Models of crack propagation

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Three simple models of steady-state crack propagation are examined in a search for clues about the nature of dynamic velocity-selection mechanisms in these systems. The first is a one-dimensional model with stick-slip friction which includes, as a special case, a model of an earthquake fault. The conclusion here is that velocity-weakening slipping friction generally causes the crack to accelerate to the limiting wave speed independent of loading strength. The second model is also one dimensional, but the dissipation mechanism is the analog of a Kelvin viscosity. In this case, steady-state solutions at large applied stresses exhibit oscillating crack-opening displacements and propagate at speeds comparable to or higher than the nominal wave speed. The third model is the two-dimensional analog of the second. Its qualitative behavior turns out to be essentially the same as in the one-dimensional version.

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I. INTRODUCTION

My purpose in the investigations to be described here is to work, via a series of simple but increasingly realistic models, toward answers to some questions in the theory of crack propagation [1]. These questions are the following.

(i) Under what circumstances is steady-state crack propagation possible?

(ii) By what physical mechanisms are steady states selected? What is the mathematical nature of the selection problem?

(iii) How are the answers to the first two questions affected by various features of the models that might be considered? What is the role of dimensionality, of dissipation mechanisms, and of specific assumptions regarding decohesion and energy balance at crack tips? In short, what, if any, are the universality classes for these phenomena?

The immediate motivation for much of this work is our recent discovery of a dynamic selection mechanism for rupture propagation in a one-dimensional model of an earthquake fault [2,3]. This ostensibly simple model of a fault exhibits shocklike rupture fronts whose properties depend on a short-wavelength cutoff; it turns out that this cutoff is necessary in order to make sense of the nonlinear differential equation which describes this system. This result suggests that there may be more variety than had previously been supposed in the range of mathematical situations that we can expect to encounter in fracture dynamics. The idea, therefore, is to explore this range of possibilities by looking at several other related models that are simple enough to be analytically tractable.

The analyses presented here by no means provide definitive answers to the questions posed in the opening paragraph, but some interesting clues seem to be emerging. In particular, we shall see that the dynamic selection mechanism encountered in the earthquake model appears to be special to situations in which displacements near the crack tip are governed by an unstable, velocity-weakening, stick-slip friction law. (Stressed interfaces in composite materials may belong in this category.) We also shall find that, in some models of ordinary fracture, steady-state modes of propagation may exist, at least mathematically, at speeds above the nominal wave speed. There is an interesting indication that an oscillatory motion sets in at some propagation speed less than the wave speed. The latter observation is intriguing in view of several outstanding puzzles in fracture dynamics—the fact that limiting speeds for crack propagation seem to be appreciably smaller than the theoretical Rayleigh maximum, and a recent observation that oscillatory instabilities may be associated with the characteristic limiting speeds [4].

The scheme of this paper is as follows. Sections II and III are devoted to the analysis of two different one-dimensional models distinguished from one another by their dissipation mechanisms. The first model, which is discussed in Sec. II, dissipates energy via velocity-dependent friction and includes our previous earthquake model as a special case. The second, described in Sec. III, is a one-dimensional version of a model of crack propagation in a material with Kelvin viscoelasticity. It is in this model that we see the interesting high-speed behavior mentioned in the preceding paragraph.

Section IV contains a discussion of a two-dimensional viscoelastic model quite similar to the one studied in an earlier paper by Barber, Donley, and myself [5]. The main reason for looking at this model here is to check whether the interesting features of the viscoelastic model in Sec. III might be artifacts of its one dimensionality. Accordingly, this two-dimensional model includes inertial effects, which were missing in the earlier work, but which are an essential ingredient for the phenomena of interest in this paper. As a compensating simplification, a scalar wave equation is substituted for the full equations of two-dimensional elasticity. The resulting model is similar in many respects to one studied by Willis [6], but there are some important differences that will be mentioned later. The analysis indicates that the one- and two-dimensional situations are qualitatively quite similar and, thus, that more detailed dynamical studies in one dimension may be very useful.

The results presented here are far from complete, even within the limited scope of the simple models being considered and the simple questions being asked about them. In particular, I restrict the discussion here to analysis that can be carried out without numerical computations, which clearly will be needed in the next stages of this project. The paper concludes, in Sec. V, with some remarks about what needs to be done next.

II. ONE-DIMENSIONAL MODELS WITH FRICTION

All of the one-dimensional models to be considered here are described by differential equations of the form

$$\ddot{U}=U''-m^2(U-\Delta)-f(U)-\phi(\dot{U}) . \tag{2.1}$$

Here, $U(x,t)$ is the displacement of the material at time t and position x along the face of the crack. Dots and primes denote differentiation with respect to t and x, respectively. We may visualize U as being either the crack-opening displacement normal to the x axis in mode I, the shear displacement parallel to the x axis in mode II, or the anti-plane displacement in mode III. We generally shall assume that the crack moves in the negative x direction. By definition, U must vanish along the unbroken region of the x axis, $x<x_{\rm tip}$.

In the absence of the terms denoted f and ϕ on the right-hand side, (2.1) is a massive wave equation in which position and time have been scaled so that the wave speed (the coefficient of U'') is unity. The quantity m^2 is a force constant which may be thought of as representing a linear elastic coupling between the fracturing material and a fixed substrate. In our earthquake model, we understood this term to be the coupling between the seismically active part of the fault and the bulk of the tectonic plate whose motion drives the system. In that case, the "mass" m is the inverse of a length whose order of magnitude is the thickness of the plate. More generally, m^{-1} plays the role of some characteristic length scale in the higher-dimensional system whose behavior we are trying to simulate in a one-dimensional model. The fully relaxed configuration is $U=\Delta$. Thus, in the unbroken region where $U=0$, the applied strain is $m\Delta$, and the elastic energy available to drive the crack is $m^2\Delta^2/2$ per unit length. Throughout this discussion we shall think of Δ as being a measure of the driving force applied to the system.

