Exact solution to the time-dependent Schrödinger equation in two dimensions

R. S. Kaushal
Department of Physics and Astrophysics, University of Delhi, Delhi 110 007, India
(Received 25 February 1992)

With a view to obtaining an exact analytic solution to the time-dependent Schrödinger wave equation for noncentral harmonic and anharmonic potentials in two dimensions, we make use of an ansatz for the eigenfunction in conjunction with the methods of Burgan *et al.* [Phys. Lett. **74A**, 11 (1979)] and Ray [Phys. Rev. A **26**, 729 (1982)]. While the method is found to work well for a restricted class of harmonic potentials, its limitations in the case of anharmonic potentials are pointed out.

PACS number(s): 03.65.Ge, 03.65.Ca, 03.20.+i

I. INTRODUCTION

In recent years the study of dynamical systems involving an explicit time dependence has attracted great interest from the point of view of both classical and quantum mechanics, particularly because such studies can prove to be useful in various branches of physics [1] and chemistry [2]. While at the classical level the studies are mainly restricted [3,4,5] to the construction of invariants (since the Hamiltonian in this case does not remain a constant of motion) and their possible physical interpretations [6], attempts have been made [7,8,9,] to use these results in obtaining exact analytic solutions to the Schrödinger wave equation at the quantum level. In other words, the invariants, if they exist and become available for a given time-dependent (TD) system, not only help in solving the equations of motion in the classical case, but also offer a deeper understanding of the solutions to the Schrödinger equation (SE).

One of the various approaches [3,4,10,11] used to investigate TD systems is the Lie-algebraic approach [10,11]. This approach, in terms of time-evolution operators introduced [5] for the TDSE, although offering a deeper mathematical insight into the problem, becomes rather difficult to deal with, especially with either Hamiltonians of order higher than quadratic or of higher dimensions. On the other hand, the method of Ray [9], a generalization of the group-transformation method of Burgan *et al.* [7], can offer an exact solution to the problem not only in higher dimensions, but also for nonquadratic Hamiltonians.

The method of Ray [9] is essentially carried out in two stages. In the first stage one performs a scale and a phase transformation of the dependent variable and a scale transformation of the independent space-time variables, thereby converting the TDSE to a more complicated form. The arbitrary functions occurring in the transformation are then fixed by setting some of the additional terms in this new equation equal to zero and subsequently by demanding the form of the TDSE to be invariant under the above transformation. This is done by modifying the potential term. In the second stage, another phase transformation of the dependent variable converts this new TDSE into a time-independent (TID) SE in one of the standard forms whose exact solutions are normally known in advance. It is interesting to note that the Hamiltonian analog of this final TIDSE turns out to be a constant of motion, which, in turn, is found [9] to have a connection with the corresponding classical Noether invariants.

Recently, we have studied [12] the quantum mechanics of noncentral TID potentials in two dimensions. In particular, the ansatz used for the eigenfunction has provided an exact solution to the TIDSE for a large variety of noncentral potentials. Coincidentally, our ansatz in its generalized form resembles that of Ray [9] or of Burgan *et al.* [7]. With a view to enlarging the list of exactly solvable TD potentials in quantum mechanics, we return to the method of Ray here, particularly with reference to the two-dimensional systems. Burgan *et al.*, however, investigated a similar problem as a special case of their studies of a multidimensional TD harmonic oscillator, but with a limited utility in the sense that they used a restricted form of the transformation. In the present work, we generalize the method of Ray to investigate a class of TD systems in two dimensions, from which the results of Burgan *et al.* can be recovered as a special case. In the next section, we employ the method to study the TD system $V(x,y,t)$. As an example, the case of a shifted TD harmonic oscillator is discussed in Sec. III. In Sec. IV, we attempt to apply the method to noncentral anharmonic TD potentials and subsequently discuss the possible limitations of the method. The results are discussed and summarized in Sec. V.

