Surface roughening and the long-wavelength properties of the Kuramoto-Sivashinsky equation

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(Received 3 December 1991)

The long-wavelength properties of the Kuramoto-Sivashinsky equation are studied in $2+1$ dimensions using numerical and analytic techniques. It is shown that this equation is *not* in the universality class of the Kardar-Parisi-Zhang model. Its roughening exponents are (up to logarithmic corrections) like those of the free-field theory, with dimension 2 being the marginal dimension for roughening. Assuming that the solution has logarithmic corrections, we derive a scaling relation for the exponents of the logarithmic terms. This solution is consistent order by order with the Dyson-Wyld diagrams. We explain why previous renormalization-group treatments failed.

PACS number(s): 05.40.+j, 64.60.Cn, 64.60.Ak, 68.35.−p

Two models for surface roughening have enjoyed tremendous attention due to their apparent simplicity and very rich nonlinear phenomenology. One is the Kardar-Parisi-Zhang (KPZ) model [1], which contains a random forcing

$$\frac{\partial h(\mathbf{x},t)}{\partial t}=\nu_0\nabla^2 h(\mathbf{x},t)+|\nabla h(\mathbf{x},t)|^2+\eta(\mathbf{x},t)\ , \tag{1}$$

where $h(\mathbf{x},t)$ is the height of a growing interface and $\eta(\mathbf{x},t)$ is a white, Gaussian random noise. The second is the Kuramoto-Sivashinsky (KS) equation [2,3], which is completely deterministic:

$$\frac{\partial h(\mathbf{x},t)}{\partial t}=\nu_0\nabla^2 h(\mathbf{x},t)-\nabla^4 h(\mathbf{x},t)+|\nabla h(\mathbf{x},t)|^2\ . \tag{2}$$

In Eqs. (1) and (2), ν_0 is a parameter that is positive in (1) and negative in (2). In both models $\mathbf{x}$ is a d-dimensional space, and we discuss the growth of the interface in $d+1$ dimensions. The KPZ equation was derived as a continuum limit of models describing random particle additions to a growing interface [4]. Without the random force $\eta(\mathbf{x},t)$ the fate of $h(\mathbf{x},t)$ is to decay to zero and stay there forever. The KS equation was derived in the context of intrinsic instabilities like flame propagation. It is linearly unstable and nonlinearly chaotic, with bounded solutions roaming on a strange attractor forever [5]. Apart from the fact that the nonlinear terms are the same, these two models seem very different, and it therefore came as a surprise when claims appeared in the literature [6,7] that as far as their long-wavelength limit is concerned, these equations have the same properties at least in $1+1$ dimensions. Our aim of this paper is to show that these equations are not in the same universality class; they exhibit very different behavior in $2+1$ dimensions. We begin by explaining how a universality class is defined here.

Both Eqs. (1) and (2) give rise to a self-affine solution [8]. Self-affine surfaces have a clear "directionality," without overhangs, and it is natural to focus on the double correlator of the height differences $S_2(\mathbf{x},t)$

$$S_2(\mathbf{x},t)=\langle[h(\mathbf{x}_0+\mathbf{x},t_0+t)-h(\mathbf{x}_0,t_0)]^2\rangle \sim x^{2\chi}\Psi(t/x^{\chi/\beta})\ . \tag{3}$$

Here the angle brackets denote an ensemble average, which is realized by an average over time t_0 and position x_0. The exponent χ is known as the static width exponent [9], which characterizes the L dependence of the saturated width W, $W=\langle[h(x_0,t_0)-\langle h\rangle]^2\rangle^{1/2}\sim L^{\chi}$. The dynamic scaling properties of self-affine surfaces are carried by the exponent β. We refer to two models as being in the same universality class if they share the same exponents χ and β in all dimensions.