The function $-f(U)$ in (2.1) is the cohesive force, that is,

$$\int_0^\infty f(U)dU=\Gamma \tag{2.2}$$

is the fracture energy. The function $-\phi(\dot{U})$ is the force of friction which, for $\dot{U}$ positive but not too large, can be assumed to have the form

$$\phi(\dot{U})=\text{const}+2\alpha\dot{U} . \tag{2.3}$$

We shall not consider situations in which $\dot{U}$ changes sign for this class of models; thus (2.3) is an adequate description of the frictional force throughout any slipping motion, that is, during the time when $\dot{U}\neq 0$; and thus we can eliminate the constant in (2.3) by absorbing it into the term $m^2\Delta$ in (2.1).

We want to consider the possibility that the sticking friction $\phi(0)$ is not the same as the zero-velocity limit of the slipping friction $\phi(\dot{U})$. In standard stick-slip models, $\phi(0)$ is allowed to assume a finite range of values up to some sticking threshold, say $\phi_{\max}$, which might normally be taken to be zero in order that $\phi(\dot{U})$ be continuous at $\dot{U}=0$. For reasons that will become clear immediately, however, we shall use the "high-school" model in which $\phi_{\max}$ is greater than zero, that is, the slipping friction is discontinuously less than the sticking threshold. None of the states of motion that appear to be physically plausible will depend upon this assumption, so the dubious reader may interpret it as a purely mathematical device that is useful for studying certain families of solutions. It is interesting to speculate, on the other hand, that there may be material properties that would produce essentially the same effect, for example, a slipping friction that behaves differently during acceleration than deceleration.

For this class of models, the detailed structure of the cohesive zone, that is, the region of the crack tip where the cohesive force $f(U)$ is nonzero, appears not to be important, at least not for determining steady-state behavior in situations where this continuum description remains valid. To see this, write $U(x,t)=U(x+vt)$ so that (2.1) becomes

$$(1-v^2)U''=m^2(U-\Delta)+2\alpha vU'+f(U) . \tag{2.4}$$

The crack is moving at speed $-v$, and its tip is at $x=0$. Let l be the end of the cohesive zone, that is, $f(U)=0$ for $x>l$. Now multiply (2.4) by U', integrate from 0 to l, and assume that l is very small. [More precisely, let $f(U)\approx\Gamma\delta(U)$.] The result is

$$\tfrac{1}{2}(1-v^2)[U'(0)]^2=\Gamma . \tag{2.5}$$

The cohesive force reduces to a simple boundary condition at the crack tip. Note that we must have $v<1$ in (2.5); the crack must move more slowly than the wave speed.

To determine an actual value for v, we must look in detail at the solutions of (2.4). These have the form, for $x>0$,

$$U(x)=\Delta-e^{q'x}(A^+e^{q''x}+A^-e^{-q''x}) , \tag{2.6}$$

where

$$q'=\frac{\alpha v}{\beta^2} , \quad q''=\frac{1}{\beta^2}(\alpha^2v^2+\beta^2m^2)^{1/2} , \tag{2.7}$$

and $\beta^2\equiv 1-v^2$. The boundary conditions (2.5) and $U(0)=0$ require

$$A^{\pm}=\left[1+\frac{1}{\left[1+\left[\frac{\beta m}{\alpha v}\right]^2\right]^{1/2}}\right]\frac{\Delta}{2}$$

$$\mp\frac{\beta}{\alpha v}\frac{\left[\frac{\Gamma}{2}\right]^{1/2}}{\left[1+\left[\frac{\beta m}{\alpha v}\right]^2\right]^{1/2}} . \qquad (2.8)$$

Suppose that we are dealing with a frictional force $\phi(\dot{U})$ that permits resticking. That is, if $\dot{U}=vU'=0$ at some point behind the tip, say, at $x=x_1$, then ϕ automatically assumes the value needed to keep $\dot{U}=0$ (so long as the slipping threshold is not exceeded) and U remains stuck at its value $U(x_1)$ for all $x>x_1$. Note that, in this case, $\phi(0)=m^2[\Delta-U(x_1)]>0$; thus, we need the "high-school" model. The value of x_1 is determined by

$$e^{-2q''x_1}=\frac{(q''+q')A^+}{(q''-q')A^-}$$

$$=\frac{D-\left[1+\left[\frac{\alpha v}{\beta m}\right]^2\right]^{1/2}-\frac{\alpha v}{\beta m}}{D+\left[1+\left[\frac{\alpha v}{\beta m}\right]^2\right]^{1/2}-\frac{\alpha v}{\beta m}}$$

$$\equiv W(v) , \qquad (2.9)$$

where $D^2\equiv m^2\Delta^2/2\Gamma$ is the ratio of the elastic energy to the fracture energy. Acceptable solutions require that

$$0<W(v)<1 . \qquad (2.10)$$

Consider first the case of positive α. So long as $D>1$, the condition (2.10) is satisfied for all v in the range $0<v<v^*$, where the maximum allowed velocity v^* is given by

$$v^*=\frac{1}{\left[1+\left[\frac{2\alpha D}{m(D^2-1)}\right]^2\right]^{1/2}} . \qquad (2.11)$$

Motion at speed v^* would be the only allowed mode if resticking were not permitted because it corresponds to the condition $A^+=0$, $x_1\rightarrow\infty$, for which $U(x)$ relaxes to Δ as $x\rightarrow\infty$. This is a perfectly satisfactory answer. The crack moves (in the forward direction) only for $\Delta>\Delta_G\equiv(2\Gamma)^{1/2}/m$, which is the analog of the Griffith criterion [7] for this system; and the velocity approaches the wave speed $v^*\rightarrow 1$ in the limit of infinite driving force Δ.