II. GENERAL TREATMENT

For the system $V(x,y,t)$ we wish to solve the TD Schrödinger equation ($\hbar=\mu=1$),

$$\left[-\frac{1}{2}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]+V(x,y,t)\right]\Psi(x,y,t)=i\frac{\partial\Psi}{\partial t}\ , \quad (1)$$

where for the function $\Psi(x,y,t)$, we make an ansatz [7,9],

$$\Psi(x,y,t)=B(t)\exp[i\phi(x,y,t)]\psi(x,y,t)\ , \quad (2)$$

with $B(t)$ a TD normalization. Now, following Ray [9], we perform the scale transformation of the space and time variables as

$$x'=\frac{x}{C_1(t)}+A_1(t),\quad y'=\frac{y}{C_2(t)}+A_2(t),\quad t'=D(t)\ . \quad (3)$$

Using the ansatz (2) and the transformation (3) in (1), one obtains

$$-\frac{1}{2}\left[iB\phi_{xx}\psi-B\phi_x^2\psi+2i\frac{B}{C_1}\phi_x\psi_{x'}+\frac{B}{C_1^2}\psi_{x'x'}+iB\phi_{yy}\psi-B\phi_y^2\psi+2i\frac{B}{C_2}\phi_y\psi_{y'}+\frac{B}{C_2^2}\psi_{y'y'}\right]+VB\psi$$
$$=i\dot{B}\psi-B\psi_t\psi+iB\dot{D}\psi_{t'}+iB\left[-x\frac{\dot{C}_1}{C_1^2}+\dot{A}_1\right]\psi_{x'}+iB\left[-y\frac{\dot{C}_2}{C_2^2}+\dot{A}_2\right]\psi_{y'} , \quad (4)$$

where various notations are the same as Ray's [9], except that A_i, C_i $(i=1,2)$ are now introduced corresponding to two space dimensions, thus increasing the number of TD arbitrary functions. Since we wish to retain [9] the form invariance of the Schrödinger equation (1) under the transformation (3), we compare the coefficients of $\psi_{x'}$ and $\psi_{y'}$, on either side of Eq. (4), implying two expressions for the arbitrary function $\phi(x,y,t)$:

$$\phi=\frac{1}{2}\frac{\dot{C}_1}{C_1}x^2-\dot{A}_1C_1x+\sigma_1(y,t) ; \quad \phi=\frac{1}{2}\frac{\dot{C}_2}{C_2}y^2-\dot{A}_2C_2y+\sigma_2(x,t) ,$$

where the arbitrary functions of integration σ_1 and σ_2 can be conveniently chosen to provide a unique expression for ϕ as

$$\phi(x,y,t)=\frac{1}{2}\left[\frac{\dot{C}_1}{C_1}x^2+\frac{\dot{C}_2}{C_2}y^2\right]-(\dot{A}_1C_1x+\dot{A}_2C_2y) . \quad (5)$$

Further use of this expression in (4) and subsequent rearrangement of various terms lead to

$$-\frac{1}{2}\left[\frac{C_2}{C_1}\psi_{x'x'}+\frac{C_1}{C_2}\psi_{y'y'}\right]+\left[VC_1C_2+\frac{1}{2}C_2\ddot{C}_1x^2+\frac{1}{2}C_1\ddot{C}_2y^2-C_1C_2(2\dot{A}_1\dot{C}_1+\ddot{A}_1C_1)x-C_1C_2(2\dot{A}_2\dot{C}_2+2\ddot{A}_2C_2)y\right.$$
$$\left.-\frac{i}{2}(\dot{C}_1C_2+C_1\dot{C}_2)+\frac{1}{2}C_1C_2(\dot{A}_1^2C_1^2+\dot{A}_2^2C_2^2)-i\frac{\dot{B}}{B}C_1C_2\right]\psi=i\dot{D}C_1C_2\psi_t . \quad (6)$$

