Continuous quantum-nondemolition measurement of photon number

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(Received 6 April 1992)

This paper presents a general theory for a continuous quantum-nondemolition measurement of photon number. This theory treats a time-distributed measurement as a sequence of measurements in which at most one photon can be detected in an infinitesimal time, and shows that the average number of photons remaining in the measured field *increases* when a photon is detected and *decreases* when no photon is detected. The state of the measured system evolves nonunitarily and reduces continuously to a number state whose eigenvalue is uniquely determined by the average rate of photodetection and whose probability distribution coincides with the initial photon-number distribution. Applying the general theory to typical quantum states — coherent, thermal, and squeezed states — shows that the continuous-state reduction towards a number state depends strongly on the initial photon statistics. Despite the nonunitarity of state evolution, an initially pure state keeps its purity: the initial density operator becomes diagonalized only if the readout information is discarded.

PACS number(s): 42.50.Ar, 03.65.Bz, 42.50.Dv

I. INTRODUCTION

State reduction in quantum measurement is one of the enigmas that has haunted physicists since the dawn of quantum mechanics. According to von Neumann's quantum theory of measurement [1], the measurement process is divided into two fundamentally different stages. In the first stage, quantum correlation between the measuring apparatus and the measured system is established through a unitary interaction; in the second stage, the apparatus meter is read out, causing nonunitary state reduction.

This picture, however, cannot be applied to continuous measurements such as photon counting because in them the apparatus meter (e.g., a sequence of photoelectric pulses) is read out continuously and the above two stages therefore proceed simultaneously [2]. To cope with this difficulty, Davies and his co-workers developed an operational theory [3–5] that has been successfully applied in several examples [6–9]. This approach has recently been extended to cover the continuous-state reduction of single-mode [10–12] and correlated [13] photon fields, leading to the prediction of such novel phenomena as measurement-induced Fano-factor oscillations [14] and the generation of the Schrödinger-cat state [15].

This paper proposes a general theory for *continuous* quantum-nondemolition (QND) measurement of photon number [16, 17]. This theory simulates a continuous measurement as a sequence of infinitesimally weak measurements. By infinitesimally weak, we mean that at most one photon can be detected within an infinitesimal time period. A brief sketch of this idea has been presented in Ref. [16], where computer simulation was used to demonstrate that the measured photon state reduces continuously to a number state. The present paper proves this analytically and substantiates the general theory in some examples.

This paper is organized as follows. Section II provides a microscopic model for continuous QND measurement of photon number. Section III develops formulas that let us use the continuous-readout information (*referring measurement*) to keep track of state reduction, and shows that the average number of photons remaining in the measured field *increases* upon photodetection and *decreases* otherwise. Furthermore, it will be shown that an initially pure state keeps its purity during the measurement process even though the state evolution is nonunitary. Section IV applies the general formulas derived in Secs. II and III to typical quantum states—coherent, thermal, and squeezed states. Section V shows that although the physical condition does not appear to differ between situations in which the readout information is used to renormalize the initial density operator and situations in which the detector is switched on but the readout information is not used, the state evolves differently in these two kinds of situations. When the readout information is not used, the state collapses into a mixture of number states. Section VI proves that the state continuously reduces to a number state whose eigenvalue is uniquely determined by the average rate of photodetection and whose probability distribution coincides with the initial photon-number distribution. Section VII shows results of computer simulation. Section VIII discusses how our proposed measurement scheme preserves the initial photon statistics and thus ensures that the photon number is a QND observable. Section IX summarizes the main results of this paper.

II. MICROSCOPIC MODEL FOR CONTINUOUS QND PHOTODETECTION

A continuous measurement can, in general, be constructed as follows. A measurement time is divided into infinitely many infinitesimal intervals. In each interval, the system to be measured is coupled to the measuring apparatus by means of a unitary interaction and the apparatus meter is read out after the interaction. Depending on the readout, the projection is made on the measured system using the probability-operator measure [18], thereby making the system evolve nonunitarily. This cycle is repeated for each infinitesimal interval, and in each measurement, the apparatus is initially prepared in a prescribed state. A continuous measurement is constructed as a sequence of these infinitesimal processes.

A previous paper [19] showed that this procedure correctly reproduces the Srinivas-Davies model for photon counting [5] if the Jaynes-Cummings Hamiltonian is used as the interaction Hamiltonian. The present paper starts from a Hamiltonian that permits the photon number to be a QND observable and constructs a continuous QND measurement of photon number.

Suppose that the system and apparatus evolve according to an interaction Hamiltonian $\hat{H}_{\rm int}$. After an interaction time Δt, the coupled system-apparatus density operator $\hat{\rho}_{s\text{-}a}(\Delta t)$ becomes

$$\hat{\rho}_{s\text{-}a}(\Delta t) = \hat{U}\hat{\rho}_s(0)\otimes\hat{\rho}_a(0)\hat{U}^\dagger, \tag{2.1}$$

where $\hat{U}\equiv\exp(-i\hat{H}_{\rm int}\Delta t/\hbar)$, and $\hat{\rho}_s(0)$ and $\hat{\rho}_a(0)$ are the initial density operators for the system and apparatus. The projection is made using the probability-operator measure $\hat{\rho}^{(\rm read)}$ which is related to the readout value X of the apparatus meter by $\hat{\rho}^{(\rm read)}\equiv\hat{1}_s\otimes|X\rangle_{a\,a}\langle X|$, where $\hat{1}_s$ is the identity operator for the system and $|X\rangle_a$ is an eigenvector for the apparatus observable $\hat{X}_a$. The system density operator $\hat{\rho}_s(\Delta t)$ immediately after the measurement is given by [18]

$$\hat{\rho}_s(\Delta t) = \frac{{\rm Tr}_a[\hat{\rho}_{s\text{-}a}(\Delta t)\hat{\rho}^{(\rm read)}]}{{\rm Tr}_{s\text{-}a}[\hat{\rho}_{s\text{-}a}(\Delta t)\hat{\rho}^{(\rm read)}]}, \tag{2.2}$$

where ${\rm Tr}_a$ and ${\rm Tr}_{s\text{-}a}$ denote traces over the apparatus and over the system and the apparatus. We assume here that in each infinitesimal process the initial state of the measuring apparatus is reset to a prescribed initial state. Thus in each infinitesimal time we repeat the following cycle: (i) initialize the measuring apparatus to $\hat{\rho}_a(0)$, (ii) use $\hat{U}$ to couple the measuring apparatus to the measured system during a time Δt, and (iii) apply Eq. (2.2) to extract the postmeasurement density operator of the measured system from the total density operator. For convenience in later discussions, we denote this measurement cycle as

$$\hat{\rho}_s(0)\to\hat{\rho}_s(\Delta t;X) = M_X(\hat{U}\hat{\rho}_s(0)\otimes\hat{\rho}_a(0)\hat{U}^\dagger), \tag{2.3}$$

where we explicitly write the readout X because the resulting state depends on it.

The continuous measurement is constructed as a sequence of these infinitesimal processes. The time evolution of the system density operator after a finite time $t=N\Delta t$ is obtained by making N successive measurement cycles:

$$\hat{\rho}_s(N\Delta t;X_1,X_2,\ldots,X_N) = M_N(\cdots M_2(\hat{U}M_1(\hat{U}\hat{\rho}_s(0)\otimes\hat{\rho}_a(0)\hat{U}^\dagger)\otimes\hat{\rho}_a(0)\hat{U}^\dagger)\otimes\cdots), \tag{2.4}$$

where $X_1,X_2,\ldots,X_N$ denote the readout values for each cycle. If we take the limits $\Delta t\to 0$ and $N\to\infty$ with $N\Delta t$ fixed at t, the sequence of readouts $X_1,X_2,\cdots,X_N$ becomes a function of time, which we denote as $X(t)$. The nonunitary state evolution of the system depends on both the initial state $\hat{\rho}_s(0)$ and the readouts $X(\tau)$ for $0\le\tau\le t$. The time evolution of the system density operator can therefore be expressed in a functional form as $\hat{\rho}_s(t,X)=f[X(\tau)\ (0\le\tau\le t),\hat{\rho}_s(0)]$.

