PHYSICAL REVIEW A VOLUME 46, NUMBER 5 1 SEPTEMBER 1992

Phase properties of real field states: The Garrison-Wong versus Pegg-Barnett predictions

Ts. Gantsog*
International Centre for Theoretical Physics, P.O. Box 586, Miramare, 34100 Trieste, Italy

A. Miranowicz and R. Tanaś
Nonlinear Optics Division, Institute of Physics, Adam Mickiewicz University, 60-780 Poznań, Poland
(Received 24 March 1992)

A comparison is made of predictions for the phase variances and the phase distribution functions obtained from the Garrison-Wong and Pegg-Barnett formalisms for real field states that include number states, coherent states, and squeezed vacuum states. It is shown that both approaches lead to qualitatively different phase distributions. The Garrison-Wong approach predicts an anisotropy of the phase distribution that is inconsistent with the symmetry of the Wigner and Q functions.

PACS number(s): 42.50.Dv

I. INTRODUCTION

The problem of correct definition in quantum theory of an operator corresponding to the phase of a one-mode quantum field has a long history and has provoked many discussions and controversies. There have been numerous attempts to solve this problem, and the best known solution is that of Susskind and Glogower [1] who introduced the cosine and sine operators of the phase (see also [2]).

In 1970 Garrison and Wong [3] constructed the self-adjoint operator, canonically conjugated to the number operator on the dense set of the state vectors in the Hilbert space. Later on, a family of similar phase operators was proposed by Popov and Yarunin [4], Damaskinsky and Yarunin [5], as well as Galindo [6]. These operators, however, have not attracted much attention because of their rather complex structure that made them impractical.

Recently, Pegg and Barnett [7-9] have suggested an alternative approach using the states of a well defined phase as a starting point. To construct the Hermitian phase operator they restrict the state space to a finite $(s+1)$-dimensional state space Ψ spanned by the first $s+1$ number states. The main idea of this approach is to calculate all relevant physical quantities such as means, variances, etc. in this finite-dimensional state space first, and only after all calculations were performed s is allowed to tend to infinity. The Pegg-Barnett phase formalism has been widely applied to calculate the phase properties of a number of single- and two-mode field states (see [10] and papers cited therein). The predictions of the Pegg-Barnett formalism for the phase fluctuations were compared with the existing measurements [11, 12] (modified by Nieto [13]) of the phase fluctuations of a coherent laser beam exhibiting good agreement with the experimental data [14–16].

After the successes of the Pegg-Barnett approach, the Garrison-Wong phase operator, which is equivalent to the Popov–Yarunin and Galindo phase operators, was compared with the Pegg-Barnett operator by Bergou and Englert [17] and Popov and Yarunin [18]. In both the latter papers one can find statements that the Pegg-Barnett phase operator is an "approximation" to the Garrison-Wong phase operator. Both approaches give the same results for highly excited states, but there are essential differences for the states with few photons. These differences have definite physical implications and raise the question: which approach gives a more acceptable physical interpretation? The definite answer to this question can be supplied by the experiment, but the predictions based on the two approaches can be confronted with the information available from other sources (like the Wigner or Q functions).

In this paper we compare the Garrison-Wong (GW) and Pegg-Barnett (PB) phase approaches. In Secs. II and III we briefly review both approaches and explain the main differences between them. In Sec. IV we calculate the phase variances and the phase distributions from the two alternative approaches for the number states, coherent states, and squeezed vacuum states. We show that the polar plots of the phase distributions for the number states and the squeezed vacuum states obtained from the GW formulas have asymmetric shape which is inconsistent with the symmetry of the Wigner function and the Q function for such states. The distributions obtained according to the PB approach do not suffer from such asymmetry. For large numbers of photons predictions from both formalisms, as expected, are indistinguishable.

II. THE GARRISON-WONG PHASE OPERATOR

Garrison and Wong [3] constructed the phase operator $\hat{\phi}_{\rm GW}$ using the relation

$$\langle g|\hat{\phi}_{\rm GW}|f\rangle = \int_{\theta_0}^{\theta_0+2\pi} d\theta\, g^*(e^{-i\theta})\,\theta\, f(e^{-i\theta}), \quad (1)$$

for any $g, f \in H^2$, where H^2 is the Hilbert space in the unit disk of the complex plane, and θ_0 is arbitrary. Here, we have changed the sign with respect to the original GW paper and introduced arbitrary θ_0. The inner product in H^2 is defined by

$$\langle g|f\rangle = \int_{\theta_0}^{\theta_0+2\pi} d\theta\, g^*(e^{-i\theta})\, f(e^{-i\theta}). \tag{2}$$

The boundary value of f is given by a convergent Fourier series

$$f(e^{-i\theta}) = \sum_{n=0}^{\infty} f_n\, e^{-in\theta}, \tag{3}$$

which does not contain coefficients f_n with negative n.