Much work was devoted to the computation of χ and β in the context of KPZ [10]. In $1+1$ dimensions it is stated that [1] $\chi=\frac{1}{2}$ and $\beta=\frac{1}{3}$. In $2+1$ dimensions numerical evidence [4] indicates that $\chi\approx 0.4$ and $\beta\approx 0.25$. There is also an exact scaling relation that characterizes the KPZ problem, i.e., $\chi+\chi/\beta=2$. In the context of KS, the dynamical renormalization-group analysis of Ref. [6] culminated in the conclusion that the long-wavelength limit of Eq. (2) is identical to Eq. (1) in $1+1$ dimensions. Indeed, the numerical work of Ref. [7] corroborated these claims on the basis of simulations of KS in $1+1$ dimensions. Reference [6] has also concluded that Eq. (2) does not possess scale-invariant solutions in $2+1$ dimensions.

It is our contention that the KS equation does have scale-invariant solutions in $2+1$ dimensions. We first show that the results of extensive numerical simulations point toward a value of χ that is $\chi=0$ in $2+1$ dimensions. The behavior of the KS equation in $2+1$ dimensions is closer to that of the free-field theory

$$\frac{\partial h(\mathbf{x},t)}{\partial t}=\nabla^2 h(\mathbf{x},t)+\eta(\mathbf{x},t) \tag{4}$$

than to the behavior of the KPZ model. Why this is so will be explained after the examination of the numerics.

In Fig. 1(a), we display the Fourier spectrum of the spatial autocorrelation function in (2+1) D, $\langle h_{\mathbf{k}} h_{-\mathbf{k}}\rangle$, obtained from real-space integration of Eq. (2) with $L^2=512\times512$. One sees a bump in the range of the fastest growing linearly unstable modes, and then a long tail towards small-k components that can be very well fit by a $1/k^2$ behavior except for logarithmic corrections that are discussed below. Panel (b) shows convincingly that the width versus L dependence is weaker than a power, and indeed $\chi=0$. Panel 1(c) shows a fit of the function $W\sim\ln[\ln(L/L^*)]$, with L^* being a typical length. This fit is suggested by the theory that we describe next.

To understand these numerical findings, we should ponder for a moment the properties of the KS equation. First, we recognize that the range of linearly unstable modes plays a crucial role in the nonlinear dynamical behavior of the equation. Even if we are interested in the IR properties at very small k, it is not likely that we can disregard this dynamically dominant range of $\mathbf{k}$ vectors, which are also seen in the prominent bumps in the spectra, cf. Fig. 1. In fact, we shall see that we need to carefully respect the nonlocal interactions in $\mathbf{k}$ space; they are responsible to the observed behavior. This remark will be soon turned into an analytic tool. Second, we realize that in terms of perturbative analyses of Eq. (2), we are in an awkward situation. The linear part is unstable, and any perturbation theory that expands around the bare linear propagator is bound to have uncontrolled divergences. This is in fact one of the errors of Ref. 6. On the other hand, we know that a renormalized propagator must exist since there are rigorous proofs for the existence and boundedness of the solutions of the KS equation [5]. Accordingly, we should develop our perturbative treatment around the *renormalized* rather than the bare propagator. Thus we assume from the start that such a renormalized propagator exists, and we only need to find a self-consistent scheme to examine its properties. The natural scheme to do so is the Wyld diagrammatic technique, which we turn to now.

Consider, in $\mathbf{q}=\mathbf{k},\omega$ representation, the dressed propagator $G_{\mathbf{q}}=G_{\mathbf{k},\omega}$, defined as the response of the nonlinear