Now let us calculate this functional for the interaction Hamiltonian

$$\hat{H}_{\rm int} = \hbar g\hat{a}^\dagger\hat{a}(\hat{\sigma}+\hat{\sigma}^\dagger), \tag{2.5}$$

where g is the coupling constant between the system and the measuring apparatus, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of the photon field, and $\hat{\sigma}$ represents an operator that makes a bistable degenerate device transit from one state to the other. A good example here is a molecule with two degenerate potential wells; the transition from the left valley to the right, or vice versa, is caused by a photon, but the photon is not absorbed. We will henceforth call this device simply an "atom." It is easy to show that the Hamiltonian (2.5) satisfies the QND conditions for the photon number [20]:

$$\begin{aligned}
&\text{(a)}\quad [\hat{n}(0),\hat{n}(t)]=0,\\
&\text{(b)}\quad [\hat{H}_{\rm int}(t),\hat{n}(t)]=0,\\
&\text{(c)}\quad [\hat{H}_{\rm int}(t),\hat{\sigma}(t)]\neq 0,\\
&\text{(d)}\quad \hat{H}_{\rm int}(t)\ \text{should be a function of}\ \hat{n}(t),
\end{aligned} \tag{2.6}$$

where $\hat{n}(t)=\hat{a}^\dagger(t)\hat{a}(t)$ is the interaction-picture representation of the photon-number operator. Condition (a) shows that the photon number is a QND observable, and condition (b) ensures that the interaction Hamiltonian (2.5) is of the "back-action evading" type. The time evolution of the coupled density operator can be described as

$$\hat{\rho}(t)=\hat{\rho}(t_0)+\sum_{m=1}^{\infty}\left(\frac{1}{i\hbar}\right)^m\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{m-1}}dt_m[\hat{H}_{\rm int}(t_1),[\hat{H}_{\rm int}(t_2),\cdots[\hat{H}_{\rm int}(t_m),\hat{\rho}(t_0)]\cdots]], \tag{2.7}$$

where

$$\hat{\rho}(t_0)=\hat{\rho}_f(t_0)\otimes|l\rangle_{a\,a}\langle l| \tag{2.8}$$

is the initial density operator for the system and the atom [21]. Here the subscript f stands for the photon field, and l indicates that the atom is initially prepared in the "left" valley. Substituting Eqs. (2.5) and (2.8) into Eq. (2.7) and keeping the terms up to the second-order in Δt gives

$$\begin{aligned}\hat{\rho}(t_0+\Delta t)=\hat{\rho}(t_0)&+ig\Delta t[\hat{\rho}_f(t_0)\hat{n}\otimes|l\rangle_{a\,a}\langle r|-\hat{n}\hat{\rho}_f(t_0)\otimes|r\rangle_{a\,a}\langle l|]\\&+\frac{(ig\Delta t)^2}{2}\{[\hat{\rho}_f(t_0)\hat{n}^2+\hat{n}^2\hat{\rho}_f(t_0)]\otimes|l\rangle_{a\,a}\langle l|-2\hat{n}\hat{\rho}_f(t_0)\hat{n}\otimes|r\rangle_{a\,a}\langle r|\},\end{aligned} \tag{2.9}$$

where r indicates that the atom is in the "right" valley. If the atom is found in the right valley after the interaction, one photon has been detected; if the atom is found in the left valley, no photon has been detected. We will refer to these processes as the one-count process and the no-count process.

For the one-count process, $\hat{\rho}^{(\rm read)}(t_0)$ is given by $\hat{1}_f\otimes|r\rangle_{a\,a}\langle r|$. Substituting this $\hat{\rho}^{(\rm read)}$ and Eq. (2.9) into Eq. (2.2), we obtain

$$\hat{\rho}_f(t_0+\Delta t)=\frac{\mathcal{J}[\hat{\rho}_f(t_0)]}{{\rm Tr}_f\{\mathcal{J}[\hat{\rho}_f(t_0)]\}}, \tag{2.10}$$

where

$$\mathcal{J}[\hat{\rho}_f(t_0)]=g^2\Delta t\,\hat{n}\hat{\rho}_f(t_0)\hat{n}. \tag{2.11}$$

For the no-count process, $\hat{\rho}^{(\rm read)}(t_0)$ is given by $\hat{1}_f\otimes|l\rangle_{a\,a}\langle l|$. Using Eqs. (2.2) and (2.9), we obtain

$$\hat{\rho}_f(t_0+\Delta t)=\frac{\mathcal{S}_{\Delta t}[\hat{\rho}_f(t_0)]}{{\rm Tr}_f\{\mathcal{S}_{\Delta t}[\hat{\rho}_f(t_0)]\}}, \tag{2.12}$$

where

$$\begin{aligned}\mathcal{S}_{\Delta t}[\hat{\rho}_f(t_0)]&=\hat{\rho}_f(t_0)-\frac{(g\Delta t)^2}{2}[\hat{\rho}_f(t_0)\hat{n}^2+\hat{n}^2\hat{\rho}_f(t_0)]\\&=\exp\left[-\frac{g^2\Delta t}{2}\hat{n}^2\Delta t\right]\hat{\rho}_f(t_0)\\&\quad\times\exp\left[-\frac{g^2\Delta t}{2}\hat{n}^2\Delta t\right]+O((\Delta t)^4).\end{aligned} \tag{2.13}$$

N successive operations of Eq. (2.13) yield

$$\begin{aligned}\mathcal{S}_\tau[\hat{\rho}_f(t_0)]=&\exp\left[\left(-i\omega-\frac{g^2\Delta t}{2}\right)\hat{n}^2\tau\right]\hat{\rho}_f(t_0)\\&\times\exp\left[\left(i\omega-\frac{g^2\Delta t}{2}\right)\hat{n}^2\tau\right],\end{aligned} \tag{2.14}$$

where $\tau=N\Delta t$, and we have transformed the expression of the superoperator to the Schrödinger picture. The factor $g^2\Delta t$ in Eqs. (2.11) and (2.14) represents the magnitude of coupling between the field and the detector, and in the following discussion we will use the symbol λ for this quantity. That is,

$$\lambda\equiv g^2\Delta t. \tag{2.15}$$

Since the coupling constant λ is proportional to Δt, the probability of more than one photon being detected can be neglected in the limit $\Delta t\to 0$.

III. STATE EVOLUTION IN A REFERRING MEASUREMENT PROCESS

In the preceding section we obtained two superoperators, $\mathcal{J}$ and $\mathcal{S}$, that describe the one-count and no-count processes. As we shall see, these superoperators give both the probability for each process and the state immediately after the process. In the following two subsections we examine the effects of one-count and no-count processes on the state change of the measured field, and we formulate continuous measurement as a time sequence of these two processes. We also show that the measured state reduces continuously to a number state rather than a mixture of number states. Since we treat only the photon field in the following, we will henceforth omit the subscript f, which stands for the photon field.