To ensure this equation again represents the standard TDSE in terms of primed space-time coordinates (i.e., the form of the SE remains invariant) along with a modified real potential V', we should have

$$\frac{C_1}{C_2}=\frac{C_2}{C_1}=1, \quad \dot{D}C_1C_2=1, \quad \frac{\dot{B}}{B}=-\frac{\dot{C}_1C_2+C_1\dot{C}_2}{2C_1C_2} ,$$

which, in turn, imply, say,

$$C_1(t)=C_2(t)=C(t) , \quad (7a)$$

and subsequently

$$t'=D(t)=\int dt\frac{1}{C^2(t)} , \quad (7b)$$

$$B(t)=1/C(t) , \quad (7c)$$

$$\phi(x,y,t)=\frac{1}{2}\frac{\dot{C}}{C}(x^2+y^2)-C(\dot{A}_1x+\dot{A}_2y) . \quad (7d)$$

Note the difference in the expression for $B(t)$ as compared to that of Ray [9]. Finally, Eq. (6) reduces to the form

$$-\tfrac{1}{2}(\psi_{x'x'}+\psi_{y'y'})+V'(x',y',t')\psi=i\psi_{t'} , \quad (8)$$

where the potential V' is now given by

$$V'=VC^2+\tfrac{1}{2}C\ddot{C}(x^2+y^2)-C^2(\ddot{A}_1C+2\dot{A}_1\dot{C})x$$
$$-C^2(\ddot{A}_2C+2\dot{A}_2\dot{C})y+\tfrac{1}{2}C^4(\dot{A}_1^2+\dot{A}_2^2) . \quad (9)$$

By setting $y=0$ and $A_2(t)=0$ in (3) and (9), one can easily recover the results of Ref. [9] for one dimension. If, in addition, $A_1(t)=0$, then one can arrive at the results of Burgan *et al.* [7]. In the next section, we use these results to solve the TDSE for a shifted rotating harmonic-oscillator potential in two dimensions.

III. EXAMPLE: $V(x,y,t)=b_{20}x^2+b_{02}y^2+b_{10}x+b_{01}y+b_0$

Here we apply the results obtained in Sec. II to the potential

$$V(x,y,t)=b_{20}(t)x^2+b_{02}(t)y^2$$
$$+b_{10}(t)x+b_{01}(t)y+b_0(t) , \quad (10)$$

which is a case of a shifted rotating harmonic oscillator in two dimensions. Using the inverse of the transformation (3) in (10), the potential $V'(x',y',t')$ can be computed from (9) as

$$V'=C^3(b_{20}C+\tfrac{1}{2}\ddot{C})x'^2+C^3(b_{02}C+\tfrac{1}{2}\ddot{C})y'^2$$
$$+C^3[(b_{10}-\ddot{A}_1C-2\dot{A}_1\dot{C})-2A_1(b_{20}C+\tfrac{1}{2}\ddot{C})]x'$$
$$+C^3[(b_{01}-\ddot{A}_2C-2\dot{A}_2\dot{C})$$
$$-2A_2(b_{02}C+\tfrac{1}{2}\ddot{C})]y'+F(t') , \quad (11)$$

with

$$F(t')=A_1C^3[A_1(b_{20}C+\tfrac{1}{2}\ddot{C})-(b_{10}-\ddot{A}_1C-2\dot{A}_1\dot{C})]$$
$$+A_2C^3[A_2(b_{02}C+\tfrac{1}{2}\ddot{C})-(b_{01}-\ddot{A}_2C-2\dot{A}_2\dot{C})]$$
$$+b_0C^2+\tfrac{1}{2}C^4(\dot{A}_1^2+\dot{A}_2^2) . \quad (12)$$