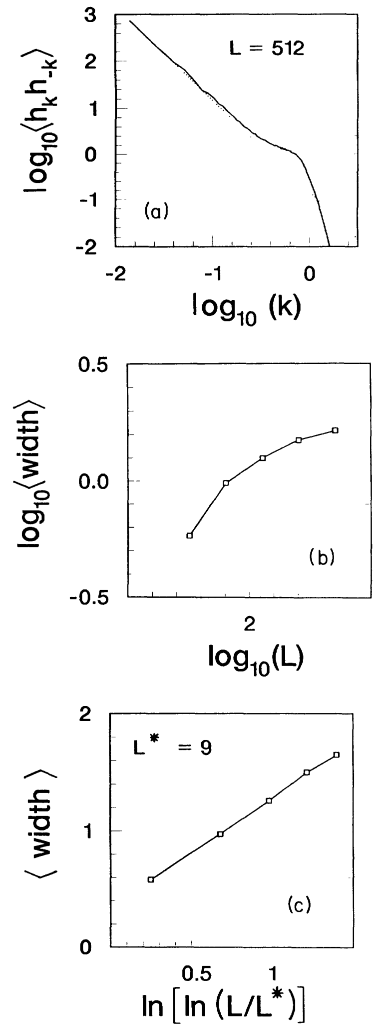

FIG. 1. Numerical results for Eq. (2) in $2+1$ dimensions. (a) The simultaneous correlator in a system of size 512×512. The dotted line is const/k^2. (b) The width W vs L in a log-log plot for systems of sizes between 32×32 and 512×512. (c) The width W vs $\ln[\ln(L/L^*)]$ with L^* being 9.

system to a vanishingly small external perturbation $\delta f_{\mathbf{q}'}$: $G_{\mathbf{q}}\delta(\mathbf{q}-\mathbf{q}')=\langle \delta h_{\mathbf{q}}/\delta f_{\mathbf{q}'}\rangle$. A consequence of the existence [5] of a bounded strange attractor and Lyapunov exponents for the KS equation is that $G_{\mathbf{q}}$ exists. It is also standard knowledge that if $f_{\mathbf{q}}$ is a white, Gaussian noise obeying $\langle f_{\mathbf{q}} f^*_{\mathbf{q}'}\rangle=\delta(\mathbf{q}-\mathbf{q}')f^2_{\mathbf{q}}$, then $G_{\mathbf{q}}\delta(\mathbf{q}-\mathbf{q}')=\langle h_{\mathbf{q}} f^*_{\mathbf{q}'}\rangle/f^2$. In terms of $G_{\mathbf{q}}$ and the dressed correlator $n_{\mathbf{q}}\delta(\mathbf{q}-\mathbf{q}')=\langle h_{\mathbf{q}} h^*_{\mathbf{q}'}\rangle$, one derives [11,12] the Dyson-Wyld equations

$$G_{\mathbf{k},\omega}=\frac{1}{\omega-i\gamma_{\mathbf{k}}-\Sigma(\mathbf{k},\omega)}\ , \tag{5}$$

$$n_{\mathbf{k},\omega}=|G_{\mathbf{k},\omega}|^2[\Phi(\mathbf{k},\omega)+f^2_{\mathbf{k},\omega}],\quad n_{\mathbf{k}}=\int n_{\mathbf{k}\omega}d\omega/2\pi\ . \tag{5'}$$

In these equations $\gamma_{\mathbf{k}}$ is the bare linear part, which in the case at hand reads $\gamma_{\mathbf{k}}=(-\nu_0 k^2-k^4)$. The mass operators $\Sigma(\mathbf{k},\omega)$ and $\Phi(\mathbf{k},\omega)$ are the "self-energy" and the

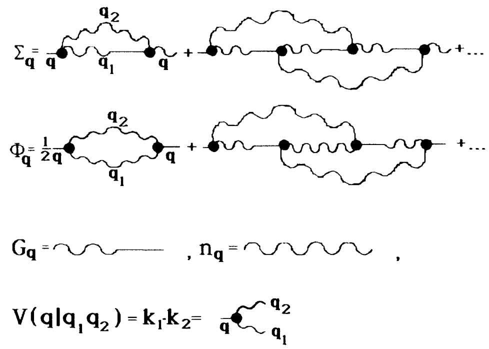

FIG. 2. The first diagrams in the Wyld expansion for the mass operator, together with the symbols used for the propagator, correlator, and vertex.