A. One-count process

The super-operator $\mathcal{J}$ gives both the probability for the one-count process and the state of the photon field immediately afterward. We use Eq. (2.11) to express the probability $P(\mathcal{J})dt$ that the one-count process occurs within a time dt as

$$P(\mathcal{J})dt\equiv{\rm Tr}[\mathcal{J}\hat{\rho}(t)]dt=\lambda\langle n^2(t)\rangle dt, \tag{3.1}$$

where

$$\langle n^k(t)\rangle\equiv{\rm Tr}[\hat{\rho}(t)(\hat{a}^\dagger\hat{a})^k] \tag{3.2}$$

is the kth photon-number moment just before the one-count process. The superoperator $\mathcal{J}$ also gives the density operator immediately after the one-count process in terms of the premeasurement density operator. Combining Eq. (2.10) with Eq. (2.11) yields

$$\hat{\rho}(t^+)=\frac{\mathcal{J}\hat{\rho}(t)}{{\rm Tr}[\mathcal{J}\hat{\rho}(t)]}=\frac{\hat{n}\hat{\rho}(t)\hat{n}}{\langle n^2(t)\rangle}, \tag{3.3}$$

where t^+ denotes a time infinitesimally later than t. The average photon number immediately after the one-count process is thus given as

$$\langle n(t^+)\rangle \equiv \mathrm{Tr}[\hat\rho(t^+)a^\dagger a] = \frac{\langle n^3(t)\rangle}{\langle n^2(t)\rangle}. \quad (3.4)$$

This equation shows that the average photon number of the postmeasurement state depends on higher-order moments of the premeasurement photon statistics. The difference between the average photon numbers before and after the one-count process is therefore given by

$$\langle n(t^+)\rangle - \langle n(t)\rangle = \frac{\langle n^3(t)\rangle - \langle n^2(t)\rangle\langle n(t)\rangle}{\langle n^2(t)\rangle}. \quad (3.5)$$

If the field is not in a number state, it is easy to verify that $\langle n^3(t)\rangle > \langle n^2(t)\rangle\langle n(t)\rangle$. This means that after the one-count process, the average number of photons remaining in the field is *increased* even though the photon was neither absorbed nor emitted by the detector atom. If the field is in the number state, however, the photon number does not change during the one-count process. This feature is characteristic of a nondemolition measurement of the photon number. Table I lists, for typical initial states, the change in the average photon number for the one-count process.

TABLE I. Change in the average photon number by the one-count process.

Initial value	After one-count process	States
$\langle n\rangle$	$\langle n\rangle$	number
$\langle n\rangle$	$\langle n\rangle + 2 - \frac{1}{\langle n\rangle+1}$	coherent
$\langle n\rangle$	$3\langle n\rangle + \frac{3\langle n\rangle+1}{2\langle n\rangle+1}$	thermal

B. No-count process

The no-count process is characterized by the superoperator $\mathcal{S}_\tau$. The probability $P(\mathcal{S}_\tau)$ of no count being registered in the interval from t to $t+\tau$ is given by

$$P(\mathcal{S}_\tau) = \mathrm{Tr}[\mathcal{S}_\tau\hat\rho(t)]. \quad (3.6)$$

The superoperator $\mathcal{S}_\tau$ also gives the density operator immediately after the no-count process. Using Eqs. (2.14) and (2.15), we express this operator as

$$\hat\rho(t+\tau) = \frac{\mathcal{S}_\tau\hat\rho(t)}{\mathrm{Tr}[\mathcal{S}_\tau\hat\rho(t)]} = \frac{\exp[-(i\omega+\lambda/2)\hat n^2\tau]\hat\rho(t)\exp[(i\omega-\lambda/2)\hat n^2\tau]}{\mathrm{Tr}[\hat\rho(t)\exp(-\lambda\hat n^2\tau)]}. \quad (3.7)$$

Expanding the exponentials of Eq. (3.7) with respect to small $\lambda\tau$ yields a differential equation describing the time evolution of the density operator during the no-count process:

$$\frac{d}{dt}\hat\rho(t) = -(i\omega+\lambda/2)\hat n^2\hat\rho(t) + (i\omega-\lambda/2)\hat\rho(t)\hat n^2 + \lambda\langle n^2(t)\rangle\hat\rho(t). \quad (3.8)$$

From this immediately follows the time evolution of the average photon number during the no-count process:

$$\frac{d}{dt}\langle n(t)\rangle = -\lambda\left[\langle n^3(t)\rangle - \langle n^2(t)\rangle\langle n(t)\rangle\right]. \quad (3.9)$$

Since $\langle n^3(t)\rangle > \langle n^2(t)\rangle\langle n(t)\rangle$ (except for number states), we find that the average photon number *decreases* in time even though the photon number is nondestructively measured. For the number state, though, the photon number remains the same during the no-count process. Again, this preservation of photon number for number states is a characteristic feature of nondemolition measurement.

C. Continuous measurement

Having characterized the one-count process and the no-count process, we are now in a position to describe the time evolution of the photon field when QND photodetection is being carried out *continuously* throughout the measurement period. We refer to such a process as continuous quantum nondemolition (CQND) photodetection. Suppose that the measurement process started at $t=0$ and ended at $t=T$, and that m photons were registered at times τ_j $(j=1,2,\ldots,m)$ with no further photons registered during the measurement period. Then, the density operator of the photon field, $\hat\rho_m^{\mathrm{CQND}}(\tau_1,\tau_2,\ldots,\tau_m;0,T)$, immediately after the measurement process is given by [10]

$$\hat\rho_m^{\mathrm{CQND}}(\tau_1,\tau_2,\ldots,\tau_m;0,T) = \frac{\mathcal{S}_{T-\tau_m}\mathcal{J}\mathcal{S}_{\tau_m-\tau_{m-1}}\mathcal{J}\cdots\mathcal{J}\mathcal{S}_{\tau_1}\hat\rho(0)}{\mathrm{Tr}[\mathcal{S}_{T-\tau_m}\mathcal{J}\mathcal{S}_{\tau_m-\tau_{m-1}}\mathcal{J}\cdots\mathcal{J}\mathcal{S}_{\tau_1}\hat\rho(0)]}, \quad (3.10)$$

where $\hat\rho(0)$ is the initial density operator of the photon field. The denominator here is sometimes called the probability distribution of forward recurrence times (PDF) [10, 22]. This distribution gives the probability per (unit time)^m, $P_m^{(\mathrm{forward})}(\tau_1,\tau_2,\ldots,\tau_m;0,T)$, that from time 0 to T, one-count processes occur only at times t_j $(j=1,2,\ldots,m)$:

$$P_m^{(\mathrm{forward})}(\tau_1,\tau_2,\ldots,\tau_m;0,T) = \mathrm{Tr}[\mathcal{S}_{T-\tau_m}\mathcal{J}\mathcal{S}_{\tau_m-\tau_{m-1}}\mathcal{J}\cdots\mathcal{J}\mathcal{S}_{\tau_1}\hat\rho(0)]. \quad (3.11)$$

Equations (2.11) and (2.14) can be used to show that

$$\mathcal{S}_{T-\tau_m}\mathcal{J}\mathcal{S}_{\tau_m-\tau_{m-1}}\mathcal{J}\cdots\mathcal{S}_{\tau_1}\hat\rho(0) = \lambda^m\exp\left[-\left(i\omega+\frac{\lambda}{2}\right)\hat n^2T\right] \times \hat n^m\hat\rho(0)\hat n^m\exp\left[\left(i\omega-\frac{\lambda}{2}\right)\hat n^2T\right]. \quad (3.12)$$

Substituting Eq. (3.12) into Eq. (3.10) yields

$$\hat{\rho}_m^{\rm CQND}(\tau_1,\tau_2,\ldots,\tau_m;0,T)=\frac{\exp[-(i\omega+\frac{\lambda}{2})\hat{n}^2T]\hat{n}^m\hat{\rho}(0)\hat{n}^m\exp[(i\omega-\frac{\lambda}{2})\hat{n}^2T]}{{\rm Tr}[\hat{\rho}(0)\hat{n}^{2m}\exp(-\lambda\hat{n}^2T)]}. \quad (3.13)$$

This equation gives the nonunitary time evolution of the photon density operator during the CQND measurement of the photon number. Since the right-hand side (rhs) of this equation does not depend on the times at which photons were detected, we may denote the quantity on the left-hand side of Eq. (3.13) simply as $\hat{\rho}_m^{\rm CQND}(T)$. On the other hand, the PDF is given if we substitute Eq. (3.12) into Eq. (3.11):

$$P_m^{\rm (forward)}(\tau_1,\tau_2,\ldots,\tau_m;0,T)=\lambda^m{\rm Tr}[\hat{\rho}(0)\hat{n}^{2m}\exp(-\lambda\hat{n}^2T)]. \quad (3.14)$$

The rhs of this equation, too, does not depend on the times of photocount registration. This is in sharp contrast to conventional photon counting, where the PDF depends explicitly on the times of photocount registration [10]. In conventional photon counting, the photon field attenuates over time [2] and the PDF therefore depends on times of photocount registration. In CQND photon counting, however, the photon field does not attenuate, and the PDF therefore does not depend on times of photocount registration.