Subsequently, Popov and Yarunin [4] established the connection of this operator to the Susskind and Glogower [1] "exponential" phase operators $E_\pm$, which has the form

$$\hat{\phi}_{\rm GW} = \theta_0 + \pi + i\,\left[\ln(1 - e^{i\theta_0}\, E_+) \; - \; \ln(1 - e^{-i\theta_0}\, E_-)\right]. \tag{4}$$

The operators E_- and $E_+ = (E_-)^\dagger$ are defined by the annihilation and creation operators a and $a^\dagger$ of the mode

$$\begin{aligned} E_- &= a\,(a^\dagger a)^{-1/2},\\ E_+ &= (a^\dagger a)^{-1/2}\, a,\\ [E_-, E_+] &= |0\rangle\langle 0|, \end{aligned} \tag{5}$$

where $|0\rangle$ is the vacuum state.

Let us consider the "phase states"

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \exp(in\theta)\, |n\rangle, \tag{6}$$

which are the right and left eigenstates of the operators E_- and E_+,

$$\begin{aligned} E_-\, |\theta\rangle &= \exp(i\theta)\, |\theta\rangle\,,\\ \langle\theta|\, E_+ &= \exp(-i\theta)\, \langle\theta|. \end{aligned} \tag{7}$$

The states (6) are not orthogonal but they allow for the resolution of the identity operator

$$\int_{\theta_0}^{\theta_0+2\pi} d\theta\, |\theta\rangle\langle\theta| = \hat{1}. \tag{8}$$

Applying Eqs. (7) and (8) to the operator (4) we can rewrite it into the form

$$\begin{aligned} \hat{\phi}_{\rm GW} &= \theta_0 \; + \; \pi + i \int_{\theta_0}^{\theta_0+2\pi} d\theta\, |\theta\rangle\langle\theta|\, \{\ln[1 - e^{-i(\theta-\theta_0)}] \\ &\quad - \ln[1 - e^{i(\theta-\theta_0)}]\} \\ &= \int_{\theta_0}^{\theta_0+2\pi} d\theta\, |\theta\rangle\theta\langle\theta|. \end{aligned} \tag{9}$$

Since the states (6) are not orthogonal, they are not the eigenstates of the GW phase operator.

From Eq. (9) we have

$$\langle g|\hat{\phi}_{\rm GW}|f\rangle = \int_{\theta_0}^{\theta_0+2\pi} d\theta\, \langle g|\theta\rangle\theta\langle\theta|f\rangle. \tag{10}$$

If we take the field states $|f\rangle$ in the form

$$|f\rangle = \sum_{n=0}^{\infty} f_n\, |n\rangle, \tag{11}$$

we then have

$$\langle\theta|f\rangle = \; f(e^{-i\theta}) = \; \sum_{n=0}^{\infty} f_n e^{-in\theta}, \tag{12}$$

which has the same form as Eq. (3), and we can consider the phase operators (1) and (4) as equivalent. However, we should keep in mind that the GW phase operator is defined on the dense set of the state vectors which, for mathematical consistency and the requirement that the number-phase commutator should be $-i$, imply $f(-1) = 0$.

Since the states (6) are not orthogonal, we have

$$\hat{\phi}^k_{\rm GW} \neq \int_{\theta_0}^{\theta_0+2\pi} d\theta\, \theta^k |\theta\rangle\langle\theta| \quad (k > 1), \tag{13}$$

and for the expectation values

$$\langle f|\hat{\phi}^k_{\rm GW}|f\rangle \neq \int_{\theta_0}^{\theta_0+2\pi} d\theta\, \theta^k |\langle\theta|f\rangle|^2 \quad (k > 1). \tag{14}$$