"intrinsic noise," respectively, which are written as an infinite expansion in terms of $G_{\mathbf{k},\omega}$ and $n_{\mathbf{k},\omega}$. The first diagrams appearing in the expansion of $\Sigma(\mathbf{k},\omega)$ and $\Phi(\mathbf{k},\omega)$, which are the same for KPZ and KS, are shown in Fig. 2. These diagrams are completely standard, and the techniques to derive Eq. (5) and (5′) can be found in many places [11,12]. Our contribution is in analyzing these diagrams in light of the comments offered above, taking the physics of KS into account. The analysis of Eq. (5′) is different for KPZ and KS. In the former case, $f_{\mathbf{q}}^2$ can be taken as the bare noise correlator $\eta_{\mathbf{q}}^2$. In the latter case, $\eta_{\mathbf{q}}$ is absent and $f_{\mathbf{q}}^2$ can be put to zero. The equation generates its own intrinsic noise even in the absence of external perturbations. We shall show that the sign of ν_0 distinguishes the solutions of the two cases.

We write now the two integrals that are represented by the lowest-order diagrams for $\Sigma(\mathbf{k},\omega)$ and $\Phi(\mathbf{k},\omega)$:

$$\Sigma^{(2)}(\mathbf{k},\omega)=\int V(\mathbf{k}_1,\mathbf{k}_2)V(\mathbf{k}_2,\mathbf{k})\delta(\mathbf{q}+\mathbf{q}_1-\mathbf{q}_2)\times G_{\mathbf{q}_1}n_{\mathbf{q}+\mathbf{q}_1}d\mathbf{q}_1 d\mathbf{q}_2 \,, \tag{6}$$

$$\Phi^{(2)}(\mathbf{k},\omega)=\int V^2(\mathbf{k}_1,\mathbf{k}_2)\delta(\mathbf{q}+\mathbf{q}_1-\mathbf{q}_2)n_{\mathbf{q}_1}n_{\mathbf{q}_2}d\mathbf{q}_1 d\mathbf{q}_2 \,. \tag{7}$$

As a first step we take into account these diagrams only. In the context of quantum field theory such an approximation is known as a "one-loop approximation," or an approximation of nonrenormalized interactions. In the theory of hydrodynamic turbulence this is known as Kraichnan's direct interaction approximation [12]. In the context of KPZ it is referred to as the mode-coupling technique [13,14]. It is important to emphasize that in our context the approximation [6,7] is not a naive second-order perturbation theory. The latter would be obtained if in (6) and (7) one used the bare propagators. We use the renormalized Green's function and correlation function from the start. This allows us to expect results that are at least qualitatively correct. Nevertheless, this is an approximation that has to be checked later for self-consistency. We shall explain later how the results of this analysis can be tested to all orders in the diagrammatic expansion. As this point we shall seek solutions for Eqs. (5)–(7) using the ansatz

$$G_{k,\omega}=\frac{1}{\nu k^z}g\left[\frac{\omega}{\nu k^z}\right],\quad n_{\mathbf{k},\omega}=\frac{n_{\mathbf{k}}}{\nu k^z}f\left[\frac{\omega}{\nu k^z}\right],\quad n_{\mathbf{k}}=\frac{n}{k^y}\,, \tag{8}$$

where νk^z has the dimension of ω, and (n/k^y) has the dimension of the simultaneous double correlator $n_{\mathbf{k}}$. As long as the integrals (6) and (7) converge, the exponents y and z satisfy a scaling relation. Denoting by t the scaling exponent of the bare vertex ($t=2$ in this case), power counting leads to

$$2z+y=2t+d\,. \tag{9}$$

This is a well-known scaling relation for models with strong quadratic nonlinearities. In the language of χ and β, $y=d+2\chi$ and $z=\chi/\beta$. Equation (9) translates to $\chi/\beta+\chi=t$, which is the relation mentioned above. A failure of this scaling relation indicates either nonconvergence of the integrals, or a free-field theory in which $G_{\mathbf{q}}$ remains undressed, or a renormalization of the vertex.