We are left, however, with the first of many unanswered questions to be encountered here: What mode is selected if resticking is allowed? Because all of the resticking modes are slower than v^* for $\alpha>0$, it seems likely that v^* is selected. Any slow, "restuck" mode should be unstable against faster perturbations that would move the tip of the crack out ahead of the sticking point. But this argument is not a proof.

Now consider the case of velocity-weakening friction $\alpha<0$. Examination of (2.9) and (2.10) indicates that the fully relaxed solution ($A^+=0$ for $v<1$) exists only for $D<1$ ($\Delta<\Delta_G$), which is mathematically possible because the friction acts as if it were antidissipative at small values of $\dot{U}$. The associated propagation speed is still v^* as given by (2.11), but v^* is now a decreasing function of the driving force—a strong indication of instability. Moreover, v^* is now the lower, rather than the upper, bound of the range of allowed velocities, which is $v^*<v<1$ for $D<1$ and $0<v<1$ for $D>1$. Note that the upper bound for both cases is the wave speed $v=1$.

For a variety of reasons, it seems likely that the crack with unstable friction $\alpha<0$ accelerates all the way to the wave speed $v\rightarrow 1$ at any driving force Δ. This must be the case if, as argued above, the system naturally selects the fastest mode accessible to it. As $v\rightarrow 1$, $U'(0)$ diverges according to (2.5), and the width of the tip region $(q''-q')^{-1}$ vanishes; but $A^+=0$ and $x_1\rightarrow\infty$ according to (2.8) and (2.9).

This picture of an infinitely sharp rupture front moving at the wave speed is entirely consistent with the results obtained in Ref. [2], in which Tang and I described an analytic and numerical study of the unstable case $\alpha<0$ with zero fracture energy. In order to understand the dynamics of that system, we needed to introduce a short-wavelength cutoff, which we chose to be the grid spacing Δx in a finite-difference approximation. In the case where the system is everywhere at the slipping threshold, we found that the rupture speed approaches the wave speed—from above for $\Gamma=0$—like $(\Delta x)^{2/3}$ as $\Delta x\rightarrow 0$, and that the width of the front also vanishes like $(\Delta x)^{2/3}$. The latter result means that, although the front becomes infinitely sharp in the continuum limit, it contains infinitely many grid points.

For initial states further away from the slipping threshold, we found that propagation speeds are smaller but still approach the wave speed, possibly from below, in the continuum limit. In none of these situations did we need to invoke a resticking friction $\phi(0)$ with physically questionable values greater than zero. The discrete elements, or "blocks," always come to rest at the end of a slipping motion in configurations where the required sticking friction is below the original threshold; but this behavior is strictly dependent on the assumed discreteness of the system. It seems reasonable to guess that something similar happens in the more general situation with nonzero fracture energy; there is no reason to expect anything other than a smooth crossover in taking the two limits $\Delta x\rightarrow 0$ and $\Gamma\rightarrow 0$.

In summary, it appears that the differential equation (2.1) is not, by itself, a mathematically complete statement of the rupture-propagation problem for the case of unstable, velocity-weakening friction. While not complete, the evidence obtained here indicates that any crack described by an initially smooth displacement $U(x,t)$ will accelerate to the wave speed; its tip will contract to a point; and some new, smallest length scale must be invoked in order to determine the dynamics in detail.

III. ONE-DIMENSIONAL MODEL WITH VISCOUS DISSIPATION

Consider now the case where the function $\phi(\dot{U})$ in (2.1) has the form of a Kelvin viscosity in one dimension:

$$\phi(\dot{U}) = -\eta \dot{U}'' \ . \tag{3.1}$$

This situation is intrinsically different from the one considered in the preceding section, as can be seen by looking at the equation for steady-state motion at velocity $-v$; $U = U(x+vt)$:

$$\eta v U''' + \beta^2 U'' - m^2(U-\Delta) = f(U) \ , \tag{3.2}$$

where $\beta^2 = 1 - v^2$ as before. Because viscous dissipation enters as a third derivative, it is a singular perturbation in (3.2). The "energy method" used to derive (2.5) does not work, and we shall not be able to approximate the cohesive force $f(U)$ by a zero-range δ function.

The simple device that makes this intrinsically nonlinear problem into an analytically solvable, piecewise linear one is the assumption

$$f(U) = \begin{cases} f_0 & \text{for } 0 < U \le \delta \\ 0 & \text{for } \delta < U \ , \end{cases} \tag{3.3}$$

so that

$$\Gamma = f_0 \delta \tag{3.4}$$

is the fracture energy. Let the width of the cohesive zone be l; that is, the crack tip is at $x=0$,

$$U(l) = \delta \ , \tag{3.5}$$

and the cohesive force vanishes for $x > l$.

Within the cohesive zone $0 < x < l$ we can write

$$U(x) = \Delta - \frac{f_0}{m^2} + \sum_{j=1}^{3} A_j e^{q_j x} \ , \tag{3.6}$$

where the q_j are solutions of

$$\eta v q_j^3 + \beta^2 q_j^2 - m^2 = 0 \ . \tag{3.7}$$

The coefficients A_j are easily obtained in terms of the q's by requiring that $U(0) = U'(0) = U''(0) = 0$. The result is

$$A_j = \frac{\left[\dfrac{f_0}{m^2} - \Delta\right]}{\mathcal{D}} q_k q_i (q_i - q_k) \ , \tag{3.8}$$

where the determinant $\mathcal{D}$ can be written

$$\mathcal{D} = \sum_j q_j q_k (q_k - q_j) \tag{3.9}$$

where $(j,k,i) = (1,2,3)$ and cyclic permutations.