This means that the quantity $|\langle\theta|f\rangle|^2$ cannot be interpreted as a phase-distribution function. To find the Garrison-Wong phase-distribution function we have to calculate the quantity

$$P_{\rm GW}(\theta) = \; |{}_{\rm GW}\langle\theta|f\rangle|^2 = \left|\sum_{n=0}^{\infty} f_n\; {}_{\rm GW}\langle\theta|n\rangle\right|^2, \tag{15}$$

where the vector $|\theta\rangle_{\rm GW}$ is the eigenvector of the GW phase operator. The function ${}_{\rm GW}\langle\theta|n\rangle$ has a quite complex structure [3, 4, 18], but it can be found from the recursive formulas given by Garrison and Wong [3], which are

$${}_{\rm GW}\langle\theta|n\rangle = \left[\frac{1}{\pi}\, \sin\left(\frac{\theta-\theta_0}{2}\right)\right]^{1/2} \phi_n(\theta), \tag{16}$$

where, for $n \geq 1$,

$$\phi_n(\theta) = - \sum_{m=0}^{n-1} \left(1 - \frac{m}{n}\right)\, \gamma_{n-m}(\theta)\, \phi_m(\theta), \tag{17}$$

$$\begin{aligned} \gamma_n(\theta) \; &= \; \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+2\pi} d\theta'\, \ln|\theta' - \theta|\, e^{in\theta'} \\ &\quad -\frac{1}{2n}\left[e^{in\theta_0} \; + \; e^{in\theta}\right], \end{aligned} \tag{18}$$

and

$$\phi_0(\theta) = e^{-\gamma_0(\theta)}, \tag{19}$$

$$\gamma_0(\theta) = -\tfrac{1}{2} + \frac{1}{4\pi}[\,(2\pi+\theta_0-\theta)\ \ln(2\pi+\theta_0-\theta) + (\theta-\theta_0)\ \ln(\theta-\theta_0)]. \tag{20}$$

We shall use formulas (16)–(20) to calculate the GW phase distribution for some real states of the field.

Substituting (6) into (9) and performing the integration over θ yields the following number-states expansion for the Garrison-Wong phase operator:

$$\hat{\phi}_{\rm GW} = \theta_0 + \pi + \sum_{\substack{n,n'\\ n\neq n'}} \frac{\exp[i(n-n')\theta_0]\,|n\rangle\langle n'|}{i\,(n-n')}, \tag{21}$$

which leads to the number-phase commutator

$$[a^\dagger a, \hat{\phi}_{\rm GW}] = -i\,(1 - |\theta_0\rangle\langle\theta_0|), \tag{22}$$

and for the states for which $\langle\theta_0|f\rangle = 0$, the second term on the right-hand side vanishes giving the value demanded by Garrison and Wong.

III. THE PEGG-BARNETT PHASE FORMALISM

Pegg and Barnett [7-9] introduced the Hermitian phase formalism, which is based on the observation that in a finite-dimensional state space the states with the well-defined phase exist [19]. Thus, they restrict the state space to a finite $(s+1)$-dimensional space Ψ spanned by the number states $|0\rangle, |1\rangle, ..., |s\rangle$. In this space they define a complete orthonormal set of phase states by

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(in\theta_m)\,|n\rangle, \quad m = 0, 1, ..., s \tag{23}$$

where the values of θ_m are given by

$$\theta_m = \theta_0 + \frac{2\pi m}{s+1}. \tag{24}$$

The value of θ_0 is arbitrary and defines a particular basis set of $(s+1)$ mutually orthogonal phase states. The Pegg-Barnett (PB) Hermitian phase operator is defined as

$$\hat{\phi}_{\rm PB} = \sum_{m=0}^{s} \theta_m\,|\theta_m\rangle\langle\theta_m|. \tag{25}$$

Of course, the phase states (23) are eigenstates of the phase operator (25) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and $\theta_0 + 2\pi$. The Pegg-Barnett prescription is to evaluate any observable of interest in the finite basis (23) and only after that take the limit $s \to \infty$.

Since the phase states (23) are orthonormal, $\langle\theta_m|\theta_{m'}\rangle = \delta_{mm'}$, the kth power of the Pegg-Barnett phase operator (25) can be written as

$$\hat{\phi}^k_{\rm PB} = \sum_{m=0}^{s} \theta^k_m\,|\theta_m\rangle\langle\theta_m|, \tag{26}$$

and the expectation value of the kth power of the phase operator can be calculated as

$$\langle f|\hat{\phi}^k_{\rm PB}|f\rangle = \sum_{m=0}^{s} \theta^k_m\,|\langle\theta_m|f\rangle|^2, \tag{27}$$

where the quantity $\langle\theta_m|f\rangle|^2$ gives a probability of being found in the phase state $|\theta_m\rangle$.

