Generalized linear input-output theory for quantum fluctuations

Jean-Michel Courty and Serge Reynaud
Laboratoire de Spectroscopie Hertzienne, Université Pierre et Marie Curie, Case 74, 4 place Jussieu, F-75252 Paris CEDEX 05, France
(Received 10 February 1992)

Scattering of quantum fluctuations by a linear system is entirely characterized by impedance functions that also allow the determination of the system fluctuations. We present a generalization of this linear input-output theory to the case of a nonlinear scattering system. We define response functions that are similar to the susceptibility functions of linear response theory, but behave as noncommuting quantities. These functions contain a "dynamical" part determined by the relaxation of the system and a "structural" one related to the commutators between the system observables. The generalized linear input-output theory provides us with a complete description of the system fluctuations as well as of the output reservoir fields. The consistency of the results is ensured by general relations existing between the response functions and the correlation functions.

PACS number(s): 42.50.Lc, 05.40.+j

I. INTRODUCTION

It is now well established that nonlinear-optical sources are able to squeeze the quantum fluctuations of one quadrature component of the electromagnetic field below the noise level of vacuum fluctuations [1] and large squeezing factors have been demonstrated experimentally [2].

In standard theoretical methods of quantum optics [3,4], the computation of such effects is based on a coherent-state representation of the fields (a semiclassical representation where the phase-space momenta correspond to the normally ordered quantum momenta). The evolution of the semiclassical distribution is given by a Fokker-Planck equation with nonpositive diffusion coefficients. A linearization procedure is used for computing the field correlation functions. The correlation functions of the output field are then deduced from the intracavity ones [5]. For nonlinear-optical systems containing atoms excited not too far from resonance, it is important to treat properly the atomic fluctuations [6]. This can be done by making use of the coherent-state representation [7–9].

For the optical parametric amplifiers, when there are no atomic fluctuations, it is possible to derive the output fluctuations by a much simpler technique, studying directly the linear transformation of the incident fluctuations by the system [10]. This semiclassical technique is quite analogous to the Nyquist theory of noise in electrical systems [11,12].

The same intuitive approach can be used for oscillators operated above threshold (optical parametric oscillators [13]). As in the "linear stability analysis" [14], one first obtains a working point by neglecting the effect of the fluctuations. Then, one describes the evolution of the field fluctuations by the classical equations linearized in the vicinity of this working point. Finally, one uses the classical reflection-transmission equations to compute the output fluctuations by regarding all fluctuations as driven by classical random fields incident on the mirrors. These incident fluctuations are chosen to fit the Wigner representation [15,16] (the phase-space momenta correspond to the symmetrically ordered quantum momenta) rather than the coherent-state representation.

The semiclassical linear input-output theory has been used for computing the generation of twin photon beams by the nondegenerate optical parametric oscillator [17], by bistable devices built on one-photon [18] or two-photon [19] parametric processes. It has been shown to be equivalent to the standard approach (using a coherent-state representation) for any pure Kerr processes (parametric processes without atomic fluctuations) in the limiting case of a high-Q cavity [20]. It has also been used for studying low-Q cavities [18]. Related techniques, using semiclassical Langevin equations, have allowed study of the statistical properties of diode lasers with controlled current noise [21] and of electrical systems containing nonlinear components [12,22]. Extra classical fluctuations in the pump fields such as those associated with the Schallow-Townes diffusion process have also been treated by using this method [23].

It has been shown recently [24,25] that the treatment of atomic fluctuations can be done in the same spirit by using the techniques of linear-response theory [26,27]. The parametric transformation of the fields by the atomic medium is described by susceptibility functions while the noise generation is described by atomic correlation functions. The correlation functions of the output fields are deduced from these susceptibility and correlation functions which are themselves calculated by using the techniques of resonance fluorescence theory [28,29]. As in the semiclassical method, these functions are calculated at a stable working point obtained from a mean-field analysis. An interesting property of this technique is that it does not rely upon a particular semiclassical representation of the fields; it deals directly with the quantum Langevin equations, as is usual in resonance fluorescence theory [28].

In the particular case where the scattering system is a harmonic oscillator, it is well known that it is entirely characterized by some impedance functions which are closely related to the susceptibility functions of linear-response theory. The knowledge of these functions provides us with a complete and consistent description of the system and reservoir fluctuations. The linear input-output theory introduced previously [24,25] has been used in situations where the scatterer is a nonlinear system. However, the susceptibility functions were still behaving as classical quantities (commuting numbers).

The purpose of the present paper is to show that the development of linear- response techniques can be pushed one step further. We introduce response functions behaving as noncommuting operators when applied to the various observables of the nonlinear scatterer. These functions contain a "dynamical" part and a "structural" one. The dynamical part is determined by relaxation theory and it contains all the frequency dependence of the response functions. The structural part is related to the noncommutativity of the commutators between system observables.

Simplification occurs in the limiting cases (i) of a linear scatterer where the commutators between system observables are classical numbers; (ii) of the usual linear-response theory [26] which depends only upon the mean value of the response operators; (iii) when one studies quantum fluctuations of the fields emitted in a "weakly coupled" reservoir [30].

In a first part of the paper, we recall some well-known properties in the particular case of a linear scatterer and their description in terms of impedances (Sec. II). Then we come to the problem of a nonlinear system coupled to one or two harmonic reservoirs. We introduce "tensorial" notations which will allow us to write the results for any nonlinear scatterer. We give the expressions which are unchanged when shifting from the linear scatterer to the nonlinear one (Sec. III).

We write the relaxation equations for the nonlinear system coupled to the reservoirs and we compute the correlation functions from resonance fluorescence theory. We deduce the susceptibility functions describing the linear response of the mean values of the system observables to a classical modulation of the incident reservoir fields. Then, we show that the linear response of the system to the quantum fluctuations of the incident reservoir fields is described by noncommuting response operators which have the same dynamical behavior as the susceptibility functions (Sec. IV). We compute the fluctuations of the output fields and derive the input-output transformation. We analyze the particular case of a weakly coupled reservoir (Sec. V). During these derivations, we check that the generalized linear input-output theory provides us with a consistent description of the system and field fluctuations (Secs. IV and V).

In a last part, we summarize the results obtained in this paper (Sec. VI). This conclusion can be used as a starting point for the application of the method to specific problems.

II. CASE OF A LINEAR SYSTEM

In this section, we collect the well-known results corresponding to the damped harmonic oscillator. We present these results for an electrical system ("*RLC* oscillator") where they have an intuitive interpretation in terms of circuit theory and impedance functions [11,12].

A. Damped harmonic oscillator

The Hamiltonian for a harmonic electrical oscillator ("*LC* oscillator") is

$$H_S = Q^2/(2C) + U(\Phi) \ , \quad U(\Phi) = \Phi^2/(2L) \ , \tag{2.1}$$

$$[\Phi, Q] = i\hbar \ , \tag{2.2}$$

where Q is the charge and Φ the magnetic flux. Indeed, the Heisenberg equations associated with this Hamiltonian are the usual equations for the voltage and current, respectively,

$$d_t\Phi = Q/C \ , \quad d_tQ = -\frac{dU}{d\Phi} = -\Phi/L \ . \tag{2.3}$$

It is well known [31] that the damping of the electrical system may be described by coupling it to a collection of harmonic oscillators:

$$H = H_S + H' \ , \tag{2.4}$$

$$H' = \int_0^\infty \frac{d\omega}{2\pi} \hbar\omega [a_\omega - \alpha_\omega \Phi]^\dagger [a_\omega - \alpha_\omega \Phi] \ . \tag{2.5}$$

The annihilation and creation operators of the reservoir describe the quantum modes of a transmission line of impedance R and they obey the usual commutation relations for a monodimensional scalar field:

$$[a_\omega, a^\dagger_{\omega'}] = 2\pi\delta(\omega - \omega') \ , \quad [a_\omega, a_{\omega'}] = [a^\dagger_\omega, a^\dagger_{\omega'}] = 0 \ . \tag{2.6}$$

The coefficients α_ω are related to the resistance R [32]:

$$\alpha_\omega = (\hbar|\omega|R/2)^{-1/2} \ . \tag{2.7}$$

We will consider in fact that the LC system is coupled to two distinct reservoirs A and B, corresponding to two resistances R_A and R_B. The aim of our work is to predict the field fluctuations that go out of the system into the reservoirs. The Hamiltonian is

$$H = H_S + H'_A + H'_B \ , \tag{2.8}$$

where H'_A (respectively, H'_B) is given by the expressions discussed previously with operators a_ω (respectively, b_ω) and coefficients α_ω (respectively, β_ω).