The structure of (3.7) is such that one of its roots, say q_1, is always real and positive; and the other two, say q_2 and q_3, have negative real parts. With this convention, we know that $\exp(q_1 x)$ must be absent for $x > l$, and therefore

$$U(x) = \Delta - \sum_{j=2}^{3} B_j e^{q_j(x-l)} \ , \quad x > l \ . \tag{3.10}$$

We can evaluate the coefficients B_j in terms of the A_j by matching U' and U'' at $x = l$. Then, by requiring that $U(l) = \delta$ in both (3.6) and (3.10), we obtain

$$\delta = \Delta - \frac{f_0}{m^2} + \sum_{j=1}^{3} A_j e^{q_j l} \tag{3.11}$$

and

$$\delta = \Delta + \frac{q_1(q_2 + q_3 - q_1)}{q_2 q_3} A_1 e^{q_1 l} + \sum_{j=2}^{3} A_j e^{q_j l} \ , \tag{3.12}$$

where the q's and A's are known functions of v via (3.7) and (3.8). These two equations suffice to determine the two unknown quantities v and l as functions of the driving force Δ and other given parameters.

As is fairly obvious, (3.11) and (3.12) cannot be solved as easily for $v(\Delta)$ as was possible previously for the case of frictional dissipation. However, we can obtain most of the important features of the solution without resorting to extensive numerical computations. One useful relationship is obtained by subtracting (3.12) from (3.11) and using the explicit formula (3.8) for A_1. The result is

$$e^{-q_1 l} = 1 - \frac{m^2 \Delta}{f_0} \ . \tag{3.13}$$

The quantity $m^2\Delta / f_0$ is the ratio of the externally applied force to the breaking force, which we generally expect to be very small. (We can assure that this happens by making f_0 large and δ small, as was done in Sec. II.) Thus we expect $q_1 l < 1$, i.e.,

$$q_1 l \simeq \frac{m^2 \Delta}{f_0} \ . \tag{3.14}$$

For completeness, let us look first at the case of very small v. To first order in v, we have

$$\begin{aligned} q_1 &\simeq \frac{m}{\beta}\left[1 - \frac{\eta m v}{2\beta^3}\right] \ , \\ q_2 &\simeq -\frac{m}{\beta}\left[1 + \frac{\eta m v}{2\beta^3}\right] \ , \\ q_3 &\simeq -\frac{\beta^2}{\eta v} \ . \end{aligned} \tag{3.15}$$

To the accuracy needed here, $\beta^2 \simeq 1$; but it will be useful later to recognize that the natural expansion parameter for solutions of (3.7) is $\eta m v / \beta^3$. With $\beta = 1$, (3.11) and (3.12) become

$$\frac{\dfrac{f_0}{m^2} - \Delta + \delta}{\dfrac{f_0}{m^2} - \Delta} \simeq \exp(-\tfrac{1}{2} m^2 \eta v l) \times \left[\cosh(ml) - \frac{\eta m v}{2}\sinh(ml)\right] \ , \tag{3.16}$$

$$\frac{\delta - \Delta}{\dfrac{f_0}{m^2} - \Delta} \simeq -\exp(-\tfrac{1}{2} m^2 \eta v l)\left[1 + \frac{\eta m v}{2}\right]\sinh(ml) \ . \tag{3.17}$$

To simplify these equations, square both sides of (3.16) and (3.17), subtract one from the other, use (3.14) to evaluate l, and then rearrange in the form of a linearized expression for $v(\Delta)$. The result is

$$v \simeq \frac{m(\Delta-\Delta_G)}{\eta f_0} , \tag{3.18}$$

where $m^2\Delta_G^2=2\Gamma$ as before. Thus the behavior near the Griffith threshold Δ_G is unremarkable, the only interesting aspect being the explicit appearance of the breaking force f_0 instead of the combination $f_0\delta=\Gamma$.

The more interesting behavior occurs at larger applied forces Δ and larger v. For values of Δ sufficiently larger than Δ_G and still much smaller than f_0/m^2, we can avoid the complexity of (3.11) and (3.12) by observing that the cohesive zone $0<x<l$ is much smaller than the overall width of the crack tip. That is, $|q_j l|<1$ for all j. Whenever this is true, we can write (3.5) in the form

$$U(l) \simeq \frac{1}{6}U'''(0)l^3 \simeq \frac{f_0 l^3}{6\eta v}=\delta , \tag{3.19}$$

where the next-to-last expression is obtained by reading the value of $U'''(0)$ directly from the original differential equation (3.2) and again assuming that $m^2\Delta<f_0$. We next use (3.14) to eliminate l and write the result in a form that will turn out to be useful in much of the following discussion:

$$\frac{\Delta}{\Delta_G} \simeq \left[\frac{3\Delta_G}{2\delta}\right]^{1/3}\left[\frac{\eta v}{m^2}\right]^{1/3} q_1(v) . \tag{3.20}$$

Because q_1 is a known function of v via (3.7), this is the desired relationship between v and Δ.

At small v, where (3.15) is valid, (3.20) can be written in the form

$$\frac{v}{\beta^3} \approx \frac{\delta(m\Delta)^3}{6\Gamma^2\eta} . \tag{3.21}$$

The range of validity of this formula is determined by the conditions

$$|q_3 l| \simeq \frac{m\Delta\beta^3}{\eta f_0 v} \ll 1 , \tag{3.22}$$

so that (3.19) is valid, and

$$\frac{mv\eta}{\beta^3} \ll 1 , \tag{3.23}$$

so that (3.15) is accurate. If we assume (3.21) to be true self-consistently, then these relations can be read in the form

$$l \ll \frac{\Delta}{\Delta_G} \ll \left[\frac{\Delta_G}{\delta}\right]^{1/3} . \tag{3.24}$$

Note that if δ is a microscopic length and Δ_G is macroscopic (of order m^{-1}), then the upper bound in (3.24) is very large.