In an actual experiment, we are often interested only in the total number of photocounts registered during a measurement period. We shall refer to this kind of process as the quantum photodetection process for the number of counts (QPN) [10]. Since the information concerning the times of photodetection is discarded, the superoperator $\mathcal{N}_T(m)$ describing the QPN is given by

$$\mathcal{N}_T(m)=\int_0^T d\tau_m\int_0^{\tau_m}d\tau_{m-1}\cdots\int_0^{\tau_2}d\tau_1\ \mathcal{S}_{T-\tau_m}\mathcal{J}\ \mathcal{S}_{\tau_m-\tau_{m-1}}\cdots\mathcal{J}\ \mathcal{S}_{\tau_1}. \quad (3.15)$$

Substituting Eq. (3.12) into Eq. (3.15) yields

$$\mathcal{N}_T(m)\hat{\rho}(0)=\frac{(\lambda T)^m}{m!}\exp\left[-\left(i\omega+\frac{\lambda}{2}\right)\hat{n}^2T\right]\hat{n}^m\hat{\rho}(0)\hat{n}^m\exp\left[\left(i\omega-\frac{\lambda}{2}\right)\hat{n}^2T\right]. \quad (3.16)$$

The probability $P(m;0,T)$ of m counts being registered in an interval $[0,T)$ is given by

$$P(m;0,T)={\rm Tr}[\mathcal{N}_T(m)\hat{\rho}(0)]. \quad (3.17)$$

Substituting Eq. (3.16) into Eq. (3.17), we have

$$P(m;0,T)=\frac{(\lambda T)^m}{m!}{\rm Tr}[\hat{\rho}(0)\hat{n}^{2m}\exp(-\lambda\hat{n}^2T)]. \quad (3.18)$$

The density operator $\hat{\rho}_m^{\rm QPN}(T)$ immediately after the QPN is therefore given by

$$\hat{\rho}_m^{\rm QPN}(T)=\frac{\mathcal{N}_T(m)\hat{\rho}(0)}{{\rm Tr}[\mathcal{N}_T(m)\hat{\rho}(0)]}. \quad (3.19)$$

Substituting Eq. (3.16) into Eq. (3.19) yields

$$\hat{\rho}_m^{\rm QPN}(T)=\frac{\exp[-(i\omega+\frac{\lambda}{2})\hat{n}^2T]\hat{n}^m\hat{\rho}(0)\hat{n}^m\exp[(i\omega-\frac{\lambda}{2})\hat{n}^2T]}{{\rm Tr}[\hat{\rho}(0)\hat{n}^{2m}\exp(-\lambda\hat{n}^2T)]}. \quad (3.20)$$

This result is identical to Eq. (3.13). That is, with respect to the postmeasurement state, the CQND and QPN give the same result.

D. Purity preservation

According to a conventional theory of measurement, which is categorized as Pauli's first-kind measurement, the measured state instantaneously collapses into a mixture of states corresponding to all possible readouts. Even if the initial state is pure, its purity is, in general, not preserved by the measurement process. When, however, a specific value of the readout is known, an initially pure state collapses into a pure state.

This feature can be generalized to our theory of continuous measurement, in which the measurement process proceeds continuously. The final state of a continuous measurement process is a mixed state if we do not keep the readout of the continuous measurement. If, however, we know the readout of the continuous measurement and use it to renormalize the initial density operator, the state evolves in a totally different way. If the initial state is a pure state, it does not collapse into a mixture but remains pure as long as we utilize all readout information to renormalize the initial density operator, that is, as long as we are considering the *referring measurement process* [10–12]. In fact, if the initial state is pure, it can be written as $\hat{\rho}(0)=|\psi\rangle\langle\psi|$. Then the square of the density operator after the CQND photodetection is given by

$$[\hat{\rho}_m^{\rm CQND}(T)]^2=\frac{\exp\left[-\left(i\omega+\frac{\lambda}{2}\right)\hat{n}^2T\right]\hat{n}^m|\psi\rangle\langle\psi|\hat{n}^me^{-\lambda\hat{n}^2T}\hat{n}^m|\psi\rangle\langle\psi|\hat{n}^m\exp\left[\left(i\omega-\frac{\lambda}{2}\right)\hat{n}^2T\right]}{\{{\rm Tr}\,[|\psi\rangle\langle\psi|\hat{n}^{2m}\exp\,(-\lambda\hat{n}^2T)]\}^2}. \quad (3.21)$$

Since $\mathrm{Tr}\,[|\psi\rangle\langle\psi|\hat{n}^m e^{-\lambda\hat{n}^2 T}\hat{n}^m] = \langle\psi|\hat{n}^m e^{-\lambda\hat{n}^2 T}\hat{n}^m|\psi\rangle$, the photon density operator satisfies the idempotency condition

$$[\hat{\rho}_m^{\mathrm{CQND}}(T)]^2 = \hat{\rho}_m^{\mathrm{CQND}}(T), \tag{3.22}$$

for arbitrary time $T \geq 0$ and for any number m of photocounts. Thus we find that, if the initial state is pure, it keeps its purity during the measurement process, even though the state evolves nonunitarily. This can be intuitively understood as follows: Since we use all readout information to renormalize the initial density operator, there is no dissipation of information. We will show in Sec. V that the density operator becomes diagonalized only if some of the available information is discarded.

IV. APPLICATIONS TO TYPICAL QUANTUM STATES

This section uses the general formulas obtained in the preceding sections to study the nonunitary time evolution of the photon density operator for three typical quantum states: coherent, thermal, and squeezed.

A. Coherent state

The density operator of the coherent state is given by $\hat{\rho}(0) = |\alpha_0\rangle\langle\alpha_0|$, where $|\alpha_0\rangle$ is a coherent-state vector with complex amplitude α_0. This state can be expressed in the number-state basis as

$$|\alpha_0\rangle = \exp\left(-\frac{|\alpha_0|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}|n\rangle. \tag{4.1}$$

The nonunitary time evolution of the initially coherent state in the CQND is obtained by substituting Eq. (4.1) into Eq. (3.13). Using the resultant density operator, we can calculate the time evolution of the initially coherent state. Since these calculations are straightforward, we do not write them out, but instead show the calculated figures. Figures 1(a)–1(c) show the time evolution of the average photon number $\langle n(t)\rangle$, the photon-number variance $\langle[\Delta n(t)]^2\rangle$, and the Fano factor $F(t) \equiv \langle[\Delta n(t)]^2\rangle/\langle n(t)\rangle$, where the one-count processes are assumed to occur at τ_j ($j = 1, 2, 3, 4, 5$).