We write this Hamiltonian as

$$H = H_0 + H_1 + H_2 \ , \tag{2.9}$$

where H_k corresponds to the kth order with respect to the coupling constants:

$$H_0 = H_S + H_A + H_B \ , \tag{2.10}$$

$$H_A = \int_0^\infty \frac{d\omega}{2\pi} \hbar\omega a^\dagger_\omega a_\omega \ , \tag{2.11}$$

$$H_1 = -I_A\Phi - I_B\Phi \ , \tag{2.12}$$

$$I_A = \int_0^\infty \frac{d\omega}{2\pi} \hbar\omega [\alpha_\omega a^\dagger_\omega + \alpha^*_\omega a_\omega] \ , \tag{2.13}$$

$$H_2=c_A\Phi^2+c_B\Phi^2 \ , \tag{2.14}$$

$$c_A=\int_0^\infty \frac{d\omega}{2\pi}\hbar\omega\alpha_\omega^*\alpha_\omega \tag{2.15}$$

(same expressions for the B reservoir with the appropriate substitutions); H_A and H_B describe the free evolution of the reservoirs; H_1 describes a linear coupling between the system and the reservoir fields I_A and I_B; H_2 is a self-interaction term.

B. Heisenberg and Langevin equations

The Heisenberg equations for the system are

$$d_t\Phi=Q/C \ , \tag{2.16}$$

$$d_tQ=-\Phi/L+I_A-2c_A\Phi+I_B-2c_B\Phi \ . \tag{2.17}$$

The Heisenberg equations for reservoir A are easily shown to give

$$I_A(t)=I_A^0(t)+\int_{t_0}^t dt' i\xi_{AA}(t-t')\Phi(t') \ , \tag{2.18}$$

$$I_A^0(t)=\int_0^\infty \frac{d\omega}{2\pi}\hbar\omega[\alpha_\omega a_\omega^\dagger e^{i\omega(t-t_0)}+\alpha_\omega^* a_\omega e^{-i\omega(t-t_0)}] \ , \tag{2.19}$$

where t_0 is any initial time. The function ξ_{AA} is given by its Fourier transform:

$$\xi_{AA}[\omega]=\tfrac{1}{2}[\Theta(\omega)-\Theta(-\omega)]\hbar\omega^2\alpha_\omega^*\alpha_\omega=\omega R_A^{-1} \ . \tag{2.20}$$

Throughout the paper, we will consider that a function f is defined in the time domain $[f(t)]$ or in the frequency domain (Kubo's notations $f[\omega]$) and that these two representations are related through

$$f(t)=\int \frac{d\omega}{2\pi}f[\omega]e^{-i\omega t} \ , \tag{2.21}$$

$$f[\omega]=\int dt f(t)e^{i\omega t} \ . \tag{2.22}$$

With the initial time t_0 either far in the past or far in the future, the expression of the reservoir field I_A becomes

$$\begin{aligned} I_A[\omega]&=I_A^{\text{in}}[\omega]+\chi_{AA}[\omega]\Phi[\omega] \\ &=I_A^{\text{out}}[\omega]-\chi_{AA}[-\omega]\Phi[\omega] \ , \end{aligned} \tag{2.23}$$

where χ_{AA} is the retarded propagator for the reservoir field:

$$\chi_{AA}(t)=2i\Theta(t)\xi_{AA}(t) \ . \tag{2.24}$$

The fields I^{in} and I^{out} which have a free evolution with initial conditions, respectively, in the past or in the future of t have to be interpreted as the input and output reservoir fields.

It has to be noted that the constant appearing in the self-energy term is the static susceptibility of the reservoir:

$$2c_A=\chi_{AA}[\omega=0] \tag{2.25}$$

and that

$$\chi_{AA}[\omega]=\chi_{AA}[\omega=0]+i\omega R_A^{-1}=2c_A+i\omega R_A^{-1} \ . \tag{2.26}$$

It follows that the reservoir fields may be written

$$\begin{aligned} I_A(t)&=2c_A\Phi(t)-R_A^{-1}d_t\Phi(t)+I_A^{\text{in}}(t) \\ &=2c_A\Phi(t)+R_A^{-1}d_t\Phi(t)+I_A^{\text{out}}(t) \ . \end{aligned} \tag{2.27}$$

Inserting these results in the Heisenberg equations for the system, one gets the following Langevin equations:

$$d_t\Phi(t)=C^{-1}Q(t) \ , \tag{2.28}$$

$$\begin{aligned} d_tQ(t)&=-L^{-1}\Phi(t)-(R_A^{-1}+R_B^{-1})d_t\Phi(t) \\ &\quad +I_A^{\text{in}}(t)+I_B^{\text{in}}(t) \\ &=-L^{-1}\Phi(t)+(R_A^{-1}+R_B^{-1})d_t\Phi(t) \\ &\quad +I_A^{\text{out}}(t)+I_B^{\text{out}}(t) \ . \end{aligned} \tag{2.29}$$

These equations describe the resistive damping (cumulative effect of the coupling depending upon the coefficients R_A and R_B and the voltage $V=d_t\Phi$) as well as the Johnson-Nyquist current noise (Langevin fluctuations of the electrical current $I=d_tQ$).

C. Input-output relations

From the relations (2.27), one deduces that the input and the output reservoir fields are different and related through the in-out relations:

$$I_A^{\text{out}}(t)=I_A^{\text{in}}(t)-2R_A^{-1}d_t\Phi(t) \ , \tag{2.30}$$

$$I_A^{\text{out}}[\omega]=I_A^{\text{in}}[\omega]+2i\xi_{AA}[\omega]\Phi[\omega] \ . \tag{2.31}$$

As the system is linear, the Langevin equations are easily solved in terms of an impedance $Z[\omega]$ which gives the voltage $V=d_t\Phi$ as a function of the input or output current fluctuations:

$$\begin{aligned} V[\omega]&=-i\omega\Phi[\omega] \\ &=C^{-1}Q[\omega] \\ &=Z[\omega]\{I_A^{\text{in}}[\omega]+I_B^{\text{in}}[\omega]\} \\ &=-Z[-\omega]\{I_A^{\text{out}}[\omega]+I_B^{\text{out}}[\omega]\} \ , \end{aligned} \tag{2.32}$$

$$Z[\omega]^{-1}=(-i\omega L)^{-1}+R_A^{-1}+R_B^{-1}-i\omega C \ . \tag{2.33}$$

This allows us to write explicitly the in-out relations

$$\begin{aligned} I_A^{\text{out}}[\omega]&=S_{AA}[\omega]I_A^{\text{in}}[\omega]+S_{AB}[\omega]I_B^{\text{in}}[\omega] \ , \\ I_B^{\text{out}}[\omega]&=S_{BA}[\omega]I_A^{\text{in}}[\omega]+S_{BB}[\omega]I_B^{\text{in}}[\omega] \ , \end{aligned} \tag{2.34}$$

$$\begin{aligned} &S_{AA}[\omega]=1+S_{AB}[\omega] \ , \quad S_{AB}[\omega]=-2Z[\omega]R_A^{-1} \ , \\ &S_{BA}[\omega]=-2Z[\omega]R_B^{-1}, \quad S_{BB}[\omega]=1+S_{BA}[\omega] \ . \end{aligned} \tag{2.35}$$