Substituting (8) in the integrals (6) and (7) and thinking about the resulting integrals in the limit $k\to 0$, we find that apparently there are two very different solutions. One is obtained if we seek a solution in which the dominant contribution to the integrals (6) and (7) comes from the range of k_1,k_2 that are of the same order of k. We refer to this solution as "local in k space" and show below that it is consistent with the universality class of KPZ. The other solution is obtained when one seeks solutions in which the dominant contribution to (6) and (7) comes from $k_1,k_2\gg k$. We refer to this solution as "nonlocal in **k** space," and show below that it leads to the observed KS phenomenology. We shall also show that only one of these solutions is tenable for KS.

In 2+1 dimensions the integrals (6) and (7) read

$$\Sigma^{(2)}(\mathbf{k},\omega)=-\int \mathbf{k}_1\cdot(\mathbf{k}+\mathbf{k}_1)(\mathbf{k}+\mathbf{k}_1)\cdot\mathbf{k}\frac{1}{\nu k_1^z}g\left[\frac{\omega_1}{\nu k_1^z}\right]\times\frac{n}{\nu|\mathbf{k}+\mathbf{k}_1|^{z+y}}f\left[\frac{\omega+\omega_1}{\nu|\mathbf{k}+\mathbf{k}_1|^z}\right]d\mathbf{k}_1 d\omega_1\,, \tag{10}$$

$$\Phi^{(2)}(\mathbf{k},\omega)=\tfrac{1}{2}\int[\mathbf{k}_1\cdot(\mathbf{k}-\mathbf{k}_1)]^2\frac{n}{\nu k_1^{z+y}}f\left[\frac{\omega_1}{\nu k_1^z}\right]\times\frac{n}{\nu|\mathbf{k}+\mathbf{k}_1|^{z+y}}f\left[\frac{\omega+\omega_1}{\nu|\mathbf{k}+\mathbf{k}_1|^z}\right]d\mathbf{k}_1 d\omega_1\,, \tag{11}$$

If we assert that the important contributions to (10) and (11) come from the high end of the k_1 range, i.e., $k_1\sim k_{\max}\gg k$, and of $\omega_1\sim\omega_{\max}\sim k_{\max}^z\gg\omega$, then we find

$$\Sigma^{(2)}(\mathbf{k},\omega)\sim C_1 n k^2 k_{\max}^{4-y-z}/\nu\,, \tag{12}$$

$$\Phi^{(2)}(\mathbf{k},\omega)\sim C_2 n^2 k^2 k_{\max}^{6-2y-z}/\nu\,, \tag{13}$$

as long as we meet the two simultaneous conditions

$$4-y-z\geq 0,\quad 6-2y-z\geq 0\,. \tag{14}$$

Thus to this order we find that for small k, ν satisfies

$$\nu = -\nu_0 + C_1 n k_{\max}^{4-y-z}/\nu \ . \tag{15}$$

Thus we get $G_q \approx 1/(\omega - i\nu k^2)$ and $n_q \approx C_2 n k_{\max}^{6-2y-z}/\nu[\omega^2+(\nu k^2)^2]$. Comparing with (8) we find $y=z=2$ [and cf. (9)], and we realize that our conditions (14) are obeyed as equalities. [This could be guessed from the fact that $2+1$ dimensions are marginal for both problems (1) and (2)]. Even though the scaling relation (9) is obeyed by the powers, the integrals diverge logarithmically. We are thus forced to assume the existence of such corrections and to redo the calculation starting with an appropriate ansatz

$$G_q = \frac{1}{\nu k^2 \ln^\alpha(k_{\max}/k)} g\left[\frac{\omega}{\nu k^2 \ln^\alpha(k_{\max}/k)}\right] , \tag{16}$$

$$n_q = \frac{n_k}{\nu k^2 \ln^\alpha(k_{\max}/k)} f\left[\frac{\omega}{\nu k^2 \ln^\alpha(k_{\max}/k)}\right], \quad n_k = \frac{n}{k^2 \ln^\beta(k_{\max}/k)} \ . \tag{17}$$

It is straightforward to check that the result of integrating (6) and (7) leads to the prediction

$$2\alpha+\beta=1 \ . \tag{18}$$

This is a second scaling relation for the exponents of the logarithms. Its existence hinges on the fact that the vertices remain unrenormalized in this problem. Since our problem has Galilean invariance, it is possible to prove that if the solution [(16),(17)] does exist, then the vertices are protected to all orders, not only that the scaling index t remains $t=2$, but also that there are no logarithmic corrections to the renormalized vertex. Double logarithmic corrections, if they arise in some diagrams, must cancel in every order of perturbation theory. As a consequence, both scaling relations (9) and (18) are consistent to all orders.