When "physical states," according to their definition by Pegg and Barnett [8, 9], are considered, we can simplify the calculation of the sum in Eq. (27) by replacing it by the integral in the limit as s tends to infinity. Since the density of phase states is $(s+1)/2\pi$, we can write Eq. (27) as

$$\langle f|\hat{\phi}^k_{\rm PB}|f\rangle = \int_{\theta_0}^{\theta_0+2\pi} d\theta\,\theta^k P_{\rm PB}(\theta), \tag{28}$$

where the continuous-phase distribution $P_{\rm PB}(\theta)$ is introduced by

$$P_{\rm PB}(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} |\langle\theta_m|f\rangle|^2, \tag{29}$$

and θ_m has been replaced by the continuous-phase variable θ.

Recent studies of phase properties of the fields generated in nonlinear optical processes show that the phase distribution (29) or the joint phase distribution (for the two-mode field) are new representations of the quantum state of the field, alternate to the Q function or the Wigner function, and they carry quite a bit of essential information characterizing the quantum state of the field. For example, when the field is a superposition of well-separated coherent states the phase distribution splits into separate peaks clearly indicating the components of the superposition [20], the phase distribution splits into separate peaks when the transition from the second-harmonic generation to the down-conversion regime takes place [21], the multiplicity of the phase distribution in the multiphoton down conversion indicates clearly the multiplicity of the process [22], etc.

Inserting (23) and (24) into (25) allows for getting the explicit number-state representation of the PB phase operator, which has the form [9]

$$\hat{\phi}_{\rm PB} = \theta_0 + \frac{s\pi}{s+1} + \frac{2\pi}{s+1} \sum_{n\neq n'} \frac{\exp[i(n-n')\theta_0]\,|n\rangle\langle n'|}{\exp[i\,(n-n')2\pi/(s+1)] - 1}. \tag{30}$$

It is easy to see that if the limit $s \to \infty$ is taken in the PB phase operator (30) one obtains the GW phase operator (21). Despite this relation, the two approaches lead to quite different predictions. The source of the differences is the noncommuting character of the operations of taking the limit $s \to \infty$ and taking the expectation value

$$\langle f(\lim_{s\to\infty}\hat{\phi}_{\rm PB})\rangle \neq \lim_{s\to\infty}\langle f(\hat{\phi}_{\rm PB})\rangle, \tag{31}$$

where the left-hand side represents the Garrison-Wong term, the right-hand side, the Pegg-Barnett term, and $f(\hat{\phi})$ is any function of the phase operator. Depending on which order we perform these two operations, we obtain different results. In the next section we compare the predictions of the two approaches for several examples of the field states.

IV. A COMPARISON OF THE GARRISON-WONG AND PEGG-BARNETT PREDICTIONS

To make the differences between the GW and PB approaches more explicit we shall calculate the variances and the phase-distribution functions, using both approaches, for the number states, the coherent states, and the squeezed vacuum. To calculate the variance of the GW operator we can use the form (21) of the operator (we assume later on $\theta_0 = -\pi$) to calculate the square of this operator,

$$\hat{\phi}^2_{\rm GW} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}|n\rangle\langle n|\sum_{k=1}^{n}\frac{1}{k^2} + \sum_{n>n'}\frac{(-1)^{n-n'}}{n-n'}\left(\frac{2}{n-n'} - \sum_{k=n'+1}^{n}\frac{1}{k}\right) \times(|n\rangle\langle n'| + |n'\rangle\langle n|). \tag{32}$$

Let us consider a field state with the number-state expansion

$$|f\rangle = \sum_n b_n e^{in\varphi}|n\rangle, \tag{33}$$

where b_n is real. In this state the GW variance is given by

$$(\Delta\hat{\phi}_{\rm GW})^2 = \langle\hat{\phi}^2_{\rm GW}\rangle - \langle\hat{\phi}_{\rm GW}\rangle^2 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}b_n^2\sum_{k=1}^{n}\frac{1}{k^2} + 2\sum_{n>n'}\frac{(-1)^{n-n'}}{n-n'}\left(\frac{2}{n-n'} - \sum_{k=n'+1}^{n}\frac{1}{k}\right)b_n b_{n'}, \tag{34}$$

where we put $\varphi = 0$ for simplicity.