The effect of the linear system upon the reservoir fields is similar to the action of a beam splitter having frequency-dependent reflectivities. It is easily checked that those reflectivities obey unitarity conditions and that the scattering may be written in terms of frequency-dependent phase shifts:

$$(I_A^{\rm out}[\omega]+I_B^{\rm out}[\omega]) = -Z[\omega]/Z[-\omega](I_A^{\rm in}[\omega]+I_B^{\rm in}[\omega])\ , \quad (2.36)$$

$$R_A I_A^{\rm out}[\omega]-R_B I_B^{\rm out}[\omega]=R_A I_A^{\rm in}[\omega]-R_B I_B^{\rm in}[\omega]\ . \quad (2.37)$$

As a consequence of the unitarity of the field scattering, the commutators of the output fields are identical to the commutators of the input ones.

D. Connection between fluctuations and impedances

The quantum fluctuations of the system and reservoirs are characterized by the correlation functions (we will always consider a stationary situation)

$$C_{XX'}(t)=\langle X(t)X'(0)\rangle\ . \quad (2.38)$$

The noise spectra are the Fourier transforms of the correlation functions:

$$\langle X[\omega]X'[\omega']\rangle=2\pi\delta(\omega+\omega')C_{XX'}[\omega]\ .$$

It will be helpful to define also the commutators $\xi_{XX'}$:

$$\xi_{XX'}(t)=\tfrac{1}{2}\langle [X(t),X'(0)]\rangle=\tfrac{1}{2}\{C_{XX'}(t)-C_{X'X}(-t)\}\ . \quad (2.39)$$

The linear response of the system when submitted to an excitation is given by the susceptibility functions of linear-response theory:

$$\chi_{XX'}(t)=2i\Theta(t)\xi_{XX'}(t)=i\Theta(t)\langle [X(t),X'(0)]\rangle\ . \quad (2.40)$$

The functions ξ_{AA} and χ_{AA} correspond to the particular case $X=X'=I_A$. The impedance function Z is the susceptibility function $\chi_{V\Phi}$ describing the response of the system voltage to an input current; as a consequence, one obtains

$$\xi_{VV}[\omega]=(2i)^{-1}\{\chi_{VV}[\omega]-\chi_{VV}[-\omega]\}=\omega\,{\rm Re}\{Z[\omega]\}\ . \quad (2.41)$$

The same expression of the commutator can also be derived directly from the solutions of the Langevin equations (2.32):

$$\xi_{VV}[\omega]=Z[\omega]Z[-\omega]\{\xi_{AA}^{\rm in}[\omega]+\xi_{BB}^{\rm in}[\omega]\} = \omega|Z[\omega]|^2{\rm Re}\{Z[\omega]^{-1}\}\ . \quad (2.42)$$

Therefore the Langevin equations provide us with a consistent description of the system fluctuations. One notes that the commutators and susceptibility functions do not depend upon the fluctuations. This is a specific property of linear systems.

The system fluctuations are characterized by the correlation functions which can be written in terms of the Nyquist current fluctuations described by the correlation functions $C_{AA}^{\rm in}$ and $C_{BB}^{\rm in}$. Assuming that the input states of the two reservoirs are uncorrelated, one deduces the system fluctuations in terms of the impedance:

$$C_{VV}[\omega]=|Z[\omega]|^2\{C_{AA}^{\rm in}[\omega]+C_{BB}^{\rm in}[\omega]\} \quad (2.43)$$

as well as the output field fluctuations in terms of the scattering coefficients:

$$C_{AA}^{\rm out}[\omega]=S_{AA}[\omega]C_{AA}^{\rm in}[\omega]S_{AA}[\omega]^* + S_{AB}[\omega]C_{BB}^{\rm in}[\omega]S_{AB}[\omega]^*\ . \quad (2.44)$$

In the particular case where the input reservoir states correspond to the vacuum:

$$C_{AA}^{\rm in}[\omega]=\xi_{AA}[\omega]2\Theta(\omega)=R_A^{-1}2\omega\Theta(\omega) \quad (2.45)$$

the system and the output fields also correspond to the vacuum state:

$$C_{VV}[\omega]=\xi_{VV}[\omega]2\Theta(\omega)\ , \quad (2.46)$$

$$C_{AA}^{\rm out}[\omega]=C_{AA}^{\rm in}[\omega]\ . \quad (2.47)$$

III. NONLINEAR SYSTEMS

The aim of the paper is to extend the results recalled in the preceding section to situations involving a nonlinear system. First, we discuss the examples of an electrical system containing a Josephson junction or an atom coupled to the electromagnetic field. Then, we introduce general notations allowing us to study any nonlinear system. We finally write the results obtained previously for a linear system which are still valid for a nonlinear one.

A. Josephson junctions and atoms

A polarized Josephson junction is an example of a nonlinear electric oscillator with a Hamiltonian:

$$H_S=Q^2/(2C)+U(\Phi)\ ,\quad U(\Phi)=I_P\Phi-I_J\varphi_0\cos(\Phi/\varphi_0)\ , \quad (3.1)$$

$$\varphi_0=\hbar/(2q)\ , \quad (3.2)$$

$$[\Phi,Q]=i\hbar\ , \quad (3.3)$$

where Q is the charge (q the elementary charge), Φ the magnetic flux (φ_0 the flux quantum), I_J the Josephson critical current, and I_P the polarization current. We will consider the situations $(0<I_P<I_J)$ where the "tilted washboard" potential U possesses minima corresponding to possible stationary working points [33]. The Heisenberg equations for this Hamiltonian coincide with the Josephson equations for the voltage and current, respectively,

$$d_t\Phi=Q/C\ ,\quad d_tQ=-\frac{dU}{d\Phi}=I_P-I_J\sin(\Phi/\varphi_0)\ . \quad (3.4)$$

In a quasiclassical regime where a great number of energy levels are involved [34], this system can be approximated as a parametric system by retaining only the cubic nonlinearity of the potential in the vicinity of the working point. More generally, the junction can be considered as a quantum system with a limited number of eigenstates of the Hamiltonian H_S involved [33]. In the present paper, we will be primarily interested in the quantum regime. Then, an operator like the magnetic flux Φ will be conveniently expressed in the basis of the eigenstates $|a_j\rangle$:

$$\Phi=\sum_{jj'}\Phi_{jj'} , \tag{3.5}$$

$$\Phi_{jj'}=|a_j\rangle\langle a_j|\Phi|a_{j'}\rangle\langle a_{j'}| . \tag{3.6}$$

The situation is quite similar for atoms which constitute another example of damped nonlinear systems. In this case, the damping is due to the coupling of the atomic dipole D with the free space electromagnetic field E. In the "electric dipole" approximation, this coupling may be written

$$H_1=-ED=-E\sum_{jj'}D_{jj'} , \tag{3.7}$$

$$D_{jj'}=|a_j\rangle\langle a_j|D|a_{j'}\rangle\langle a_{j'}| . \tag{3.8}$$

In the following, we will study situations where the system is coupled to two different reservoirs. For example, the first reservoir A represents the optical modes the output fluctuations of which we are interested in [35,36]. The second reservoir B represents a background field responsible for an extra spontaneous emission of the atomic systems [37]. For junctions, the two reservoirs correspond, for example, to two transmission lines used for coupling the junction on one hand, polarizing it on the other hand. We will make no *a priori* assumption on the ratio between the coupling strengths of the system with the two reservoirs. At the end of the paper, we will write the results of the generalized linear input-output theory in the particular case where we are interested in a weakly coupled reservoir.