For the arbitrary functions $C(t)$, $A_1(t)$, and $A_2(t)$, we make the following choices: Let $C(t)$ satisfy

$$\ddot{C}+2b_{20}C=\frac{k_1}{C^3},\quad \ddot{C}+2b_{02}C=\frac{k_2}{C^3}, \tag{13a}$$

where k_1 and k_2 are arbitrary constants related to the potential parameters b_{20} and b_{02} by

$$2(b_{20}-b_{02})C^4=k_1-k_2 . \tag{13b}$$

In other words, Eq. (13b) determines the function $C(t)$ in terms of the potential parameters and the constants k_1 and k_2. Note that for the case when $b_{20}=b_{02}$, one needs only one arbitrary constant k $(=k_1=k_2)$. For setting the arbitrary functions $A_1(t)$ and $A_2(t)$, we equate to zero the coefficients of x' and y' in Eq. (11), leading to

$$\ddot{A}_1+\frac{2\dot{A}_1\dot{C}}{C}+\frac{A_1k_1}{C^4}-\frac{b_{10}}{C}=0 , \tag{14a}$$

$$\ddot{A}_2+\frac{2\dot{A}_2\dot{C}}{C}+\frac{A_2k_2}{C^4}-\frac{b_{01}}{C}=0 . \tag{14b}$$

As a result of using Eqs. (13) and (14) in expression (11) for V', the latter takes the form

$$V'=\tfrac{1}{2}k_1x'^2+\tfrac{1}{2}k_2y'^2+F(t') , \tag{15}$$

with

$$F(t')=-\tfrac{1}{2}k_1A_1^2-\tfrac{1}{2}k_2A_2^2+\tfrac{1}{2}C^4(\dot{A}_1^2+\dot{A}_2^2)+b_0C^2 .$$

The TDSE to be solved now becomes

$$-\tfrac{1}{2}(\psi_{1x'x'}+\psi_{1y'y'})+\tfrac{1}{2}(k_1x'^2+k_2y'^2)\psi_1=i\psi_{1t} , \tag{16}$$

where ψ_1 is obtained from ψ through a phase change [9] as

$$\psi(x',y',t')=\exp\left[-\int^{t'}F(\tau)d\tau\right]\psi_1(x',y',t') . \tag{17}$$

For $k_1,k_2>0$, Eq. (16) can be realized as a TIDSE for a TID harmonic-oscillator potential in two dimensions. As a special case, the free-particle problem in two dimensions corresponding to $k_1=k_2=0$ can be discussed in the same way as Hartley and Ray [8].

As in Ref. [9], if we define the operator

$$I'=-\frac{1}{2}\left[\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}\right]+\frac{1}{2}k_1x'^2+\frac{1}{2}k_2y'^2 , \tag{18}$$

then the general solution to Eq. (16) can be written as

$$\psi_1(x',y',t')=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_{mn}e^{-i(\lambda_m+\lambda_n)t'}u_m(x')u_n(y') , \tag{19}$$

with t' given by (7b). Here c_{mn} are constants determined from

$$c_{mn}=\langle u_m(x')u_n(y'),\ \psi_1(x',y',0)\rangle ,$$

and λ_m and λ_n are given by

$$\lambda_m=(m+\tfrac{1}{2})\sqrt{k_1},\quad \lambda_n=(n+\tfrac{1}{2})\sqrt{k_2} . \tag{20}$$

Finally, the exact solution to Eq. (1) for the potential (10) can be written as

$$\Psi(x,y,t)=\frac{1}{C}\exp\left[-i\int^{(t')}F(\tau)d\tau\right]\times\exp\left[\frac{i}{2C}[\dot{C}(x^2+y^2)-C^2(\dot{A}_1x+\dot{A}_2y)]\right]\times\sum_m\sum_n\exp\left[-i(\lambda_m+\lambda_n)\int\frac{dt}{C^2}\right]\times u_m(x/C+A_1)u_n(y/C+A_2) , \tag{21}$$

where $C(t)$, $A_1(t)$, $A_2(t)$, and $F(t)$, respectively, can be obtained from Eqs. (13b), (14a), (14b), and (12). The functions u_m and u_n in (21) turn out to be the Hermite polynomials computed at the points $(x/C+A_1)$ and $(y/C+A_2)$, respectively.