For the same state (33), the phase variance calculated according to the Pegg-Barnett prescription, which says that the expectation value should be taken before the limit $s\to\infty$ is taken, is given by [9]

$$(\Delta\hat{\phi}_{\rm PB})^2 = \lim_{s\to\infty}[\langle\hat{\phi}^2_{\rm PB}\rangle - \langle\hat{\phi}_{\rm PB}\rangle^2] = \frac{\pi^2}{3} + 4\sum_{n>n'}\frac{(-1)^{n-n'}}{(n-n')^2}b_n b_{n'}. \tag{35}$$

Even superficial inspection of formulas (34) and (35) shows that the phase variances calculated according to GW and PB prescriptions are different.

For the number state $|n\rangle$, for which only $b_n = 1$ is nonzero, the variance (34) is

$$(\Delta\hat{\phi}_{\rm GW})^2 = \frac{\pi^2}{6} + \sum_{k=1}^{n}\frac{1}{k^2}, \tag{36}$$

and we have different values of the phase variance for different n in the GW approach: for the vacuum ($n = 0$) it is $\pi^2/6$ and it becomes $\pi^2/3$ in the limit $n\to\infty$. This is in a marked contrast with the PB approach which gives the value $\pi^2/3$ for any number state $|n\rangle$. So, in the PB approach all number states are states with random phase, while in the GW approach the phase of any number state (also vacuum) is not random. In Fig. 1 we show the phase variances predicted by both approaches for number states versus the photon number n. The GW phase variance asymptotically approaches the value $\pi^2/3$ for large n.

For the coherent states, we have

$$b_n = \exp(-|\alpha|^2/2)\frac{|\alpha|^n}{\sqrt{n!}}, \tag{37}$$

where $\bar{n} = |\alpha|^2$ is the mean photon number [$\alpha = |\alpha|\exp(i\phi)$, and we set $\phi = 0$ later on]. In Fig. 2 the phase variances for coherent states are plotted against the mean photon number $\bar{n}$. It is seen that as the mean

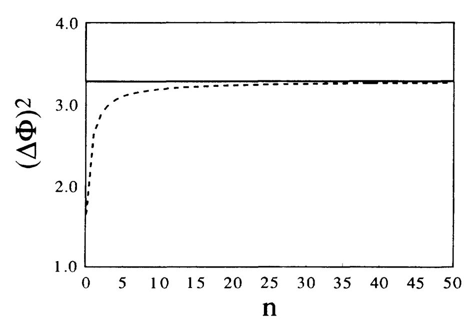

FIG. 1. Plot of the phase variances $(\Delta\hat{\phi}_{\rm PB})^2$ (solid line) and $(\Delta\hat{\phi}_{\rm GW})^2$ (dashed line) vs n for the number states.

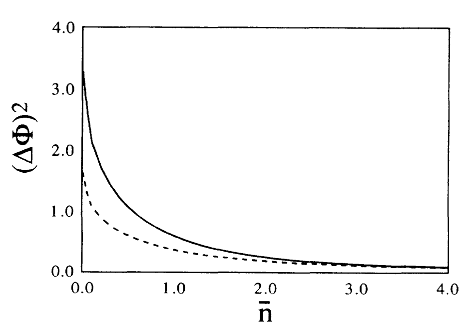

FIG. 2. Plot of $(\Delta\hat{\phi}_{\rm PB})^2$ (solid line) and $(\Delta\hat{\phi}_{\rm GW})^2$ (dashed line) vs the average photon number $\bar{n}$ for the coherent states.

number of photons increases the difference between the two variances decreases. This means that for $\bar{n} \gg 1$ the two-phase approaches give indistinguishable results, but for $\bar{n} < 1$ the differences are visible.

One more example we consider here is the squeezed vacuum for which the coefficients b_n are defined by [23]

$$b_n = \begin{cases} \frac{(-1)^{n/2}}{\sqrt{\cosh r}} \frac{\sqrt{n!}}{(n/2)!} \left(\frac{1}{2}\tanh r\right)^{n/2}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases} \tag{38}$$

The variances for the squeezed vacuum are plotted against the squeeze parameter r in Fig. 3. Asymptotically, for large r, both variances approach the value $\pi^2/4$, but they start from different values for $r = 0$ (the vacuum). Generally, it is clear that the GW phase variance is smaller than the PB phase variance. This means that the uncertainty in the GW phase measurement should be smaller than the corresponding uncertainty for the PB operator. Why is it so? To answer this question we study the phase distributions obtained from both formalisms.