Most interestingly, (3.20) determines a function $v(\Delta)$ that goes smoothly through the wave speed $v=1$ and beyond. At precisely $v=1$, the solutions of (3.7) are the complex cube roots of m^2/η, and $q_1=(m^2/\eta)^{1/3}$. From (3.20), we know that the value of Δ at this point, say $\Delta=\Delta_1$, is given by

$$\frac{\Delta_1}{\Delta_G} \simeq \left[\frac{3\Delta_G}{2\delta}\right]^{1/3} . \tag{3.25}$$

There is no indication that $v=1$ ought to be a limiting speed for crack propagation at arbitrary values of m and η.

The crucial question is whether this result implies that cracks actually propagate supersonically in this model. There is no *a priori* reason why this should not happen. For nonzero m, the actual wave speeds at long wavelengths are greater than unity, although group velocities are always smaller. Note that if we keep the applied strain $m\Delta$ constant as $m\to 0$, then Δ_1/Δ_G diverges in this limit, and we recover a situation in which the maximum propagation speed is again unity. Note also that the model is strictly one dimensional—the elastic energy being released by the moving crack is localized along a line—thus we cannot argue as in two dimensions that the rate at which energy is transported from the extended elastic field to the crack tip is limited by the speed at which elastic waves can propagate.

On the other hand, as was clear in Sec. II, the mathematical existence of steady-state solutions of an equation of motion by no means guarantees that such solutions are physically accessible to the system. We already have an indication that something interesting is happening dynamically at high speeds. The fact that q_2 and q_3 become complex $[q_2=q_3^*=(m^2/\eta)^{1/3}(-1+i\sqrt{3})/2$ at $v=1]$ means that the crack-opening displacement in the laboratory frame $U(x,t)$ undergoes underdamped oscillations as the tip passes by. This oscillatory behavior begins when q_2 and q_3 merge at some critical driving force, say $\Delta=\Delta_C$, where

$$q_2=q_3=-2q_1=-\frac{2\beta_C^2}{3\eta v_C} . \tag{3.26}$$

The corresponding propagation speed v_C is given by

$$\frac{\eta m v_C}{\beta_C^3}=\frac{2}{3\sqrt{3}} , \tag{3.27}$$

where $\beta_C^2=1-v_C^2$. For small m,

$$\frac{\Delta_C}{\Delta_G}=\left[\frac{3\Delta_G}{8\delta}\right]^{1/3} . \tag{3.28}$$

As Δ increases from Δ_c through Δ_1 and beyond, the ratio of $\mathrm{Im}q_2$ to $|\mathrm{Re}q_2|$ grows so that the damping of the oscillations becomes weaker and weaker. The interesting question, of course, is whether the appearance of such oscillations in the steady-state solution means that the actual behavior of the system might involve some periodic or even irregular motion of the tip itself.

IV. TWO-DIMENSIONAL MODEL WITH VISCOUS DISSIPATION

There are at least two respects in which one-dimensional models could be seriously and, perhaps, fatally unrealistic. We already have noted that an essential ingredient in fracture dynamics must be the mechanism by which elastic energy is transported to the crack tip, and that this mechanism cannot be represented adequately in one dimension. A second apparent difficulty in using one-dimensional models is their inherent inability to provide accurate descriptions of stress concentrations near crack tips. I therefore want to include in this paper some analysis of a two-dimensional model as a first test of whether the one-dimensional phenomena discussed in the preceding sections can occur in more realistic situations. The preliminary indication is that the one-dimensional picture is more complete than might have been expected, but remember that we are not testing truly dynamic behavior in any of these calculations.

The mathematical analysis which follows is closely parallel to that of Ref. [5], hereafter referred to as BDL, in which we studied steady-state propagation of a crack along the center of a viscoelastic strip of finite width. In fact, there is an even closer relationship between the present calculation and that of Willis [6], who also looked at steady-state propagation in a viscoelastic medium and, as we shall do here, simplified the calculation by looking at a mode-III crack where the equations of motion for linear elasticity reduce to a single scalar wave equation. There are (at least) two important differences, however. Willis assumed that the tractions on the crack faces are predetermined and fixed in the moving frame whereas, in the present case, those tractions are determined by the load applied externally to the system as a whole. On the other hand, Willis considered more realistic constitutive laws than will be discussed here.

Consider a semi-infinite material occupying the upper half of the x-y plane ($y>0$), and suppose that a crack or fault lies along the x axis. The material obeys a scalar wave equation of the form

$$\ddot{U}=\nabla^2 U - m^2(U-\Delta)+\eta\nabla^2\dot{U} \ . \tag{4.1}$$

As suggested above, $U(x,y,t)$ may most conveniently be visualized as a displacement normal to the plane, corresponding to fracture in mode III; but other interpretations are almost equally plausible. We include the mass m in (4.1) as a device that allows us to consider a finite applied strain without dealing explicitly with the outer boundaries of the system. This too is an important difference between Willis's model and ours.

Along the x axis, the stress σ,

$$\sigma\{U(x,0,t)\}=\left[1+\eta\frac{\partial}{\partial t}\right]\frac{\partial U}{\partial y}\bigg|_{y=0} , \tag{4.2}$$

must be balanced by the traction which, in general, may consist of both a cohesive stress $F\{U(x,0,t)\}$ and a frictional stress $\Phi\{\dot{U}(x,0,t)\}$. Note that F and Φ have the dimensions of stress as opposed to f and ϕ, which are forces. For present purposes, we consider only the cohesive stress F, but I hope in later applications to look at stick-slip friction as well in this two-dimensional class of models. Note that the viscous dissipation, that is, the term proportional to η in (4.1), occurs throughout the material and is not confined to the fracture surface as it was in the one-dimensional models. The viscous damping in (4.1) dissipates any energy that may be radiated from the crack tip and thus removes the need to consider far-field boundary conditions.