Having derived the scaling relation (18), we still cannot determine α and β separately. We need further considerations. As we know that G_q exists, we can conjecture that $\alpha \geq 0$, since then for small k, $\Sigma^{(2)}(\mathbf{k})$ overshadows γ_k. Moreover, below we give arguments to show that $\alpha=0$. If we accept this, then $\beta=1$, and we can integrate (17) to find $W(L)$. The leading term in the result is

$$W \sim \ln[\ln(L/L^*)] , \tag{19}$$

where L^* is of the order of $1/k_{\max}$. In Fig. 1(c) we compare this prediction to the data, and find that Eq. (19) provides a better fit than either a power or a single logarithm. We stress however that due to this weak dependence of W on L, one cannot exclude a small power of $\ln(L/L^*)$.

As was stated above, there exists another solution of the very same equations (13) and (14). This solution can be sought by asserting that the major contribution to the integrals comes from the local interactions, i,e., $k_1 \sim k_2 \sim k$. This means that the integrals are assumed to converge, and the solution has the form (8) with the scaling relation (9). This is consistent with the KPZ solution. The calculation of z and y for this solution is beyond the scope of this paper, and will be discussed elsewhere.

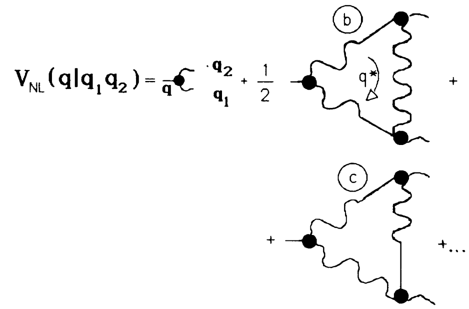

FIG. 3. The first diagrams for the vertex renormalization.

At this point we explain why our solution [(16),(17)] has to be valid to all orders in perturbation theory. This can be seen by examining the first corrections to the bare vertex V, since the nature of these higher-order corrections is the same for vertices and for mass operators. The first corrections to V are shown in Fig. 3. Consider the nonlocal UV contribution when $q^* \gg q, q_1, q_2$. The integrals in the diagrams (b) and (c) may be estimated as

$$(\mathbf{k}_1\cdot\mathbf{k}_2)\int_k^{k_{\max}} \frac{dk}{k\ln(k_{\max}/k)} \approx V(\mathbf{k}|\mathbf{k}_1\mathbf{k}_2)\ln\ln(k_{\max}/k) \ . \tag{20}$$

The double logarithmic factor is dangerous. The next corrections may contain this factor in higher powers, forming possibly an expansion of $[\ln(k_{\max}/k)]^\varepsilon$. Here ε is an unknown anomalous exponent for the logarithmic correction of the renormalized vertex: $V_{NL}=V(\ln)^\varepsilon$. However such a conclusion is untenable for the KS equation. It is possible to prove that because of the symmetry properties of the interaction in Eqs. (1) and (2) (including Galilean invariance), the double logarithmic terms are canceled and $\varepsilon=0$. This means that the solution assumed in (16) and (17) is self-consistent with all orders of perturbation theory.

It is important to note that in $1+1$ dimensions the solution found here does not exist. Beginning in the same way as in $2+1$ dimensions, we can analyze the integrals (10) and (11) using the same assumption that the major contribution comes from the higher end of the range of k_1. The only difference now is that $d\mathbf{k}_1 \sim k_1 dk_1$ is replaced by dk_1, and therefore by power counting one sees that (12) and (13) would be replaced by $\Sigma^{(2)}(k,\omega) \sim C_1 k^2 k_{\max}^{3-y-z}$ and $\Phi^{(2)}(k,\omega) \sim C_2 k_{\max}^{5-2y-z}$ if the conditions $y+z \leq 3$ and $2y+z \leq 5$ are met. In contrast to $2+1$ dimensions, these conditions are not obeyed. Comparing as before to (8) we find again $z=y=2$ and therefore the assumption that the integral diverges in the UV is not self-consistent. In fact, formally speaking, the solution found above is not available in any dimension smaller than $2+1$.