In the simplest case, the nonlinear system may be described by two states $|a_1\rangle$ and $|a_2\rangle$ separated by an energy $\hbar\omega_S$:

$$H_S=\hbar\omega_S|a_2\rangle\langle a_2| . \tag{3.9}$$

As is well known, the two-level atom coupled to electromagnetic fields can be represented as a fictitious $\frac{1}{2}$ spin interacting with magnetic fields [38]. The two spin states are associated, respectively, to the upper and lower atomic states $|a_2\rangle$ and $|a_1\rangle$ and the system operators may be expanded over the spin components:

$$\tfrac{1}{2}+\sigma_z=|a_2\rangle\langle a_2| , \quad \tfrac{1}{2}-\sigma_z=|a_1\rangle\langle a_1| , \tag{3.10}$$

$$\sigma_+=|a_2\rangle\langle a_1| , \quad \sigma_-=|a_1\rangle\langle a_2| . \tag{3.11}$$

In the rotating-wave approximation, the dipolar interaction may be written

$$H_1=-d(\sigma_+E_++\sigma_-E_-) , \tag{3.12}$$

where d is the dipole matrix element (or the flux matrix element for junctions) and E_+ and E_- are the positive and negative frequency components of the electric field.

The mean reservoir fields play the role of driving fields and they will be included in the Hamiltonian of the nonlinear system. They will therefore be treated in a nonperturbative manner whereas the effect of the field fluctuations on the relaxation of the nonlinear system will be derived from a perturbative theory (see the next section). This separation between the mean values of the reservoir fields and their fluctuations is analogous to the separation used in the dressed-atom approach [39–41].

B. Notations for the nonlinear system

The general model consists of a nonlinear system (Hamiltonian H_S) coupled to harmonic reservoirs (Hamiltonians H_A and H_B) through a linear interaction (Hamiltonian H_1):

$$H=H_0+H_1+H_2 , \quad H_0=H_S+H_A+H_B . \tag{3.13}$$

In the following, we discuss the general expressions of H_S, H_A and H_B, H_1 and of the self-energy term H_2.

The nonlinear system is described by a number r of eigenstates $|a_j\rangle$ and the operators may be expanded over the r^2 operators S_α which constitute a basis of the Liouville space associated with the system:

$$S_\alpha=|a_j\rangle\langle a_{j'}| . \tag{3.14}$$

The operator S_α corresponds to the population of one level for $j=j'$ and to a coherence between two levels otherwise. We will define I^α equal to one in the first case and zero in the second one. The identity operator I for the system is the sum over all population operators [42]:

$$I=I^\alpha S_\alpha . \tag{3.15}$$

Clearly, the product $S_{\alpha\beta}$ of the two operators S_α and S_β is either an operator S_γ or zero:

$$S_{\alpha\beta}=S_\alpha S_\beta=s_{\alpha\beta}{}^\gamma S_\gamma \tag{3.16}$$

(each coefficient $s_{\alpha\beta}{}^\gamma$ is either zero or one). We will also define the products of three (or more if necessary) operators:

$$S_{\alpha\beta\gamma}=S_\alpha S_\beta S_\gamma=s_{\alpha\beta\gamma}{}^\delta S_\delta , \tag{3.17}$$

$$s_{\alpha\beta\gamma}{}^\delta=s_{\alpha\beta}{}^\epsilon s_{\epsilon\gamma}{}^\delta=s_{\alpha\epsilon}{}^\delta s_{\beta\gamma}{}^\epsilon . \tag{3.18}$$

The commutators of basic operators will play an important role in the generalized linear-response theory and they are denoted

$$[S_\alpha,S_\beta]=S_{\alpha\beta}-S_{\beta\alpha}=(s_{\alpha\beta}{}^\gamma-s_{\beta\alpha}{}^\gamma)S_\gamma . \tag{3.19}$$

In order to avoid any ambiguity, it will be helpful to introduce a second notation for the basic operators S_α:

$$S^\alpha=S_\alpha^\dagger . \tag{3.20}$$

Raising or lowering the indices amounts either to applying Hermitian conjugation on S_α considered as an operator acting in the Hilbert space, or to permuting S_α's considered as the elements of the basis of the Liouville space:

$$S^\alpha(t)={}^S\eta^{\alpha\beta}S_\beta(t) , \quad S_\alpha(t)={}^S\eta_{\alpha\beta}S^\beta(t) \tag{3.21}$$

(each ${}^S\eta^{\alpha\beta}$ is either zero or one).

When performing explicit computations of the correlation functions, it may be convenient to translate the tensorial notations introduced in the present section to matrix notations [24]. The translation rules have to be chosen so that the matrix product operation fits the implicit summation rule for tensor indices [43].

An example of a nonlinear system is the two-level system which corresponds to the following operators S_α with their evolution frequencies ω_α:

$$\begin{aligned} &S_1=\tfrac{1}{2}+\sigma_z\ , \quad \omega_1=0\ ,\\ &S_2=\tfrac{1}{2}-\sigma_z\ , \quad \omega_2=0\ ,\\ &S_3=\sigma_+\ , \quad \omega_3=\omega_S\ ,\\ &S_4=\sigma_-\ , \quad \omega_4=-\omega_S\ . \end{aligned} \tag{3.22}$$

For this specific labeling choice, the nonzero coefficients $s_{\alpha\beta}{}^{\gamma}$ and ${}^S\eta^{\alpha\beta}$ are the following:

$$\begin{aligned} s_{11}{}^{1}&=s_{13}{}^{3}\\ &=s_{22}{}^{2}=s_{24}{}^{4}=s_{32}{}^{3}=s_{34}{}^{1}=s_{42}{}^{4}=s_{43}{}^{2}=1\ , \end{aligned} \tag{3.23}$$

$${}^S\eta^{11}={}^S\eta^{22}={}^S\eta^{34}={}^S\eta^{43}=1\ . \tag{3.24}$$

C. Coupling with the reservoirs

The linear coupling of the system with the reservoirs will be written

$$H_1=-\hbar R^{\alpha}S_\alpha\ , \quad R^{\alpha}=h_\mu{}^{\alpha}A^{\mu}\ , \tag{3.25}$$

where the A^μ are the different components of the reservoir fields and $h_\mu{}^\alpha$ some coupling constants. For the sake of simplicity, the components of the reservoirs A and B are denoted in the same manner and correspond to different values of the labeling index μ.

In the rotating-wave approximation, the components A^μ correspond to well-separated evolution frequencies ω^μ and are therefore considered as different operators. They split into two categories corresponding to positive or negative evolution frequencies ω^μ and related, respectively, to the annihilation and creation operators:

$$\begin{aligned} &A^{\mu}[\omega]=a_{\omega}\ , \quad \omega \text{ around } \omega^{\mu};\ \omega^{\mu}>0\ ,\\ &A^{\mu}[\omega]=a^{\dagger}_{-\omega}\ , \quad \omega \text{ around } \omega^{\mu};\ \omega^{\mu}<0\ . \end{aligned} \tag{3.26}$$

Some dipole components are not coupled to the reservoir fields (this is the case for the components S_1 and S_2 for a two-level system in the rotating-wave approximation). It may also occur that different dipole components are coupled to the same field component or that different field operators are coupled to the same dipole component.

Rigorously speaking, the dipole components are coupled to electric fields which are related to annihilation or creation operators through frequency-dependent factors. However, we will consider that the frequency bands corresponding to different components are narrow compared to the mean frequencies ω^μ. In this resonance approximation, it is possible to consider that the constants $h_\mu{}^\alpha$ are frequency independent.