The phase distribution for the number states that is obtained from the GW formalism can be calculated from formulas (15)–(20), and examples of such distributions were given in the Garrison and Wong paper [3]. In Fig. 4 we show some examples of the GW phase distributions for the number states. As Garrison and Wong [3] concluded, their phase distributions show oscillations with $n+1$ peaks for the n-photon state. Even for the vacuum the distribution is peaked. This means that the vacuum is anisotropic, i.e., there is a preferred phase even for the vacuum. The reason for this is the vanishing of the GW phase distribution at the ends of the phase window, which in turn is the consequence of the requirement that the number phase commutator should be $-i$ [i.e., $f(-1) = 0$]. So, forcing the phase operator to obey this commutation relation introduces anisotropy to the phase distribution. In the PB approach the phase of all number states is distributed uniformly and there is no anisotropy in the phase distribution. To visualize better this anisotropy we show in Fig. 5 the polar plots of the phase distribution for several number states. The anisotropy of the GW phase distribution is clearly seen. The PB phase distribution is, of course, isotropic. If one compares the symmetry of the phase distribution with the symmetry of the Wigner and Q functions (which for the number states are symmetrical with respect to the origin), one notices that the symmetry of the GW phase distribution is inconsistent with the symmetry of the Wigner and Q functions. Is the anisotropy a mathematical artifact, or has it a physical meaning? We leave this question open.

In Fig. 6 the phase distributions for a few coherent states with different mean numbers of photons are shown. Again it is seen that the GW phase distribution is narrower than the PB distribution, but the differences between the two rapidly disappear as the mean number of photons increases. In the case of coherent states the differences between the two approaches are less pronounced, because the phase peak in the center appears in both approaches and the fact that the GW phase distribution is

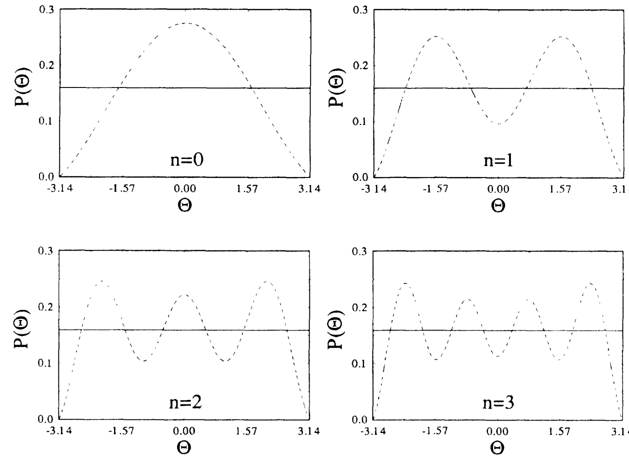

FIG. 4. Graphs of the phase distributions $P_{\mathrm{PB}}(\theta)$ (solid line) and $P_{\mathrm{GW}}(\theta)$ (dashed line) for the number states with $n = 0, 1, 2, 4$ in the rectangular coordinate system.

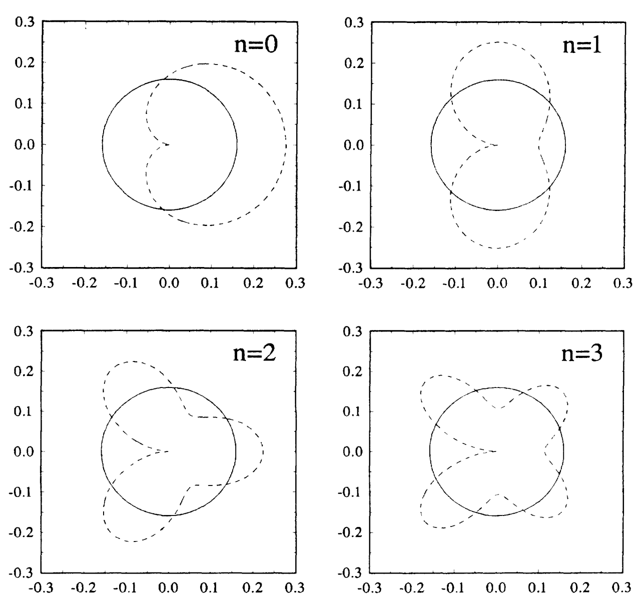

FIG. 5. Same as Fig. 4, but in the polar system.