In keeping with the rest of the discussion in this paper, we look at steady-state solutions of (4.1) in the form $U=U(x+vt,y)$. From here on, as before, we replace $x+vt$ by x.

We start by writing $U(x,y)$ in the form

$$U(x,y)=\Delta(1-e^{-my}) + \int_{-\infty}^{\infty}\frac{dk}{2\pi}\exp[ikx-\hat{K}(k)y]\hat{U}(k) \ . \tag{4.3}$$

The first term on the right-hand side is the uniformly stressed solution of (4.1) in the absence of a crack, and the second term is the perturbation caused by the crack. From (4.1) we know that

$$\hat{K}(k)=\left[\frac{m^2+\beta^2k^2+i\eta v k^3}{1+i\eta v k}\right]^{1/2} , \quad \mathrm{Re}\hat{K}>0 \ . \tag{4.4}$$

The Fourier amplitude $\hat{U}(k)$ is determined completely by the value of $U(x,y)$ on the boundary $y=0$:

$$\hat{U}(k)=\int_0^{\infty}dx\ U(x,0)e^{-ikx} , \tag{4.5}$$

where the lower limit of integration accounts for the condition that $U(x,0)=0$ for $x<0$. Then the boundary condition (4.2) for the stress at $y=0$ becomes an integral equation for $U(x,0)$:

$$-\left[1+\eta v\frac{\partial}{\partial x}\right]\int_0^{\infty}dx' K(x-x')U(x',0) = \begin{cases} F\{U(x,0)\}-m\Delta , & x>0 \\ G(x) , & x<0 , \end{cases} \tag{4.6}$$

where $G(x)$ is the as yet undetermined excess stress ahead of the crack tip, and

$$K(x)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{K}(k)e^{ikx} \ . \tag{4.7}$$

Equation (4.6) has exactly the same mathematical structure as (2.22) or (3.6) in BDL. In fact, at $v=0$, $\hat{K}=(k^2+m^2)^{1/2}$, which is the same as the approximation used in BDL for the full elastic kernel with m being inversely proportional to the width of the strip. BDL is strictly a low-velocity theory—inertial effects are neglected entirely—and thus we expect the present calculation to recover the results of BDL in the limit of small v.

As in BDL, we solve (4.6) by Wiener-Hopf methods. That is, we compute its Fourier transform and rewrite the result in such a way that terms which are analytic in the upper and lower halves of the complex k plane appear on opposite sides of the equation. The first step in this procedure is to write the kernel $\hat{K}$ in the form $\hat{K}^{(-)}/\hat{K}^{(+)}$ where the superscripts (+) and (−) mean that the corresponding functions are analytic in the upper and lower half planes, respectively. Comparing (4.4) with (3.7), we write

$$\hat{K}^{(+)}(k)=\frac{1}{(q_1-ik)^{1/2}}\,,\qquad \hat{K}^{(-)}(k)=\frac{(ik-q_2)^{1/2}(ik-q_3)^{1/2}}{\left[ik+\frac{1}{\eta v}\right]^{1/2}}\,. \tag{4.8}$$

The q's are the roots of the same cubic equation that appeared in the one-dimensional model in Sec. III. Then (4.6) becomes

$$ik\hat{K}^{(-)}(k)\hat{W}^{(-)}(k)+ik\hat{\Lambda}^{(-)}(k)=\hat{K}^{(+)}(k)[m\Delta-ik\hat{G}^{(+)}(k)]-ik\hat{\Lambda}^{(+)}(k)\,, \tag{4.9}$$

where

$$\hat{\Lambda}^{(\pm)}(k)=\pm\int_{C^{(\pm)}}\frac{dk'}{2\pi i}\frac{\hat{K}^{(+)}(k')\hat{F}^{(-)}(k')}{(k'-k)}\,. \tag{4.10}$$

Here, $\hat{G}^{(+)}(k)$ and $\hat{F}^{(-)}(k)$ are the Fourier transforms of the stresses $G(x)$ and $F\{U(x,0)\}$, respectively; the contours $C^{(+)}$ and $C^{(-)}$ go from $-\infty$ to $+\infty$ in the k' plane passing, respectively, below and above the pole at $k'=k$; and $\hat{W}^{(-)}(k)$ is the Fourier transform of

$$W(x)=\left[1+\eta v\frac{\partial}{\partial x}\right]U(x,0)\,. \tag{4.11}$$

Both sides of (4.9) must be equal to the same constant which, by looking at the limit $k\to 0$, we deduce must be $m\Delta\hat{K}^{(+)}(0)=m\Delta/(q_1)^{1/2}$. Thus

$$ik\hat{W}^{(-)}(k)=\frac{1}{\hat{K}^{(-)}(k)}[m\Delta\hat{K}^{(-)}(0)-ik\hat{\Lambda}^{(-)}(k)]\,, \tag{4.12}$$

$$ik\hat{G}^{(+)}(k)=m\Delta\left[1-\frac{\hat{K}^{(+)}(0)}{\hat{K}^{(+)}(k)}\right]-ik\left[\frac{\hat{\Lambda}^{(+)}(k)}{\hat{K}^{(+)}(k)}\right]\,. \tag{4.13}$$

Evaluation of these formulas requires more algebra than is useful to exhibit here. From (4.13), I obtain

$$G(-|x|)=\frac{m\Delta}{2(\pi q_1)^{1/2}}\int_{|x|}^{\infty}dw\frac{e^{-q_1 w}}{w^{3/2}}-\frac{1}{\pi}\int_0^l dx'e^{-q_1(|x|+x')}\left[\left[\frac{x'}{|x|}\right]^{1/2}\left[\frac{1}{|x|+x'}\right]\right]F(U(x'))\,. \tag{4.14}$$