We end this paper by explaining why only one of the solutions discussed above is tenable for KS in $2+1$ dimensions. Since we assumed that $\alpha=0$, Eq. (15) is still valid up to $\ln[\ln(k/k_{max})]$ corrections, $\nu=-\nu_0+C_1 n/\nu$ or

$$\nu^2=-\nu\nu_0+C_1 n \ . \tag{21}$$

From Eqs. (5′), (13), and (17), all integrated over frequencies, we get

$$n_k=\frac{C_2 n^2}{\nu^2 k^2 \ln^\beta(k_{max}/k)} \sim \frac{n}{k^2\ln^\beta(k_{max}/k)}$$

or

$$\nu^2=C_2 n \ . \tag{22}$$

From (21) and (22) together we find

$$(C_1-C_2)n=\nu\nu_0 \ . \tag{23}$$

Notice now that C_1 and C_2 are uniquely determined by the nonlinearity $|\nabla h|^2$, and are therefore the same for KS and KPZ. Also, n is positive definite and ν is positive by the existence of a dressed G_q. Therefore, if $C_1<C_2$, Eq. (23) is only tenable for the KS equation, in which ν_0 is negative. If $C_1>C_2$, this solution is only possible for the KPZ model. This dependence on the *bare* sign of ν_0 shows that *the KS equation and the KPZ model are not in the same universality class.* Although the analytic calculation of C_1 and C_2 is formidable since it calls for analyzing all the diagrams, our numerics show that $C_1<C_2$, since this solution is selected for the KS equation. This is in fact a computer-assisted proof that our solution is only available for the KS equation and not for the KPZ model. Note that ν_0 appears in Eq. (22) explicitly, meaning that it does not disappear even in the limit of vanishingly small k. This stems from the fact that $\alpha=0$, leaving at most $\ln\ln(k_{max}/k)$ corrections in Eq. (22). Remembering our reasoning for the cancellation of such terms due to Galilean invariance, we believe that any such corrections to C_1 and C_2, say, would cancel as well, leaving Eq. (23) unchanged.

Lastly, we show that there cannot exist solutions for almost all positive values of α. If α were positive, then for small k the nonlinear addition to the damping would exceed $\nu_0 k^2$ by a factor proportional to $\ln^\alpha(k_{max}/k)$. For k small enough, ν_0 becomes negligible, and can be put to zero. Repeating the above calculation once more, we obtain (23) again, but with different values of C_1, C_2 that depend on α. Since $\nu_0=0$, n would vanish, and the solution with $\alpha>0$ cannot exist except for an accidental additional solution for which $C_1(\alpha)=C_2(\alpha)$. Since we expect $C_1(\alpha)$ and $C_2(\alpha)$ to depend very weakly on α, we think that such an additional solution does not exist.

In summary, we have provided an analytic solution for the KS equation in 2+1 dimensions that is valid to all orders in perturbation theory, we have shown that the KS equation and the KPZ model are in different universality classes, and we exposed the errors in previous treatments of this problem. In combination with our numerics, one concludes that as far as the roughening exponents are concerned, the KS equation in 2+1 dimensions roughens like the free-field theory. Of course, our explicit calculations of the logarithmic corrections shows that it is not a free-field theory.

We benefited from discussions with T. Bohr, G. Goren, G. Grinstein, C. Jayaprakash, M. Kosterlitz, D. Mukamel, and V. Lebedev. I.P. acknowledges the partial support of the German-Israeli Foundation and the U.S.-Israel Binational Science Foundation. V.S.L. was supported by the Weizmann Institute.

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