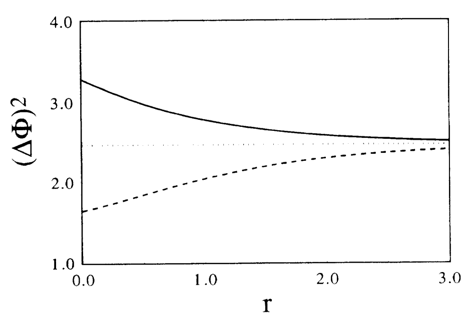

FIG. 3. Plot of $(\Delta\hat{\phi}_{\mathrm{PB}})^2$ (solid line) and $(\Delta\hat{\phi}_{\mathrm{GW}})^2$ (dashed line) vs the squeeze parameter r for the squeezed vacuum.

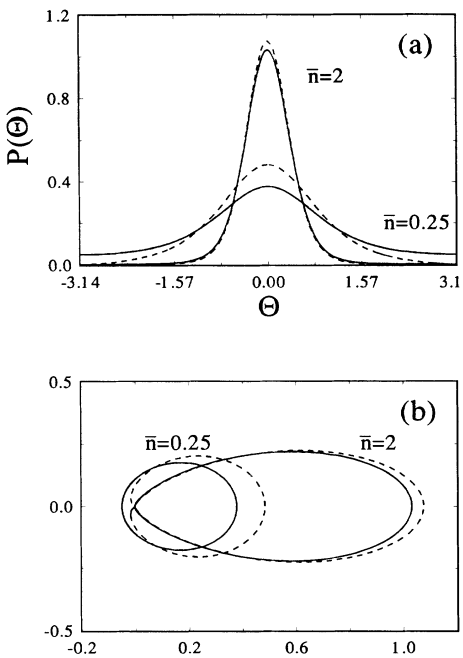

FIG. 6. Graphs of $P_{\rm PB}(\theta)$ (solid line) and $P_{\rm GW}(\theta)$ (dashed line) for the coherent states with $\bar{n} = 0.25$, 2 in (a) the rectangular coordinate system and (b) in the polar system.

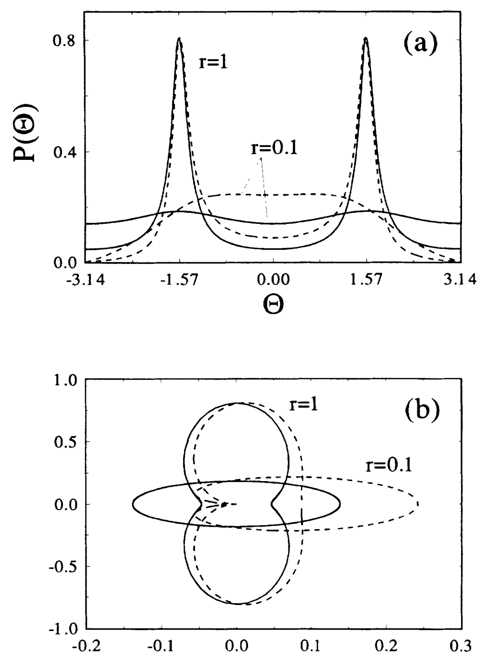

FIG. 7. Same as Fig. 6, but for the squeezed vacuum states with $r = 0.1$, 1.

zero at the ends of the window has less effect.

The phase distributions for the squeezed vacuum state are shown in Fig. 7. The two-peak structure of the phase distribution develops in both approaches, and asymptotically when the squeeze parameter r tends to infinity both GW and PB distributions approach two δ functions located at $\theta = \pm\pi/2$. However, if one considers the symmetry of the phase distribution, which is more visible from the polar plots presented in Fig. 7(b), one sees that the GW exhibits the asymmetry which is inconsistent with the elliptic shape of the contours of the Q function for such states. In Fig. 7(b) we have deliberately expanded the scale along the horizontal axis to emphasize this asymmetry. The PB phase distribution exhibits twofold rotational symmetry which is consistent with the symmetry of the Q function for the squeezed vacuum [23–25].

The above few examples show that the Garrison-Wong approach always gives the phase distributions that are narrower than the corresponding distributions obtained from the Pegg-Barnett approach. This also means that the GW phase variances will be smaller than the PB variances. The reason for this narrowing is the requirement that the number-phase commutator should be $-i$, which in effect leads to the condition $P_{\rm GW}(\pm\pi) = 0$, and because $P_{\rm GW}(\theta)$ should be normalized, there must be a phase peak even for the vacuum states. This requirement introduces an anisotropy to the phase distributions, which is inconsistent with the phase information that one would expect from the Wigner and Q functions.