Both terms in (4.14) diverge like $|x|^{-1/2}$. These divergences cancel if

$$\frac{m\Delta}{(\pi q_1)^{1/2}}=\frac{1}{\pi}\int_0^l dx\,e^{-q_1 x}\frac{1}{x^{1/2}}F\{U(x)\}\,, \tag{4.15}$$

which is the Barenblatt [8] condition for the length l of the cohesive zone. In analogy to (3.3), we can approximate the cohesive stress F by a constant F_0 for displacements U less than δ. If we further assume that $q_1 l<1$, then (4.15) becomes

$$q_1 l\simeq\frac{\pi}{4}\left[\frac{m\Delta}{F_0}\right]^2\,. \tag{4.16}$$

Note the qualitative similarity between (4.16) and (3.14). In both cases, the quantity $q_1 l$ is proportional to the ratio of the applied stress to the yield stress, raised to some dimension-dependent power.

Evaluation of (4.12) requires even more work than (4.13). For present purposes, it will be sufficient simply to present the result of this calculation for the case of the constant cohesive stress used above. I find

$$\frac{dW}{dx}=\frac{F_0}{\sqrt{\pi}}\int_{\lambda(x)}^{x}dx'\frac{\psi(x')}{(x'-x+l)^{1/2}}\,,\qquad \lambda(x)=\begin{cases}0, & x<l\\ x-l, & x>l\,,\end{cases} \tag{4.17}$$

where

$$\psi(x)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\left[\frac{\left[ik+\frac{1}{\eta v}\right]^{1/2}}{(ik-q_2)^{1/2}(ik-q_3)^{1/2}}\right]\,. \tag{4.18}$$

Note that the singularities of the term in square brackets in the integrand all lie in the upper half k plane. From here, it is a straightforward exercise to invert (4.11) to compute $U(x)$, and then to set $U(l)=\delta$ in order to compute v. The result is

$$U(l)=\frac{2F_0 l^{3/2}}{\sqrt{\pi}}\int_0^1 ds\,\psi(sl)\left[1-\exp\left[-\frac{l}{\eta v}(1-s)\right]-\frac{l}{\eta v}\int_s^1 dt\,t^{1/2}\exp\left[\frac{l}{\eta v}(s-t)\right]\right]=\delta\,. \tag{4.19}$$

As a first check on this formula, look at the limit $v \to 0$. We have $q_1 \approx m$, $q_2 \approx -m$, $q_3 \approx -1/(\eta v)$, $\psi \approx \exp(-mx)/(\pi x)^{1/2}$, and $U(l) \approx 2F_0 l/\pi$. Combining these limiting formulas with (4.16) to eliminate l, we find that the two-dimensional Griffith threshold occurs at

$$\Delta = \Delta_{G2} = \left[\frac{2F_0\delta}{m} \right]^{1/2} = \left[\frac{2\Gamma}{m} \right]^{1/2} \tag{4.20}$$

as expected. A further comparison with BDL can be made by looking at velocities large enough that $v \gg l/\eta$ but still small enough that $\beta \simeq 1$. Then $U(l) \approx F_0 l^2/(5\pi\eta v)$ and $v \sim \Delta^4$ as before. However, the range of validity of this approximation—if any—depends on the choice of the system parameters.

The more interesting behavior occurs at large driving force Δ when the velocity v becomes of order unity. As in Sec. III, all three roots q_j are of roughly the same order of magnitude for this range of v's, and $|q_j l| < 1$ for all j. The dominant contribution to $\psi(x)$ in (4.18) comes from the pair of branch points in the upper half k plane at $k = -iq_2$ and $-iq_3$, which merge at, say, $\Delta = \Delta_{C2}$ and move symmetrically off the imaginary k axis for $\Delta < \Delta_{C2}$. Thus

$$\psi(x) \approx \frac{1}{(\eta v)^{1/2}} e^{-\tilde{q}x} , \tag{4.21}$$

where $\tilde{q} \simeq \mathrm{Re}q_2 \simeq \mathrm{Re}q_3$. For use in (4.19), where we need arguments of ψ only in the range $(0, l)$, we can neglect the exponential x dependence of ψ altogether. We can also assume, and immediately check selfconsistently, that $l < \eta v$. In this limit, (4.19) becomes

$$U(l) \simeq \frac{F_0 l^{5/2}}{5\sqrt{\pi}(\eta v)^{3/2}} = \delta , \tag{4.22}$$

or, equivalently,

$$\frac{l}{\eta v} \simeq \left[\frac{5\sqrt{\pi}\delta}{\eta v F_0} \right]^{2/5} , \tag{4.23}$$

which we can make as small as we like by choosing δ small and F_0 large.

Just as (4.16) is the two-dimensional analog of (3.14), so (4.22) is the analog of (3.19). Combining (4.16) and (4.22) to eliminate l, we obtain a relation between Δ and v that can be written in a form analogous to (3.20):

$$\frac{\Delta}{\Delta_{G2}} \simeq \left[\frac{10}{\pi^2} \right]^{1/5} \left[\frac{\Delta_{G2}}{\delta} \right]^{3/5} \frac{(\eta v)^{3/10}}{m^{1/5}} [q_1(v)]^{1/2} . \tag{4.24}$$

For the special case $v = 1$, we have $q_1 = (m^2/\eta)^{1/3}$ and, therefore,

$$\frac{\Delta_{12}}{\Delta_{G2}} = \left[\frac{10}{\pi^2} \right]^{1/5} \left[\frac{\Delta_{G2}}{\delta} \right]^{3/5} (\eta m)^{2/15} , \tag{4.25}$$

where Δ_{12} is the two-dimensional analog of Δ_1 as given in (3.25). One qualitative difference between (3.25) and (4.25) is the explicit appearance of η on the right-hand side of the latter equation. In two dimensions, the excess stress required to bring the crack up to the wave speed grows with increasing viscosity. Another such difference is the dependence on the length scale m^{-1}. In one dimension, the stress $m\Delta_1$ diverges for small m like $m^{-1/3}$ whereas, in two dimensions, $m\Delta_{12} \sim m^{1/3}$ becomes small in the same limit.