Figure 1(a) shows that the average photon number does not decrease over the long term, but remains relatively constant. This is because in CQND photon counting photons are not absorbed. Figure 1(b) shows that the variance of the photon number decreases monotonically over time, which results in the monotonic decrease of the Fano factor as shown in Fig. 1(c). Thus we find that the initially coherent state continuously reduces to a number state.

B. Thermal state

The density operator of the thermal state is given by

$$\hat{\rho}(0) = \frac{1}{1+n_0}\sum_{n=0}^{\infty}\left(\frac{n_0}{1+n_0}\right)^n|n\rangle\langle n|, \tag{4.2}$$

where n_0 is the average photon number of the initial state. The nonunitary time evolution of the initially thermal state in the CQND is obtained by substituting Eq. (4.2) into Eq. (3.13). Figures 2(a)–2(c) show the time evolution of the average photon number, the photon-number variance, and the Fano factor for an initially thermal state. It is remarkable that the average

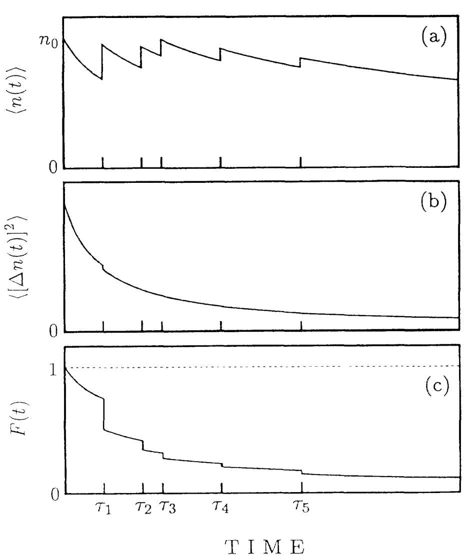

FIG. 1. Time evolution of an initially coherent state: (a) average photon number $\langle n(t)\rangle$, (b) photon-number variance $\langle[\Delta n(t)]^2\rangle$, and (c) Fano factor $F(t)$. One-count processes are assumed to occur at $\tau_1, \tau_2, \ldots$.

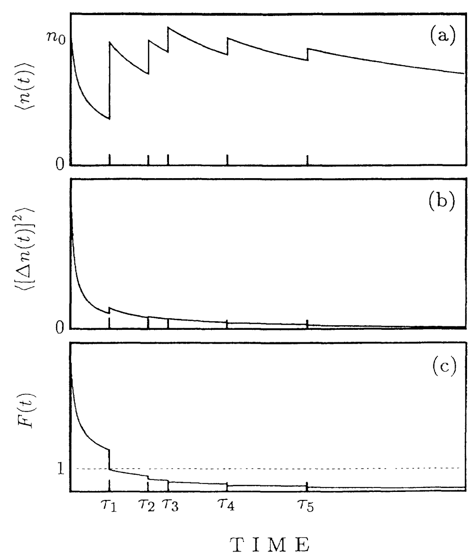

FIG. 2. Time evolution of an initially thermal state: (a) average photon number $\langle n(t)\rangle$, (b) photon-number variance $\langle[\Delta n(t)]^2\rangle$, and (c) Fano factor $F(t)$. One-count processes are assumed to occur at $\tau_1, \tau_2, \ldots$.

photon number jumps drastically upwards when the first photon is registered. This can be understood as follows. The thermal state has a power-law photon-number distribution, and therefore has a maximum probability for the vacuum state. This possibility should vanish as soon as the one-count process occurs. The vanishing probability is then redistributed over other states when the density operator is renormalized according to Eq. (3.3), thus resulting in a drastic increase in the average photon number. A similar increase in the average photon number occurs in conventional photon counting [12]. An essential difference between conventional photon counting and CQND photon counting lies in the time development of the Fano factor; in the former case the Fano factor approaches unity, reflecting the fact that the photon state eventually reduces to the vacuum state, whereas in the latter case the Fano factor approaches zero, reflecting the fact that the photon state reduces to a number state.

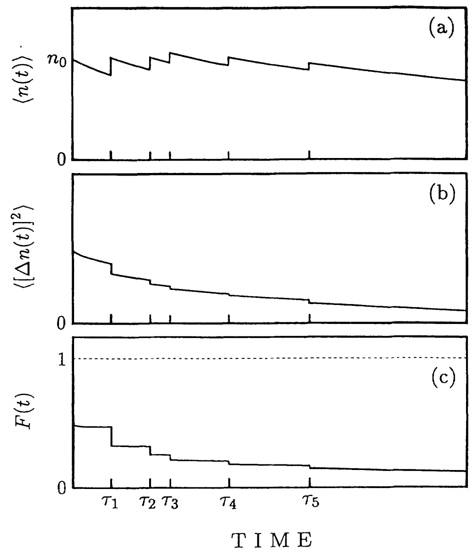

FIG. 3. Time evolution of an initially squeezed state: (a) average photon number $\langle n(t)\rangle$, (b) photon-number variance $\langle[\Delta n(t)]^2\rangle$, and (c) Fano factor $F(t)$. One-count processes are assumed to occur at $\tau_1, \tau_2, \ldots$.

C. Squeezed state

A squeezed state of light, $|\alpha, r\rangle$, can be generated from a coherent state $|\alpha\rangle$ by means of a unitary transformation [23]:

$$\begin{aligned}|\alpha,r\rangle &\equiv \exp\left\{\frac{r}{2}\left[a^2-(a^\dagger)^2\right]\right\}|\alpha\rangle \\ &= \frac{\exp\left[-\frac{|\alpha|^2}{2}+\frac{\alpha^2}{2}\tanh r\right]}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left(\frac{\tanh r}{2}\right)^{n/2}H_n\left(\frac{\alpha}{\sqrt{\sinh 2r}}\right)|n\rangle, \qquad (4.3)\end{aligned}$$

where r is a squeezing parameter and $H_n(z)$ is the nth Hermite polynomial defined as

$$H_n(z)=\sum_{m=0}^{[n/2]}\frac{(-1)^m n!}{m!(n-2m)!}(2z)^{n-2m} \qquad (4.4)$$

with $[n/2] = n/2$ for n even, $(n-1)/2$ for n odd. The nonunitary time evolution of the initially squeezed state during the CQND measurement is obtained by substituting Eq. (4.3) into Eq. (3.13). Figures 3(a)–3(c) show the time evolution of the average photon number, the photon-number variance, and the Fano factor for an initially squeezed state.

In these Figs. 1–3, the average number of photons remaining in the field increases when one photon is detected, whereas it decreases when no photon is detected. These features confirm the general arguments in Sec. III. Although each of these states eventually reduces to a number state, the intermediate-state evolution greatly differs from one state to another. Thus we find that the way the state reduces towards a number state depends strongly on the initial photon statistics.

V. STATE EVOLUTION IN A NONREFERRING MEASUREMENT PROCESS

A. General formalism

So far we have considered the situation in which we use all readout information to renormalize the density operator of the field. On the other hand, there also is a situation where the detector is switched on but the readout information is discarded (we shall refer to this kind of process as a *nonreferring measurement process*). Is there any difference in state evolution between these two kinds of measurement processes, even though the physical situation seems to be the same? The answer is yes, as we shall show in the following.