It may be mentioned that if we have a coupling term of the type $b_{11}xy$ present in the potential (10), then a term of the type $C^4b_{11}x'y'$ also appears finally in the potential V' [cf. Eq. (15)] and hence does not allow the TDSE to reduce to a TID form like (16) even after carrying out the phase transformation (17). No doubt the transformations (2), (3), and (17) have converted the TD problem to a TID one, but only for a limited class of harmonic potentials. For example, if the inverse harmonic and/or coupling terms are also present in the harmonic oscillator, then the present method fails. Also, the method does not allow the inclusion of anharmonic terms in the potential. We highlight some of these difficulties encountered with the method in the next section.

IV. CASE OF ANHARMONIC POTENTIALS

In this section, we demonstrate the limitations of the present method with reference to anharmonic potentials. In particular, we consider the solution to the TDSE for a potential with quartic type anharmonicity, namely

$$V(x,y,t)=b_{40}x^4+b_{04}y^4+b_{22}x^2y^2+b_{20}x^2+b_{02}y^2+b_{11}xy+b_{10}x+b_{01}y+b_0 , \tag{22}$$

where b_0 and the b_{ij}'s are functions of t. In this case the potential V' from (9), after using the inverse of transformation (3), becomes

$$V'(x',y',t')=C^6(b_{40}x'^4+b_{04}y'^4+b_{22}x'^2y'^2-4b_{40}A_1x'^3-4b_{04}A_2y'^3-2b_{22}A_1x'y'^2-2b_{22}A_2x'^2y') +C^4(4C^2b_{22}A_1A_2+b_{11})x'y'+C^3(6C^3b_{40}A_1^2+C^3b_{22}A_2^2+Cb_{20}+\tfrac{1}{2}\ddot{C})x'^2 +C^3(6C^3b_{04}A_2^2+C^3b_{22}A_1^2+Cb_{02}+\tfrac{1}{2}\ddot{C})y'^2-C^3(2C^3b_{40}A_1^3+2C^3b_{22}A_1A_2^2+Cb_{11}A_2 +2Cb_{20}A_1+A_1\ddot{C}+C\ddot{A}_1+2\dot{A}_1\dot{C}-b_{10})x'-C^3(2C^3b_{04}A_2^3+2C^3b_{22}A_1^2A_2+Cb_{11}A_1+2Cb_{02}A_2 +\ddot{C}A_2+C\ddot{A}_2+2\dot{A}_2\dot{C}-b_{01})y'+C^3[C^3b_{40}A_1^4+C^3b_{04}A_2^4+C^3b_{22}A_1^2A_2^2+Cb_{11}A_1A_2 +A_1^2(Cb_{20}+\tfrac{1}{2}\ddot{C})+A_2^2(Cb_{02}+\tfrac{1}{2}\ddot{C})+A_1(C\ddot{A}_1+2\dot{A}_1\dot{C}-b_{10}) +A_2(\ddot{A}_2C+2\dot{A}_2\dot{C}-b_{01})+\tfrac{1}{2}C(\dot{A}_1^2+\dot{A}_2^2)-Cb_{11}A_1A_2]+b_0C^2 \tag{23}$$

In dealing with this form (or for that matter any other anharmonic form in general), the difficulties in reducing Eq. (8) to a TID form of the type (16) can be outlined as follows: Since the potential now necessarily contains anharmonic terms, it is not possible to reduce the TDSE with the potential (23) to a TIDSE with a harmonic potential unless the transformation (3) in space variables becomes nonlinear. This will, however, lead to other complications in applying the present method. Alternatively, one can assume the coefficients of the anharmonic terms in $V(x,y,t)$ to be TID and the time dependence mainly appears in the quadratic and linear terms only. In this case, however, one will need the exact solution to the TIDSE with the corresponding anharmonic potential, and such a solution is not often available in the literature. Although the exact solution to TIDSE for a limited class of anharmonic potentials is now known [12], the difficulty in applying the present method to these cases still remains. This is mainly because these anharmonic potentials are found to admit an exact normalizable solution, provided they necessarily contain inverse harmonic terms and/or cross-terms. In such a situation the present method for the available exactly solvable anharmonic potentials does not remain as transparent as for the harmonic potentials.