An expression similar to (4.25) can be obtained for Δ_{C2}, the analog of Δ_C in (3.28), the only difference being that the numerical prefactor in (4.25) is reduced by a factor $1/2^{1/3}$. Moreover, q_1 in (4.24) is a monotonically increasing function of v (for large enough v). Thus, once again, nothing special happens at $v = 1$. The function $v(\Delta)$ rises smoothly and monotonically through the wave speed. For $\Delta > \Delta_{C2}$, the branch points in the integrand in (4.18) no longer lie on the imaginary k axis, so that $\psi(x)$ has an oscillatory part. According to (4.17), these oscillations must also be present in $U(x)$ but, because of the several smoothing integrations implied by (4.11) and (4.17) in going from ψ to U, they will not be so visible as in the one-dimensional case.

Finally, a remark is in order regarding the stress near the crack tip. The left-hand side of (4.15) is the stress-intensity factor, usually denoted by the symbol K_{III} for a mode-III crack. K_{III} is the coefficient of the term in $G(-|x|)$ which appears to diverge like $|x|^{-1/2}$ when observed at values of $|x|$ much larger than l, the size of the cohesive zone, and much smaller than the macroscopic length scale m^{-1}. If, as is sometimes done in the literature, we think of K_{III} as having been computed for fixed applied stress $m\Delta$ and arbitrary propagation speed v, then we find that it decreases with increasing v. For v (or η) small enough that the leading term in (3.15) is accurate, $K_{\mathrm{III}} \sim \sqrt{\beta}$. This ostensibly paradoxial behavior is not indicative of an instability of the kind we found for velocity-weakening friction in Sec. II. The relation between v, η, and Δ must necessarily be taken into account in order to discuss the dynamics of this system. For example, in the interesting regime in which (4.24) is valid, $K_{\mathrm{III}} \sim (\eta v)^{3/10}$, which is an increasing function of both the propagation speed v and the dissipation strength η.

V. SUMMARY AND CONCLUSIONS

The following brief summary of the principal conclusions reached in this paper includes some remarks and speculations about future directions for inquiry.

(i) One-dimensional model with unstable stick-slip friction. The most important feature of this class of models, which includes the simple model of an earthquake fault discussed in earlier papers [2,3], is that the dynamic instability associated with velocity-weakening friction destabilizes the crack tip and causes it to accelerate toward the wave speed. As presented here, however, the continuum description of these models is incomplete. In order to determine the actual speed of propagation and the detailed structure of the crack tip, one must introduce some additional dynamic mechanism, generally involving a new, small-length scale, that acts to control the instability. We know (from the previous work) that the stabilized steady-state motion consists of a propagating narrow pulse whose width is determined by the new length scale.

This behavior appears to be closely similar to the "self-healing pulse of slip in earthquake rupture" proposed recently by Heaton [9]. It will be interesting to explore that possibility in future studies.

The analyses presented here include no detailed investigation of the tip instability, and the conclusions about the dynamically selected propagation mode are based on earlier analysis of the earthquake model in which fracture energy was neglected. Moreover, the cohesive force near the crack tip, which gives rise to the fracture energy, has so far been taken to be infinitesimally short ranged for use in the friction model. Thus a number of technical details remain to be sorted out. The most interesting outstanding questions, however, pertain to the physical nature of the required new length scale. (A finite-ranged cohesive force does not suffice because it does not provide the necessary stabilization.) There are a number of possibilities, most of them being more realistic than the *ad hoc* introduction of a lattice spacing as in Ref. [2]. Clearly, a more detailed understanding of this mechanism would be necessary in order to argue that this propagation mode is a "Heaton pulse."

(ii) One-dimensional model with viscous dissipation. The simple Kelvin viscosity used in this class of models is a singular perturbation which forces the use of a more complete description of the Barenblatt cohesive zone than was needed in the models with frictional dissipation. At large driving forces and correspondingly high propagation speeds, the material within the cohesive zone acts as if it were a fluid; only the viscous force is operative in balancing the cohesive force in this region. [See Eq. (3.19) and the discussion surrounding it.] The elastic force plays no role and, as a result, the propagation speed becomes greater than the elastic wave speed at a finite value of the applied force. Interestingly, the steady-state crack-opening displacement acquires an oscillatory component at high velocities.

One obvious question is whether a more realistic constitutive relation than the Kelvin viscosity might produce a qualitative change in this behavior. For example, a creep compliance that exhibits some instantaneous elasticity as well as slow viscous response to a sudden change in the stress would seem likely to produce a bound on the propagation speed for large driving forces. Another question concerns the oscillations. Nothing in the present analysis indicates that there is any instability associated with this phenomenon, and preliminary numerical analyses [10], seem to confirm that these are stable and accessible propagation modes. Nevertheless, the oscillations could play an important role at high propagation speeds in a variety of more complicated and realistic situations.

(iii) Two-dimensional model with viscosity. The principal reason for looking at the two-dimensional model in the present context was to learn whether dimensionality is crucial for the qualitative dynamic behavior seen in the one-dimensional cases. For the models with purely viscous dissipation, it apparently is not. Power laws in the relations between velocity and driving force are dimension dependent, and even the dependence on viscosity and sample size are different in two dimensions than in one; but the basic phenomena remain closely analogous in the two cases. A particularly interesting question, specially relevant for earthquake dynamics, is whether dimensionality might make more important differences in the models with unstable stick-slip friction. It should also be useful to explore the role of dimensionality in connection with more realistic constitutive laws.

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