As for the system operators, raising or lowering the indices amounts either to applying Hermitian conjugation on the field operators:

$$A_\mu(t)=A^{\mu}(t)^{\dagger}\ , \tag{3.27}$$

$$A_\mu[\omega]=A^{\mu}[-\omega]^{\dagger}\ , \quad \omega_\mu=-\omega^{\mu} \tag{3.28}$$

or, equivalently, to permuting the field components:

$$A^{\mu}(t)={}^A\eta^{\mu\nu}A_\nu(t)\ , \quad A_\mu(t)={}^A\eta_{\mu\nu}A^{\nu}(t)\ . \tag{3.29}$$

As discussed previously, the self-energy term H_2 cancels the static susceptibility of the reservoir fields. In the resonance approximation, it is checked that those static susceptibilities are zero so that one can forget H_2. The terms which are similar to the Lamb shifts for atomic systems cannot be obtained in the resonance approximation, but they may be included in the system Hamiltonian H_S (any modification of these terms due to the coupling is neglected).

D. Correlation and susceptibility functions

The system fluctuations are characterized by the correlation functions ${}^SC_{\alpha\beta}$ or their Fourier transforms, the noise spectra:

$${}^SC_{\alpha\beta}(t)=\langle S_\alpha(t)S_\beta(0)\rangle\ , \tag{3.30}$$

$$\langle S_\alpha[\omega]S_\beta[\omega']\rangle=2\pi\delta(\omega+\omega')\,{}^SC_{\alpha\beta}[\omega]\ . \tag{3.31}$$

We also define the commutators ${}^S\xi_{\alpha\beta}$:

$$\begin{aligned} {}^S\xi_{\alpha\beta}(t)&=\tfrac{1}{2}\langle[S_\alpha(t),S_\beta(0)]\rangle\\ &=\tfrac{1}{2}[{}^SC_{\alpha\beta}(t)-{}^SC_{\beta\alpha}(-t)] \end{aligned} \tag{3.32}$$

and the retarded susceptibility functions which will determine the linear response of the system to an external perturbation:

$${}^S\chi_{\alpha\beta}(t)=2i\Theta(t)\,{}^S\xi_{\alpha\beta}(t)\ . \tag{3.33}$$

In the next part of the paper, we will calculate all these functions by using methods derived from the resonance fluorescence theory. It has to be stressed that the system commutators and susceptibilities are no longer independent of the fluctuations for a nonlinear system.

The fluctuations of the reservoir fields A^μ are characterized in the same manner:

$${}^AC^{\mu\nu}(t)=\langle A^{\mu}(t)A^{\nu}(0)\rangle\ , \tag{3.34}$$

$${}^A\xi^{\mu\nu}(t)=\tfrac{1}{2}\langle[A^{\mu}(t),A^{\nu}(0)]\rangle\ , \tag{3.35}$$

$${}^A\chi^{\mu\nu}(t)=2i\Theta(t)\,{}^A\xi^{\mu\nu}(t)\ . \tag{3.36}$$

The Heisenberg equations for the fields are still linear so that the field commutators are the same as previously (they remain state independent). One obtains, from the definition of the field components,

$${}^A\xi^{\mu\nu}[\omega]=\tfrac{1}{2}\epsilon^{\mu\nu}\ , \quad \epsilon^{\mu\nu}=\mathrm{sgn}\,(\omega^{\mu})\,{}^A\eta^{\mu\nu}\ , \tag{3.37}$$

$${}^A\chi^{\mu\nu}[\omega]=\tfrac{1}{2}i\epsilon^{\mu\nu}\ . \tag{3.38}$$

Solving the Heisenberg equations for the fields and choosing the initial time far in the past or far in the future, one obtains the fields in terms of the reservoir susceptibility and of the input fields:

$$A^{\mu}[\omega]=A^{\mu\,\mathrm{in}}[\omega]+\tfrac{1}{2}i\epsilon^{\mu\nu}h_{\nu}{}^{\alpha}S_{\alpha}[\omega]$$
$$=A^{\mu\,\mathrm{out}}[\omega]-\tfrac{1}{2}i\epsilon^{\mu\nu}h_{\nu}{}^{\alpha}S_{\alpha}[\omega]\,, \tag{3.39}$$

$$A^{\mu\,\mathrm{out}}[\omega]=A^{\mu\,\mathrm{in}}[\omega]+i\epsilon^{\mu\nu}h_{\nu}{}^{\alpha}S_{\alpha}[\omega]\,. \tag{3.40}$$

This operatorial expression of the linear response of the reservoir field to the coupling to the dipole is still valid because of the linearity of the field equations.

IV. GENERALIZED LINEAR-RESPONSE THEORY

In the preceding section, we have written the results which have the same form for a nonlinear system and for a linear one. We will now study the relaxation of the nonlinear system and compute the correlation and susceptibility functions by using the techniques of resonance fluorescence theory. We use a perturbative theory of relaxation. We show that the linear-response properties valid for a linear system can be generalized to nonlinear systems with appropriate modifications. We demonstrate that they provide us with a consistent description of the system fluctuations.

A. Relaxation equations

In order to obtain the relaxation equations, we write the Heisenberg equations for the system operators:

$$d_t S_{\alpha}(t)=-i\omega_{\alpha}S_{\alpha}(t)-i[R^{\beta}(t)S_{\beta}(t),S_{\alpha}(t)]\,. \tag{4.1}$$

We have supposed that the different field components are treated separately and that the corresponding frequency bands are narrow compared to the mean frequencies (resonance approximation). It follows that the noise spectrum characterizing the reservoir fluctuations can be regarded as flat on each frequency band taken separately, with different values for the different frequency bands. This is the condition for the so-called Markov approximation [36].

In these conditions, a perturbative solution up to the second order in the coupling leads to the following relaxation equations:

$$d_t\langle S_{\alpha}(t)\rangle=[-iL_{\alpha}{}^{\beta}-K_{\alpha}{}^{\beta}]\langle S_{\beta}(t)\rangle\,, \tag{4.2}$$

where the coefficients $L_{\alpha}{}^{\beta}$ are related to the system eigenfrequencies:

$$L_{\alpha}{}^{\beta}=\omega_{\alpha}\delta_{\alpha}{}^{\beta} \tag{4.3}$$

($\delta_{\alpha}{}^{\beta}$ are the Kronecker symbols) and the relaxation coefficients $K_{\alpha}{}^{\beta}$ are given by

$$-K_{\alpha}{}^{\beta}\langle S_{\beta}\rangle={}^{R}C^{\gamma\delta\,\mathrm{in}}\langle S_{\gamma\alpha\delta}-(S_{\gamma\delta\alpha}+S_{\alpha\gamma\delta})/2\rangle\,, \tag{4.4}$$

that is, using the expression of products of basic operators,

$$K_{\alpha}{}^{\beta}={}^{R}C^{\gamma\delta\,\mathrm{in}}[(s_{\gamma\delta\alpha}{}^{\beta}+s_{\alpha\gamma\delta}{}^{\beta})/2-s_{\gamma\alpha\delta}{}^{\beta}]\,. \tag{4.5}$$

The noise spectra ${}^{R}C^{\gamma\delta\,\mathrm{in}}$ characterize the fluctuations of the input reservoir fields:

$${}^{R}C^{\gamma\delta\,\mathrm{in}}=h_{\mu}{}^{\gamma}h_{\nu}{}^{\delta}\,{}^{A}C^{\mu\nu\,\mathrm{in}}[\omega^{\mu}]\,. \tag{4.6}$$