Although the readout of the apparatus meter (one-count or no-count) is not referred in the nonreferring process, either the one-count process or the no-count process must occur in each infinitesimal interval dt. The state evolution in the nonreferring process can therefore be expressed as a statistical summation of these two possible processes:

$$\hat{\rho}(t+dt)=P(\mathcal{J})dt\frac{\mathcal{J}\hat{\rho}(t)}{\mathrm{Tr}[\mathcal{J}\hat{\rho}(t)]}+P(\mathcal{S}_{dt})\frac{\mathcal{S}_{dt}\hat{\rho}(t)}{\mathrm{Tr}[\mathcal{S}_{dt}\hat{\rho}(t)]}, \qquad (5.1)$$

where $P(\mathcal{J})dt$ at and $P(\mathcal{S}_{dt})$ are the statistical weights for the one-count process and the no-count process, and $\hat{\rho}(t)$ is the photon density operator that has evolved from the initial state $\hat{\rho}(0)$ in a nonreferring measurement process. By using the formulas in Sec. III for $P(\mathcal{J})$, $\mathcal{J}\hat{\rho}$, $P(\mathcal{S}_{dt})$, and $\mathcal{S}_{dt}\hat{\rho}$, we can rewrite Eq. (5.1) as

$$\hat{\rho}(t+dt) = \hat{\rho}(t) - \frac{\lambda}{2} dt \hat{n}^2 \hat{\rho}(t) - \frac{\lambda}{2} dt \hat{\rho}(t) \hat{n}^2 + \lambda dt \hat{n} \hat{\rho}(t) \hat{n}. \tag{5.2}$$

This leads to a differential equation

$$\frac{d\hat{\rho}}{dt} = -\frac{\lambda}{2} \left(\hat{\rho}\hat{n}^2 + \hat{n}^2 \hat{\rho} \right) + \lambda \hat{n} \hat{\rho} \hat{n}. \tag{5.3}$$

B. Diagonalization of the density operator

The solution of this master equation is given by [24]

$$\rho_{kl}(t) = \exp\left[-\frac{\lambda}{2} (k-l)^2 t \right] \rho_{kl}(0), \tag{5.4}$$

which shows that the photon density operator becomes diagonalized as the nonreferring measurement process proceeds. This is in sharp contrast to the referring measurement process in which an initially pure state keeps its purity (see Sec. III D).

C. State evolution in operator form

Let us define a super-operator $\mathcal{T}_\tau$ describing the time evolution of the photon density operator during a time τ

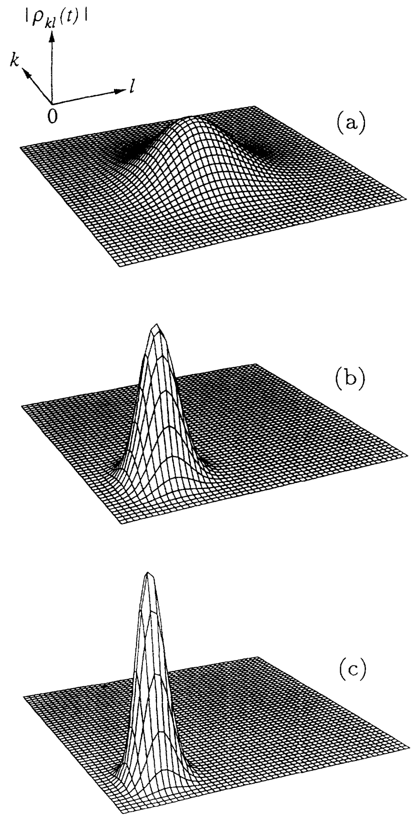

FIG. 4. State reduction to a number state in the referring measurement process. As time progresses, the density matrix in the number-state basis $[\rho_{kl}(t) = \langle k|\hat{\rho}_m^{\rm CQND}(t)|l\rangle]$ tends to be sharply peaked. The initial state is chosen to be a coherent state with $|\alpha_0|^2 = 23$: (a) $\lambda t = 0$, $m = 0$, (b) $\lambda t = 0.05$, $m = 2$; and (c) $\lambda t = 0.1$, $m = 3$.

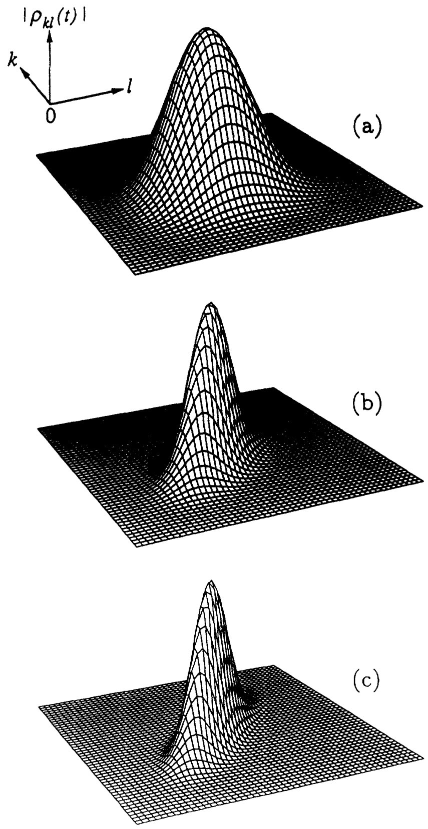

FIG. 5. Diagonalization of the photon density matrix in the number-state basis $[\rho_{kl}(t) = \langle k|\hat{\rho}^{\rm NMP}(t)|l\rangle]$ in the nonreferring measurement process. The initial state is chosen to be a coherent state with $|\alpha_0|^2 = 23$: (a) $\lambda t = 0$, (b) $\lambda t = 0.05$, and (c) $\lambda t = 0.1$.

in the nonreferring process such that

$$\hat{\rho}(t+\tau) = \frac{\mathcal{T}_\tau \hat{\rho}(t)}{{\rm Tr}[\mathcal{T}_\tau \hat{\rho}(t)]}. \tag{5.5}$$

Since we do not refer to the number of counts, we have

$$\mathcal{T}_\tau = \sum_{m=0}^{\infty} \mathcal{N}_\tau(m). \tag{5.6}$$

Clearly, $\mathcal{T}_\tau$ satisfies the following identity:

$${\rm Tr}[\mathcal{T}_\tau \hat{\rho}(t)] = 1. \tag{5.7}$$

From Eq. (3.16), we find that the state evolution in the nonreferring measurement process (NMP) is given in operator form as

$$\hat{\rho}^{\rm NMP}(T) = \sum_{m=0}^{\infty} \frac{(\lambda T)^m}{m!} \exp\left[-\left(i\omega + \frac{\lambda}{2} \right) \hat{n}^2 T \right] \hat{n}^m \hat{\rho}(0) \hat{n}^m \times \exp\left[\left(i\omega - \frac{\lambda}{2} \right) \hat{n}^2 T \right]. \tag{5.8}$$

Figures 4 and 5 illustrate the time evolution of the photon density operator for an initially coherent state in the referring and nonreferring measurement processes. We see that the state in Fig. 4 reduces to a number state, whereas the state in Fig. 5 collapses to a mixture of number states.

VI. ESTIMATION OF PHOTON NUMBER

Equations (3.3) and (3.7) show that if the initial state is a number state, that is,

$$\hat{\rho}(0) = |n_0\rangle\langle n_0|, \tag{6.1}$$

the state changes neither at the one-count process nor at the no-count process. Hence Eq. (3.1) shows that the one-count process occurs in each infinitesimal interval dt with the same probability $\lambda n_0^2 dt$. Denoting by $m(T)$ the number of counts that have been registered during the period from $t = 0$ to T, we can see from the law of large numbers that $m(T)/T$ converges to λn_0^2 as $T \to \infty$, and therefore we can estimate the photon number n_0 by $\sqrt{m(T)/\lambda T}$ when T is sufficiently large. This suggests that even for an arbitrary initial state $\hat{\rho}(0)$, the quantity

$$n^{\text{estimate}}(T) \equiv \sqrt{\frac{m(T)}{\lambda T}}. \tag{6.2}$$

can be regarded as an estimate of photon number. We justify this in the following two subsections by showing that in the limit of $T \to \infty$ both the probability distribution and the state reduction for the measurement of n^{estimate} coincide with those for the first-kind measurement of the photon number. These results reveal that the measurement scheme proposed in this paper actually provides a QND measurement of photon number.

