $\tau_0=130$ ms. The behavior of the circles in both Figs. 8 and 9 remains the same. In Fig. 10 we fix $H=1.2$ V, $U=0.08$ V, $m=40$, and plot I_{NL} and I_L vs τ_0. As we expected, I_L is not sensitive to τ_0 while I_{NL} is monotonically increased by increasing τ_0.

The difference in our experiment compared to the previous study of stochastic resonance is that we use an arbitrary signal consisting of a sequence of pulses instead of a purely sinusoidal signal. For the periodic signal there are many linear approaches, including various optimal filters, amplifiers, and integrators to enhance the signal-to-noise ratio effectively and to reconstruct the signal out of strong noise. Nevertheless, for an arbitrary signal the linear approaches become much less effective. In this case, the advantage of our nonlinear treatment turns out to be very useful. Our experiment has suggested an alternative way towards a practical application of the nonlinear systems, especially the application of stochastic resonance.

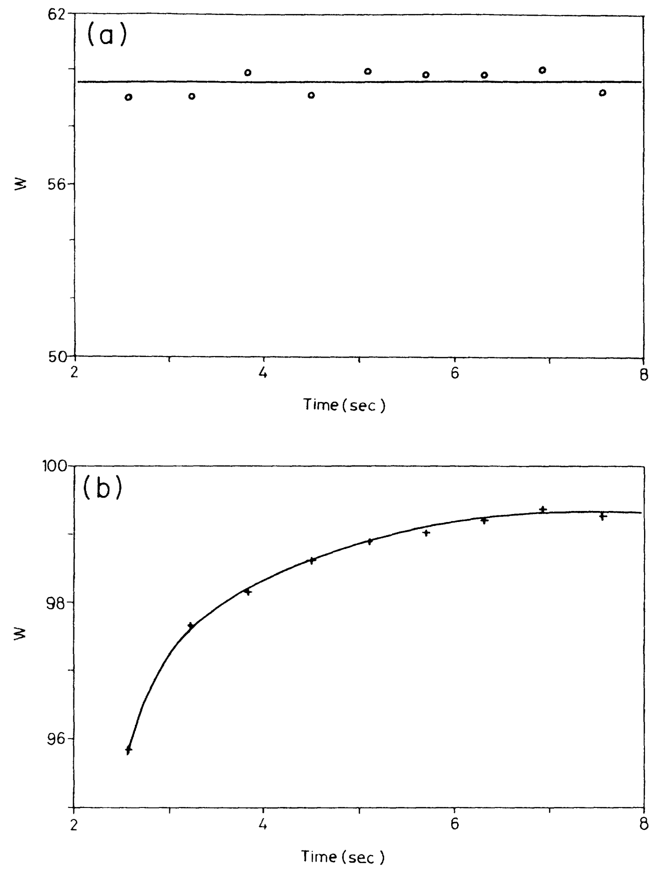

FIG. 10. I_L (a) and I_{NL} (b) plotted against τ_0. $H=1.2$ V, $U=0.08$ V, and $m=40$. I_L is insensitive to τ_0 while I_{NL} can be monotonically increased by increasing τ_0. The time indicated in the horizontal axis is the time length of 100 pulses.

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[1] R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, 453 (1981).
[2] G. Nicolis and G. Nicolis, Tellus **33**, 225 (1981).
[3] S. Fauve and F. Heslot, Phys. Lett. **97A**, 5 (1983).
[4] B. McNamara, K. Weisenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988).
[5] B. McNamara and K. Weisenfeld, Phys. Rev. A **39**, 4854 (1989).
[6] R. F. Fox, Phys. Rev. A **39**, 4148 (1989).
[7] G. Hu, G. Nicolis, and C. Nicolis, Phys. Rev. A **40**, 2030 (1990).
[8] W. M. Zheng, Phys. Rev. A **44**, 6443 (1991).
[9] Gong De-chun, Hu Gang, Wen Xiao-dong, Yang Chun-yan, Qin Gang-rong, and Li Rong, preceding paper, Phys. Rev. A **46**, 3243 (1992).
[10] L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. Lett. **62**, 349 (1989).
[11] L. Gammaitoni, E. Menichella-Saetta, S. Santucci, F. Marchesoni, and C. Presilla, Phys. Rev. A **40**, 2114 (1989).
[12] T. Zhou and F. Moss, Phys. Rev. A **41**, 4255 (1990).
[13] G. Hu, G. R. Qing, D. C. Gong, and X. D. Weng, Phys. Rev. A **44**, 6414 (1991).