The evolution of the mean values $\langle S_{\alpha}(t)\rangle$ from an initial state $\langle S_{\alpha}(0)\rangle$ are given by the retarded Green functions $G_{\alpha}{}^{\beta}$:

$$\Theta(t)\langle S_{\alpha}(t)\rangle=G_{\alpha}{}^{\beta}(t)\langle S_{\beta}(0)\rangle \tag{4.7}$$

which are obtained in the frequency domain as the inverse of the evolution matrix:

$$G_{\alpha}{}^{\gamma}[\omega][-i\omega\delta_{\gamma}{}^{\beta}+iL_{\gamma}{}^{\beta}+K_{\gamma}{}^{\beta}]=\delta_{\alpha}{}^{\beta}\,. \tag{4.8}$$

The preservation of the sum of the populations

$$I^{\alpha}d_t\langle S_{\alpha}\rangle=0 \tag{4.9}$$

is ensured by the relations

$$I^{\alpha}L_{\alpha}{}^{\beta}=I^{\alpha}K_{\alpha}{}^{\beta}=0\,,\quad I^{\alpha}G_{\alpha}{}^{\beta}(t)=I^{\beta}\Theta(t)\,. \tag{4.10}$$

This property is associated with the existence of a steady state, which will be denoted

$$\bar{S}_{\alpha}=\langle S_{\alpha}(\infty)\rangle\,,\quad [iL_{\alpha}{}^{\beta}+K_{\alpha}{}^{\beta}]\bar{S}_{\beta}=0\,. \tag{4.11}$$

We will consider that the steady state is nondegenerate, which eliminates the configurations with several trap levels. The steady state may be derived algebraically from the asymptotic values of the Green functions for long times, that is, from the residues of the Green functions at zero frequency:

$$\bar{S}_{\alpha}I^{\beta}=\lim_{\omega\to0}(-i\omega G_{\alpha}{}^{\beta}[\omega])=G_{\alpha}{}^{\beta}(t=\infty)\,. \tag{4.12}$$

B. Correlation functions for the system observables

The correlation functions ${}^{S}C_{\alpha\beta}$ evaluated at equal times are readily deduced from the structure properties of the basic operators:

$${}^{S}C_{\alpha\beta}(0)=\langle S_{\alpha\beta}(\infty)\rangle=\bar{S}_{\alpha\beta}=s_{\alpha\beta}{}^{\gamma}\bar{S}_{\gamma}\,. \tag{4.13}$$

For positive delays, the correlation functions ${}^{S}C_{\alpha\beta}(t)$ may be evaluated by using the quantum regression theorem [44] which states that they have the same evolution as the mean values $\langle S_{\alpha}(t)\rangle$ in a transient regime:

$$\Theta(t)\,{}^{S}C_{\alpha\beta}(t)=G_{\alpha}{}^{\gamma}(t)\,{}^{S}C_{\gamma\beta}(0)=G_{\alpha}{}^{\gamma}(t)\bar{S}_{\gamma\beta}\,. \tag{4.14}$$

The noise spectra and the susceptibility functions are deduced from these expressions:

$${}^{S}C_{\alpha\beta}(t)=\Theta(t)\,{}^{S}C_{\alpha\beta}(t)+\Theta(-t)\,{}^{S}C_{\alpha\beta}(t)$$
$$=G_{\alpha}{}^{\gamma}(t)\bar{S}_{\gamma\beta}+G_{\beta}{}^{\gamma}(-t)\bar{S}_{\alpha\gamma}\,, \tag{4.15}$$

$${}^{S}\chi_{\alpha\beta}(t)=i\Theta(t)[\,{}^{S}C_{\alpha\beta}(t)-{}^{S}C_{\beta\alpha}(-t)]$$
$$=iG_{\alpha}{}^{\gamma}(t)[\bar{S}_{\gamma\beta}-\bar{S}_{\beta\gamma}]\,, \tag{4.16}$$

that is, in the frequency range

$${}^{S}C_{\alpha\beta}[\omega]=G_{\alpha}{}^{\gamma}[\omega]\bar{S}_{\gamma\beta}+G_{\beta}{}^{\gamma}[-\omega]\bar{S}_{\alpha\gamma}\,, \tag{4.17}$$

$${}^{S}\chi_{\alpha\beta}[\omega]=iG_{\alpha}{}^{\gamma}[\omega][\bar{S}_{\gamma\beta}-\bar{S}_{\beta\gamma}]\,. \tag{4.18}$$

For any specific configuration of the nonlinear system, these results ensure that the correlation and susceptibility functions are related through

$$^{S}C_{\alpha\beta}[\omega]-{}^{S}C_{\beta\alpha}[-\omega]=-i\{{}^{S}\chi_{\alpha\beta}[\omega]-{}^{S}\chi_{\alpha\beta}[-\omega]\}$$
$$=2\,{}^{S}\xi_{\alpha\beta}[\omega]\ . \qquad (4.19)$$

Using the fact that S_α and S^α are conjugate to each other, one shows that the correlation functions may be written as the elements of a hermitian matrix:

$$^{S}C_{\beta\alpha}[\omega]^*={}^{S}C^{\alpha\beta}[\omega]={}^{S}\eta^{\alpha\gamma}\,{}^{S}\eta^{\beta\delta}\,{}^{S}C_{\gamma\delta}[\omega]\ , \qquad (4.20)$$

$$^{S}C_{\beta}{}^{\alpha}[\omega]^*={}^{S}C_{\alpha}{}^{\beta}[\omega]\ . \qquad (4.21)$$

C. Linear response to classical modulations

We consider now that the system, being in its stationary state, is submitted to a weak classical modulation of the reservoir fields:

$$\delta H_1=-\hbar\mathcal{R}^\alpha S_\alpha\ ,\quad \mathcal{R}^\alpha=h_\mu{}^\alpha\mathcal{A}^\mu\ . \qquad (4.22)$$

Using the Heisenberg equations, it appears that the variations of the mean values are given by

$$d_t\langle\delta S_\alpha(t)\rangle=[-iL_\alpha{}^\beta-K_\alpha{}^\beta]\langle\delta S_\beta(t)\rangle+\mathcal{F}_\alpha(t)\ , \qquad (4.23)$$

$$\mathcal{F}_\alpha(t)=i(\bar{S}_{\alpha\beta}-\bar{S}_{\beta\alpha})\mathcal{R}^\beta(t)\ . \qquad (4.24)$$

The solution of these classical Langevin equations provides us with the usual linear-response expressions:

$$\langle\delta S_\alpha[\omega]\rangle=G_\alpha{}^\gamma[\omega]\mathcal{F}_\gamma[\omega]={}^{S}\chi_{\alpha\beta}[\omega]\mathcal{R}^\beta[\omega]\ , \qquad (4.25)$$

where $^{S}\chi_{\alpha\beta}$ are the retarded susceptibility functions.

D. Linear response to quantum fluctuations

In the case of a linear system, we have seen that the linear response theory also allows us to compute the quantum fluctuations of the system in its stationary state [see Eq. (2.43)]. We show now that this linear response property can be generalized to nonlinear systems by introducing new response functions.

First, we transform the Heisenberg equations to quantum Langevin equations. These equations are obtained by solving the Heisenberg equations up to the second order in the coupling, keeping in mind that the input fields have quantum fluctuations [45]:

$$d_tS_\alpha(t)=[-iL_\alpha{}^\beta-K_\alpha{}^\beta]S_\beta(t)+F_\alpha(t)\ , \qquad (4.26)$$

$$F_\alpha(t)=i(S_{\alpha\beta}-S_{\beta\alpha})R^{\beta^{\text{in}}}(t)\ . \qquad (4.27)$$

The classical Langevin equations may be recovered as mean values of the quantum ones, by replacing the operators S_α and $S_{\alpha\beta}$ by their stationary mean values $\bar{S}_\alpha$ and $\bar{S}_{\alpha\beta}$ and the quantum fluctuations R^β by the classical perturbations $\mathcal{R}^\beta$.