A. Probability distribution

This subsection shows that the probability distribution of n^{estimate} converges to the initial photon-number distribution as $T \to \infty$, or equivalently, that the probability distribution of $m(T)/\lambda T$ converges to that of n^2. We prove this by using characteristic functions of probability distributions [25, 26].

The characteristic function for the variable $m(T)/\lambda T$ is defined as

$$C_T(\omega) = \sum_{m=0}^{\infty} \exp\left(i\omega \frac{m}{\lambda T}\right) P(m; 0, T). \tag{6.3}$$

Substituting Eq. (3.18) into this yields

$$C_T(\omega) = \sum_{n=0}^{\infty} \exp\left\{\lambda T n^2 \left[\exp\left(i\frac{\omega}{\lambda T}\right) - 1\right]\right\} \rho_{nn}(0). \tag{6.4}$$

Taking the limit $T \to \infty$ yields

$$\lim_{T\to\infty} C_T(\omega) = \sum_{n=0}^{\infty} \exp\left(in^2\omega\right) \rho_{nn}(0), \tag{6.5}$$

which is identical to the characteristic function for the probability distribution of n^2 for the initial state. Thus we find that the quantity defined in Eq. (6.2) serves as an estimate of photon number for an arbitrary initial state.

B. Collapse to a number state

The result of the preceding subsection shows that in each sequence of the measurement process, n^{estimate} converges to a natural number n_0 as $T \to \infty$, although n_0 is not predictable at the beginning of each sequence but distributes stochastically according to the initial photon-number distribution. In this subsection we show that in each sequence the state $\hat{\rho}_{m(T)}^{\text{CQND}}(T)$ reduces continuously to a number state whose eigenvalue is determined by

$$n_0 = \lim_{T\to\infty} \sqrt{\frac{m(T)}{\lambda T}}. \tag{6.6}$$

Here we can assume that

$$\rho_{n_0,n_0}(0) > 0 \tag{6.7}$$

because an event of probability 0 can be ignored. Using Eq. (3.13), we can express the (k,l)-matrix element of $\hat{\rho}_{m(T)}^{\text{CQND}}(T)$ as

$$[\hat{\rho}_{m(T)}^{\text{CQND}}(T)]_{kl} = \frac{\gamma_k(T)\gamma_l^*(T)\rho_{kl}(0)}{\sum_{n=0}^{\infty}|\gamma_n(T)|^2\rho_{nn}(0)}, \tag{6.8}$$

where

$$\gamma_k(T) = k^{m(T)} \exp\left[-\left(i\omega + \frac{\lambda}{2}\right)k^2 T\right]. \tag{6.9}$$

We now show that if $k \neq n_0$, Eq. (6.6) leads to

$$\lim_{T\to\infty} \frac{\gamma_k(T)}{\gamma_{n_0}(T)} = 0. \tag{6.10}$$

It follows immediately from Eqs. (6.7), (6.8), and (6.10) that $\lim_{T\to\infty}[\hat{\rho}_{m(T)}^{\text{CQND}}(T)]_{kl}$ vanishes unless $k = l = n_0$. The convergence of $\hat{\rho}_{m(T)}^{\text{CQND}}$ to $|n_0\rangle\langle n_0|$ is thus proved.

For $n_0 \geq 1$, we have

$$\left|\frac{\gamma_k(T)}{\gamma_{n_0}(T)}\right| = \exp\left[-\frac{\lambda T}{2}(k^2 - n_0^2) + m(T)\ln\frac{k}{n_0}\right]. \tag{6.11}$$

Equation (6.6) can be rewritten as

$$m(T) = \lambda T n_0^2 + o(T), \tag{6.12}$$

and substituting this into Eq. (6.11) yields

$$\left|\frac{\gamma_k(T)}{\gamma_{n_0}(T)}\right| = \exp[-AT + o(T)], \tag{6.13}$$

where

$$A = \frac{\lambda}{2}\left\{(k^2 - n_0^2) - n_0^2 \ln\left(\frac{k}{n_0}\right)^2\right\}. \tag{6.14}$$

From a well-known inequality

$$x - 1 > \ln x \qquad \text{if} \quad x > 0 \quad \text{and} \quad x \neq 1, \tag{6.15}$$

it is easily shown that $A > 0$ if $k \neq n_0$. Thus Eq. (6.10) is obtained from Eq. (6.13).

We now consider two cases in which $n_0 = 0$: (i) $m(T) = 0$ for $\forall T$, and (ii) $\exists t_0$ such that $m(T) \geq 1$ for $\forall T \geq t_0$.

In case (i), we have $\gamma_0(T) = 1$ for $\forall T$ and $\lim_{T\to\infty} \gamma_k(T) = 0$ if $k \geq 1$. Thus Eq.(6.10) for $n_0 = 0$ is verified.

In case (ii), on the other hand, we have

$$\gamma_0(T) = 0 \quad \text{for} \quad \forall T \geq t_0, \tag{6.16}$$

and Eq. (6.10) does not hold for $n_0 = 0$. Indeed, it can be shown that in this case Eq. (6.10) holds for $n_0 = 1$ and hence $\hat{\rho}^{\mathrm{CQND}}_{m(T)}$ converges to $|1\rangle\langle 1|$ as $T \to \infty$. This might seem to contradict our claim in this subsection, but case (ii) can be neglected for the following reason. Equation (6.6) for $n_0 = 0$ implies that

$$\lim_{T\to\infty} \sqrt{\frac{m(T) - m(t_0)}{\lambda(T - t_0)}} = 0. \tag{6.17}$$

According to the result of the preceding subsection, the probability that this convergence occurs is $[\hat{\rho}^{\mathrm{CQND}}_{m(t_0)}(t_0)]_{00}$, which according to Eqs. (6.8) and (6.16), vanishes in case (ii).

VII. COMPUTER SIMULATION

In the foregoing analysis, we implicitly assumed that we know the initial state and that the times of one-count processes are given. It is, on the other hand, interesting to simulate the continuous QND measurement process that we may actually observe.

Figure 6 illustrates some examples of computer simulations for an initially coherent state, where the counting pulses are produced by a random noise generator according to the probabilities of one-count and no-count processes given by Eqs. (3.1) and (3.6). The ordinate shows the photon number estimated according to Eq. (6.2), where $N_{\mathrm{count}}(t)$ is the number of one-count processes that have occurred from time 0 to t. This number is a value that can actually be obtained by observation, and can therefore be used by an observer as an estimated photon number (because the observer does not know the initial state).

Each path in Fig. 6 corresponds to a different initial value (seed value) for the random noise generator, and traces an individual continuous QND measurement of photon number in the referring measurement process. This figure shows that the estimated photon number fluctuates immediately after the measurement starts, but that these fluctuations diminish as the measurement proceeds. Figures 7(a) and 7(b) show the probability distribution of the photon number at time 0 and at a later time for one path. We see that the photon-number distribution becomes narrower, indicating the reduction to a number state. This result is consistent with the curves shown in Fig. 1.

The final values of n^{estimate} become distributed when we change the initial value of the random noise generator. This reflects a stochastic nature of light, but the computer simulations confirm that, as the measurement proceeds, the distribution of n^{estimate} tends to coincide with the initial photon-number distribution.

VIII. DISCUSSION

We have shown that in the referring measurement process, any initial state eventually reduces to a number state whose eigenvalue depends on the rate of photodetection and differs from one sequence of measurement to another. Although these eigenvalues are distributed, we have shown that their probability distribution coincides with the initial photon-number distribution.