V. DISCUSSION AND SUMMARY

The group-transformation method of Burgan *et al.* [7] as generalized by Ray [9] is applied to obtain an exact analytic solution to the TDSE in two dimensions. The case of the shifted rotating TD harmonic oscillator is discussed in detail as an example. As far as the connection of the solution with TD invariants for this case is concerned, it can be deduced in the same way as Ray [9] for the one-dimensional case. An invariant has also been recently constructed [13] for a particular type of shifted rotating TD harmonic oscillator i.e., with $b_{02}=b_{20}$. It is noticed that, while the present method works successfully for TD harmonic potentials in one or higher dimensions, it exhibits limitations, not only for TD anharmonic potentials, but also for other varying forms of the harmonic potentials involving inverse harmonic and/or coupling terms. This latter class of potentials has been studied [14] recently at the classical level.

The difficulties in obtaining exact solutions for the anharmonic potentials within the framework of the present method are explicitly pointed out in Sec. IV. As far as the extension of the present method to three-dimensional TD systems is concerned, it can easily be carried out in a straightforward manner. But, again, restricted class of TD harmonic potentials, as in two dimensions, can be studied. For the case of anharmonic potentials, it becomes desirable to look for other alternative methods to solve the problem.

ACKNOWLEDGMENTS

The author wishes to thank Dr. D. Parashar for several helpful discussions. University Grants Commission, India, is gratefully acknowledged for providing the Research Scientist Scheme under which the present work has been carried out.

[1] G. Dattoli, A. Reniere, A. Torre, and J. C. Gallardo, Phys. Rev. A **35**, 4175 (1987); F. Ciocci, G. Dattoli, A. Reniere, and A. Torre, Phys. Rep. **141**, 1 (1986); J. M. Cervero and J. D. Lejarreta, Quant. Opt. **2**, 333 (1990).
[2] B. Gazdy and D. A. Micha, J. Chem. Phys. **82**, 4926 (1985); I. Benjamin, *ibid.* **85**, 1511 (1986).
[3] M. Lutzky, Phys. Lett. A **68**, 3 (1978).
[4] J. R. Ray and J. L. Reid, J. Math. Phys. **20**, 2054 (1979).
[5] F. Wolf and H. J. Korsch, Phys. Rev. A **37**, 1934 (1988); F. M. Fernandez, J. Math. Phys. **30**, 1522 (1989).
[6] H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math Phys. **10**, 1458 (1969); C. J. Eliezer AND A. Gray, SIAM J. Appl. Math. **30**, 463 (1976); S. C. Mishra, Ph.D. thesis, Delhi University, 1985 (unpublished).
[7] J. R. Burgan, M. R. Feix, E. Figalkow, and A. Munier, Phys. Lett. A **74**, 11 (1979).
[8] J. G. Hartley and J. R. Ray, Phys. Rev. A **24**, 2873 (1981); D. C. Khandekar and S. V. Lawande, J. Math. Phys. **20**, 1870 (1979).
[9] J. R. Ray, Phys. Rev. A **26**, 729 (1982).
[10] J. R. Ray and J. L. Reid, Phys. Lett. A **71**, 317 (1979); J. L. Reid and J. R. Ray, J. Phys. A **15**, 2751 (1982); C. Athorne, Phys. Lett. A **159**, 375 (1991).
[11] R. S. Kaushal and H. J. Korsch, J. Math. Phys. **22**, 1904 (1981).
[12] R. S. Kaushal, Ann. Phys. (N.Y.) **206**, 90 (1991).
[13] R. S. Kaushal, D. Parashar, and S. C. Mishra (unpublished).
[14] R. S. Kaushal and D. Parashar, Mod. Phys. Lett. A **6**, 2887 (1991).