Then, the quantum fluctuations of the system can be derived by solving the quantum Langevin equations:

$$S_\alpha[\omega]=G_\alpha{}^\gamma[\omega]F_\gamma[\omega]=X_{\alpha\beta}[\omega]R^{\beta\text{in}}[\omega]\ , \qquad (4.28)$$

$$X_{\alpha\beta}[\omega]=G_\alpha{}^\gamma[\omega]i(S_{\gamma\beta}-S_{\beta\gamma})\ . \qquad (4.29)$$

The usual susceptibility functions are equal to the mean values of these response operators:

$$\chi_{\alpha\beta}[\omega]=\langle X_{\alpha\beta}[\omega]\rangle=G_\alpha^\gamma[\omega]i(\bar{S}_{\gamma\beta}-\bar{S}_{\beta\gamma})\ . \qquad (4.30)$$

Two types of terms appear in the expression (4.29) of the response operator $X_{\alpha\beta}[\omega]$. On one hand, the terms $G_\alpha{}^\gamma[\omega]$ are dynamical expressions which are determined by the relaxation equations and contain all the frequency dependence of the response operators. They play exactly the same role as in the expression (4.30) of the susceptibility functions. On the other hand, the commutators $(S_{\gamma\beta}-S_{\beta\gamma})$ are related to the structure of the nonlinear system and do not depend upon the dynamics. For linear systems, they are pure numbers and the linear response formulas are also valid for quantum fluctuations. This is not the case for nonlinear systems and the quantities $X_{\alpha\beta}[\omega]$ have to be treated as noncommuting response operators.

This property allows us to give a precise definition of the semiclassical limit when quantum fluctuations are studied. This limit corresponds to situations where the response operators commute and can be considered as classical numbers. This occurs for a linear scatterer or for a nonlinear scatterer in its highly excited states. We will see later on that a similar simplification is obtained also when a weakly coupled reservoir is studied.

E. Consistency of the generalized linear-response theory

The correlation functions of the system can be obtained in two different manners. In Sec. IV, they have been deduced from the quantum regression theorem [see Eq. (4.17)]. Now they can also be inferred from the generalized linear-response formula (4.28). In order to be sure that the linear-response method is consistent, it is interesting to check that the two derivations provide us with the same results.

From the Langevin equations (4.28), one infers the following correlation functions:

$$\langle S_\alpha[\omega]S_\beta[\omega']\rangle=\langle X_{\alpha\gamma}[\omega]X_{\beta\delta}[\omega']\rangle$$
$$\times\langle R^{\gamma\,\text{in}}[\omega]R^{\delta\,\text{in}}[\omega']\rangle\ , \qquad (4.31)$$

$$^{S}C_{\alpha\beta}[\omega]=\langle X_{\alpha\gamma}[\omega]X_{\beta\delta}[-\omega]\rangle\,{}^{R}C^{\gamma\delta\,\text{in}}[\omega]\ . \qquad (4.32)$$

The comparison with Eq. (2.43) suggests that the response operators $X_{\alpha\beta}[\omega]$ may be considered as noncommuting impedances describing the linear response of the nonlinear system to the quantum fluctuations of the input reservoir fields.

Using the expressions (4.29) of the response functions, we first transform the expression of these fluctuations to

$$^{S}C_{\alpha\beta}[\omega]=G_\alpha{}^\gamma[\omega]G_\beta{}^\delta[-\omega]\,{}^{R}C^{\epsilon\varphi\,\text{in}}$$
$$\times[\bar{S}_{\gamma\epsilon\varphi\delta}+\bar{S}_{\epsilon\gamma\delta\varphi}-\bar{S}_{\epsilon\gamma\varphi\delta}-\bar{S}_{\gamma\epsilon\delta\varphi}]\ . \qquad (4.33)$$

Using the expressions (4.5) of the relaxation coefficients and the equations obeyed by the steady state, one shows that

$$^{S}C_{\alpha\beta}[\omega]=G_\alpha{}^\gamma[\omega]G_\beta{}^\delta[-\omega]$$
$$\times\{[i\omega\delta_\delta{}^\varphi+iL_\delta{}^\varphi+K_\delta{}^\varphi]\bar{S}_{\gamma\varphi}$$
$$+[-i\omega\delta_\gamma{}^\epsilon+iL_\gamma{}^\epsilon+K_\gamma{}^\epsilon]\bar{S}_{\epsilon\delta}\}\ . \qquad (4.34)$$

Using the definition (4.8) of the Green functions, it finally appears that these expressions are identical to the correlation functions (4.17) derived directly from the quantum regression theorem.

This establishes that the generalized linear response theory provides us with a consistent account of the system fluctuations.

V. GENERALIZED LINEAR INPUT-OUTPUT THEORY

In this section, we study the fluctuations of the output reservoir fields. We derive their expressions and check that the output fields obey the same commutation relations as the input ones. We write the output correlation functions in a form which facilitates the comparison with the results obtained in a coherent-state representation. We analyze the case of a weakly coupled reservoir where the input-output transformation can be described in terms of a classical transfer function and of an added noise.

A. Quantum transfer functions

The fluctuations of the output reservoir fields are easily obtained from the operatorial expression (3.40) of the output fields and the operatorial expression (4.28) of the system fluctuations:

$$A^{\mu\,\mathrm{out}}[\omega]=A^{\mu\,\mathrm{in}}[\omega]+i\epsilon^{\mu\nu}h_\nu{}^{\alpha}X_{\alpha\beta}[\omega]R^{\beta\,\mathrm{in}}[\omega]\ . \tag{5.1}$$

The output fluctuations can therefore be written in terms of quantum transfer functions:

$$A^{\mu\,\mathrm{out}}[\omega]=\Lambda^{\mu}{}_{\nu}[\omega]A^{\nu\,\mathrm{in}}[\omega]\ , \tag{5.2}$$

$$\Lambda^{\mu}{}_{\nu}[\omega]=\delta^{\mu}{}_{\nu}+i\epsilon^{\mu\rho}h_\rho{}^{\alpha}X_{\alpha\beta}[\omega]h_\nu{}^{\beta}\ . \tag{5.3}$$

The transfer functions $\Lambda^{\mu}{}_{\nu}$ are noncommuting system operators which generalize the frequency-dependent reflectivities describing the scattering upon a harmonic scatterer [see Eqs. (2.35)].

In the following, we check that these quantum transfer functions preserve the commutation relations of the reservoir fields (unitary scattering).

B. Correlation between system and reservoir fluctuations

For that purpose, we have first to study the correlations between the system operators and the input fields. Using the operatorial expression (4.28) of the system fluctuations, one readily obtains

$$\begin{aligned}\langle S_\alpha[\omega]A^{\nu\,\mathrm{in}}[\omega']\rangle&={}^{S}\chi_{\alpha\gamma}[\omega]\langle R^{\gamma\,\mathrm{in}}[\omega]A^{\nu\,\mathrm{in}}[\omega']\rangle\ ,\\ \langle A^{\mu\,\mathrm{in}}[\omega]S_\beta[\omega']\rangle&={}^{S}\chi_{\beta\delta}[\omega']\langle A^{\mu\,\mathrm{in}}[\omega]R^{\delta\,\mathrm{in}}[\omega']\rangle\ .\end{aligned} \tag{5.4}$$

The correlations between system and reservoir fluctuations are deduced in a simple manner from the input fluctuations and from the susceptibility functions. This property has already been used for studying the characterization of quantum non demolition (QND) measurements [46].