We have also shown that the average photon number remaining in the field *increases* when a photon is detected and *decreases* otherwise. One may then wonder whether the scheme proposed here is really a QND measurement

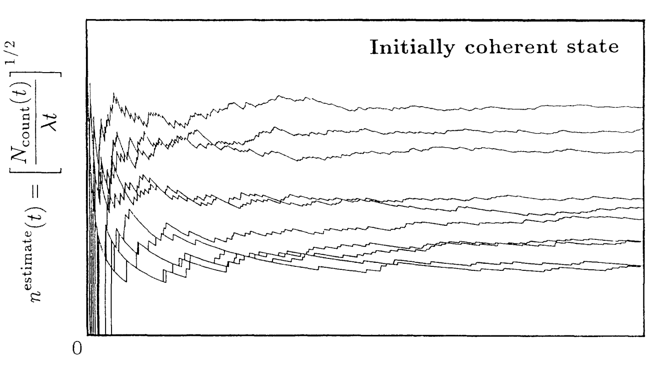

FIG. 6. Computer simulation of the continuous QND measurement of photon number. Each path corresponds to a different seed value for the random-noise generator.

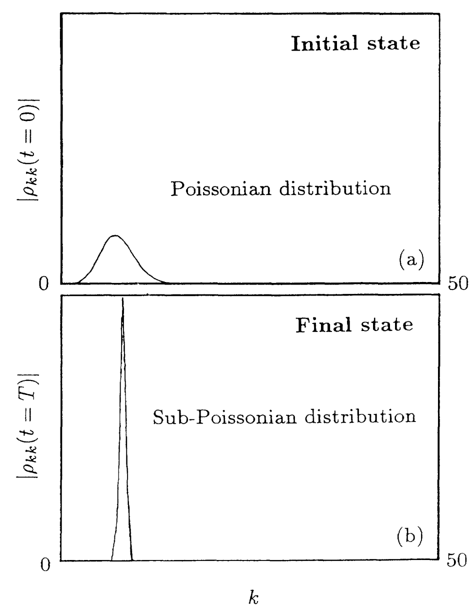

FIG. 7. Initial and final photon-number distribution calculated by computer simulation.

of photon number at an intermediate state of measurement. Let us show that the photon number is indeed nondestructively measured throughout the measurement process. Let $\langle n^k(T)\rangle_m$ be the kth-order photon-number moment at time T when m photons have been detected. Since the photon density operator at time T is given by $\hat{\rho}_m^{\rm CQND}(T)$, we have

$$\langle n^k(T)\rangle_m = {\rm Tr}[\hat{\rho}_m^{\rm CQND}(T)\hat{n}^k]. \tag{8.1}$$

Substituting Eq. (3.13) into the rhs of Eq. (8.1) yields

$$\langle n^k(T)\rangle_m = \frac{{\rm Tr}[\hat{\rho}(0)\hat{n}^{2m+k}\exp(-\lambda\hat{n}^2T)]}{{\rm Tr}[\hat{\rho}(0)\hat{n}^{2m}\exp(-\lambda\hat{n}^2T)]}. \tag{8.2}$$

Depending on the number of counts and the measurement time, these quantities can, in general, be smaller or larger than their corresponding initial quantities $\langle n^k(0)\rangle_m$. However, this contradicts neither any QND condition nor the conservation laws in quantum mechanics because m is a result for a single measurement. The quantities to be compared in quantum mechanics are those which are *ensemble-averaged* over possible m. The ensemble averaging can be carried out using the probability distribution $P(m;0,T)$ in Eq. (3.18), and it can be shown that

$$\sum_{m=0}^{\infty} P(m;0,T)\langle n^k(T)\rangle_m = \langle n^k(0)\rangle. \tag{8.3}$$

This result shows that photon statistics are conserved during the nonreferring measurement process.

IX. CONCLUSIONS

We have proposed an operational theory for continuous QND measurement of photon number and studied how the measured photon field evolves during the measurement process. When the readout information is used to renormalize the initial density operator, the average photon number remaining in the field increases when a photon is detected and decreases when none is detected, but the state eventually reduces to a number state whose eigenvalue is uniquely determined by the rate of photodetection. Although the final state differs from one measurement to another and is not predictable, its probability distribution coincides with the initial photon-number distribution. Furthermore, if the initial state is a pure state, it remains pure throughout the measurement process. The measured state collapses into a mixture of number states only when the available information is discarded.

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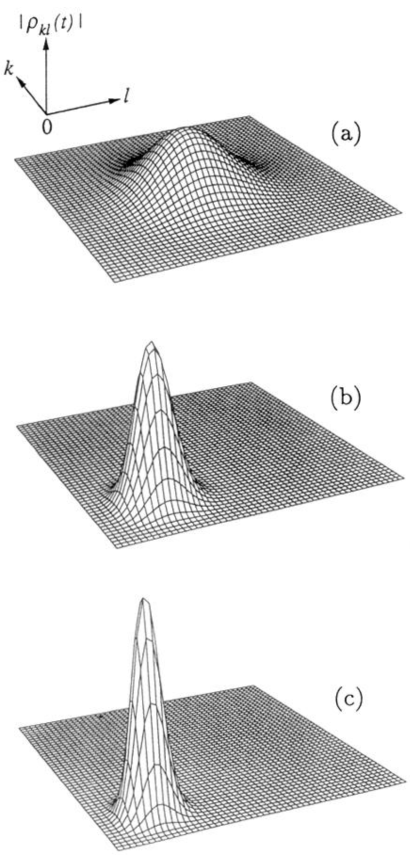

FIG. 4. State reduction to a number state in the referring measurement process. As time progresses, the density matrix in the number-state basis $[\rho_{kl}(t) = \langle k|\hat{\rho}_m^{\mathrm{CQND}}(t)|l\rangle]$ tends to be sharply peaked. The initial state is chosen to be a coherent state with $|\alpha_0|^2 = 23$: (a) $\lambda t = 0,\ m = 0$, (b) $\lambda t = 0.05,\ m = 2$; and (c) $\lambda t = 0.1,\ m = 3$.

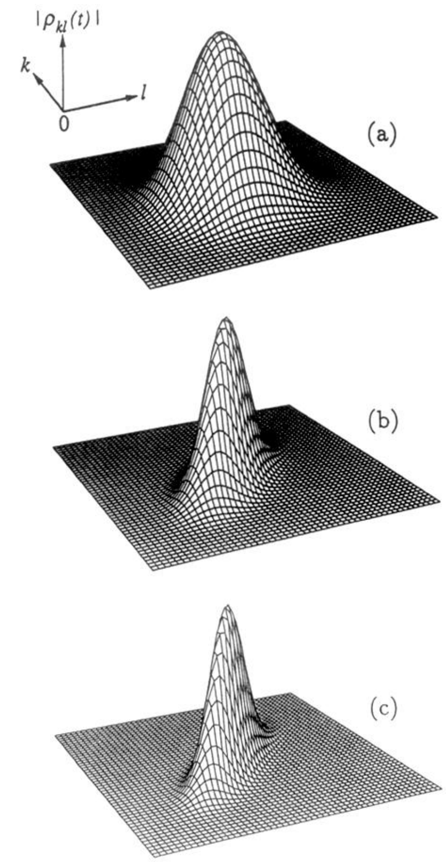

FIG. 5. Diagonalization of the photon density matrix in the number-state basis $[\rho_{kl}(t) = \langle k|\hat{\rho}^{\rm NMP}(t)|l\rangle]$ in the nonreferring measurement process. The initial state is chosen to be a coherent state with $|\alpha_0|^2 = 23$: (a) $\lambda t = 0$, (b) $\lambda t = 0.05$, and (c) $\lambda t = 0.1$.