V. CONCLUSIONS

In this paper we have compared predictions of two quantum phase formalisms: the Garrison-Wong formalism and the Pegg-Barnett formalism. The phase variances and the phase distributions have been calculated for the number states, coherent states, and the squeezed vacuum state. We have shown that the Garrison-Wong phase distribution is narrower than the Pegg-Barnett distribution, although for real physical states the quantitative differences between the two can be irrelevant. There is, however, one important qualitative difference between the GW and PB formalisms. The GW formalism introduces the anisotropy into phase distributions, and even the vacuum is anisotropic. This anisotropy is a consequence of their requirement that the number-phase commutator should be $-i$, i.e., the requirement that the number and phase operators are a Heisenberg pair. This in turn gives zero values of the phase distribution at the ends of the phase window, and since the distribution is normalized the phase peak must appear even for the vacuum state. This striking anisotropy of the phase distribution is best visible when polar plots of the phase distribution are made. Such asymmetry is inconsistent, however, with the symmetry of the Wigner and Q functions of field states.

The Pegg-Barnett formalism does not suffer from any "symmetry-breaking" problem. The symmetry of the PB phase distribution is consistent with the symmetry of the

Wigner and Q functions. So, there is a qualitative difference between the two formalisms in predicting phase properties of optical fields. This difference has its origin in a different understanding of the canonically conjugate variables [26], and it has basic character.

ACKNOWLEDGMENTS

One of us (Ts.G.) thanks Professor G.S. Agarwal for illuminating discussions on the subject of this paper, and the Centre for Theoretical Physics, Trieste, Italy, for the hospitality extended to him during his stay in Trieste.

* Permanent address: Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Head Post Office P.O. Box 79, Moscow 101000, Russia.

[1] L. Susskind and J. Glogower, Physics **1**, 49 (1964).
[2] P. Carruthers and M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).
[3] J. C. Garrison and J. Wong, J. Math. Phys. **11**, 2242 (1970).
[4] V. N. Popov and V. S. Yarunin, Vestn. Leningr. Univ. **N22**, 7 (1973).
[5] E. V. Damaskinsky, V. S. Yarunin, Izv. Vyssh. Uchebn. Zaved. Tomsk Univ. **6**, 59 (1978).
[6] A. Galindo, Lett. Math. Phys. **8**, 495 (1984); **9**, 263 (1985).
[7] D. T. Pegg and S. M. Barnett, Europhys. Lett. **6**, 483 (1988).
[8] S. M. Barnett and D. T. Pegg, J. Mod. Opt. **36**, 7 (1989).
[9] D. T. Pegg and S M Barnett, Phys. Rev. A **39**, 1665 (1989).
[10] R. Tanaś, J. Sov. Laser Res. **12**, 395 (1991).
[11] H. Gerhardt, U. Büchler, and G. Litfin, Phys. Lett. **49A**, 119 (1974).
[12] H. Gerhardt, H. Welling, and D. Frölich, Appl. Phys. **2**, 91 (1973).
[13] M. M. Nieto, Phys. Lett. **60A**, 401 (1974).
[14] R. Lynch, Phys. Rev. A **41**, 2841 (1990).
[15] C. C. Gerry and K. E. Urbański, Phys. Rev. A **42**, 662 (1990).
[16] A. Bandilla, Opt. Commun. **80**, 267 (1991).
[17] J. Bergou and B.-G. Englert, Ann. Phys. (N.Y.) **209**, 479 (1991).
[18] V. N. Popov and V. S. Yarunin, J. Mod. Opt. (to be published).
[19] R. Loudon, *The Quantum Theory of Radiation* (Oxford University Press, Oxford, 1973) p. 143.
[20] Ts. Gantsog and R. Tanaś, Phys. Rev. A **44**, 2086 (1991).
[21] Ts. Gantsog, R. Tanaś, and R. Zawodny, Phys. Lett. A **155**, 1 (1991).
[22] R. Tanaś and Ts. Gantsog, Phys. Rev. A **45**, 5031 (1992).
[23] R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987).
[24] N. Grønbech-Jensen, P. L. Christiansen, and P. S. Ramanujam, J. Opt. Soc. Am. B **6**, 2423 (1989).
[25] J. A. Vaccaro and D. T. Pegg, Opt. Commun. **70**, 529 (1989).
[26] D. T. Pegg, J. A. Vaccaro, and S. M. Barnett, J. Mod. Opt. **37**, 1703 (1990).