One deduces

$$\begin{aligned}\langle S_\alpha[\omega]A^{\nu\,\mathrm{in}}[\omega']\rangle&=2\pi\delta(\omega+\omega')\,{}^{S}\chi_{\alpha\beta}[\omega]\\ &\quad\times h_\rho{}^{\beta}\,{}^{A}C^{\rho\nu\,\mathrm{in}}[\omega]\\ \langle A^{\mu\,\mathrm{in}}[\omega]S_\beta[\omega']\rangle&=2\pi\delta(\omega+\omega')\,{}^{S}\chi_{\beta\delta}[\omega']\\ &\quad\times h_\rho{}^{\delta}\,{}^{A}C^{\mu\rho\,\mathrm{in}}[\omega]\ .\end{aligned} \tag{5.5}$$

One can now compute the correlation functions characterizing the output field fluctuations by using these expressions and the system correlation function (4.17):

$$\begin{aligned}{}^{A}C^{\mu\nu\,\mathrm{out}}[\omega]={}&{}^{A}C^{\mu\nu\,\mathrm{in}}[\omega]\\ &+i\epsilon^{\mu\rho}h_\rho{}^{\alpha}\,{}^{S}\chi_{\alpha\beta}[\omega]\,h_\sigma{}^{\beta}\,{}^{A}C^{\sigma\nu\,\mathrm{in}}[\omega]\\ &+i\epsilon^{\nu\sigma}h_\sigma{}^{\beta}\,{}^{S}\chi_{\beta\alpha}[-\omega]\,h_\rho{}^{\alpha}\,{}^{A}C^{\mu\rho\,\mathrm{in}}[\omega]\\ &+i\epsilon^{\mu\rho}h_\rho{}^{\alpha}\,i\epsilon^{\nu\sigma}h_\sigma{}^{\beta}\,{}^{S}C_{\alpha\beta}[\omega]\ .\end{aligned} \tag{5.6}$$

These equations preserve the commutation relations of the reservoir fields. As a matter of fact, the difference between the output and input field commutators may be written

$$\begin{aligned}{}^{A}\xi^{\mu\nu\,\mathrm{out}}[\omega]-{}^{A}\xi^{\mu\nu\,\mathrm{in}}[\omega]&=i\epsilon^{\mu\rho}h_\rho{}^{\alpha}\,{}^{S}\chi_{\alpha\beta}[\omega]\,h_\sigma{}^{\beta}\,{}^{A}\xi^{\sigma\nu}+i\epsilon^{\nu\sigma}h_\sigma{}^{\beta}\,{}^{S}\chi_{\beta\alpha}[-\omega]\,h_\rho{}^{\alpha}\,{}^{A}\xi^{\mu\rho}\\ &\quad+i\epsilon^{\mu\rho}h_\rho{}^{\alpha}\,i\epsilon^{\nu\sigma}h_\sigma{}^{\beta}\,{}^{S}\xi_{\alpha\beta}[\omega]\\ &=-2i\epsilon^{\mu\rho}\epsilon^{\nu\sigma}h_\rho{}^{\alpha}h_\sigma{}^{\beta}\{{}^{S}\chi_{\alpha\beta}[\omega]-{}^{S}\chi_{\beta\alpha}[-\omega]-2i\,{}^{S}\xi_{\alpha\beta}[\omega]\}\ .\end{aligned} \tag{5.7}$$

The functions within the brackets which characterize the system fluctuations compensate each other for any nonlinear system.

This consistency check can be considered as an *a posteriori* justification of the method.

Actually, it can be shown that the expression (5.4) of the correlations between system and reservoir fluctuations is the unique expression compatible with causality and commutator preservation.

C. General form of the output correlation functions

We can write the output correlation functions in a form which facilitates the comparison with the results of standard quantum-optical techniques [3]. Using the expressions of the correlation and susceptibility functions of the system, one transforms (5.6) into

$${}^{A}C^{\mu\nu\,\mathrm{out}}[\omega]={}^{A}C^{\mu\nu\,\mathrm{in}}[\omega]+\Delta C^{\mu\nu}[\omega]\ , \tag{5.8}$$

with

$$\Delta C^{\mu\nu}[\omega]=\Gamma^{\mu\nu}[\omega]+\Gamma^{\nu\mu}[-\omega] , \quad (5.9)$$

$$\Gamma^{\mu\nu}[\omega]=\epsilon^{\mu\rho} h_{\rho}{}^{\alpha} h_{\sigma}{}^{\epsilon} G_{\alpha}{}^{\gamma}[\omega] \times\{\bar{S}_{\epsilon\gamma}\, {}^{A}C^{\sigma\nu\,\mathrm{in}}[\omega]-\bar{S}_{\gamma\epsilon}\, {}^{A}C^{\nu\sigma\,\mathrm{in}}[-\omega]\} . \quad (5.10)$$

In the particular case where the input field is in the vacuum state, the two following properties can be used. First, the quantities $\Delta C^{\mu\nu}$ defined by Eq. (5.8) are exactly the correlation functions corresponding to normally ordered products of operators. Then, the correlation functions associated with the input fields are frequency independent (see the discussion in Secs. III and IV):

$${}^{A}C^{\mu\nu\,\mathrm{in}}=2\Theta(\omega^{\mu})\, {}^{A}\xi^{\mu\nu}=\Theta(\omega^{\mu})\epsilon^{\mu\nu} . \quad (5.11)$$

It follows that the expressions to be compared with the results of standard quantum-optical methods are

$$\Delta C^{\mu\nu}[\omega]=\epsilon^{\mu\rho}\epsilon^{\sigma\nu} h_{\rho}{}^{\alpha} h_{\sigma}{}^{\beta} D_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta}[\omega] \times\{\bar{S}_{\delta\gamma}\Theta(\omega^{\mu})-\bar{S}_{\gamma\delta}\Theta(-\omega^{\mu})\} , \quad (5.12)$$

$$D_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta}[\omega]=G_{\alpha}{}^{\gamma}[\omega]\,\delta_{\beta}{}^{\delta}-\delta_{\alpha}{}^{\gamma}\, G_{\beta}{}^{\delta}[-\omega] . \quad (5.13)$$

D. The case of a weakly coupled reservoir

We consider now that the system is coupled to two reservoirs A and B having uncorrelated input fluctuations and that we are interested only in the field emitted into reservoir A. The field components corresponding to these two reservoirs are denoted A^{μ} and B^{μ} and the coupling coefficients ${}^{A}h_{\mu}{}^{\beta}$ and ${}^{B}h_{\mu}{}^{\beta}$.

In this case, it is interesting to write the results of the generalized linear-response theory as a sum of two contributions:

$$S_{\alpha}[\omega]=X_{\alpha\beta}[\omega]\, {}^{A}h_{\mu}{}^{\beta} A^{\mu\,\mathrm{in}}+\hat{S}_{\alpha}[\omega] , \quad (5.14)$$

$$\hat{S}_{\alpha}[\omega]=X_{\alpha\beta}[\omega]\, {}^{B}h_{\mu}{}^{\beta} B^{\mu\,\mathrm{in}} . \quad (5.15)$$

The first contribution is the linear response to the fluctuations of reservoir A which we are interested in. The second contribution corresponds to extra fluctuations which are uncorrelated with the previous ones. These extra fluctuations come from reservoir B but can be considered as a proper noise of the system. In this case, the fluctuations of reservoir B appear only in the expression of the relaxation coefficients.

The output field fluctuations can also be written as a sum of two contributions:

$$A^{\mu\,\mathrm{out}}[\omega]=\Lambda^{\mu}{}_{\nu}[\omega] A^{\nu\,\mathrm{in}}[\omega]+i\epsilon^{\mu\nu}\, {}^{A}h_{\nu}{}^{\alpha}\hat{S}_{\alpha}[\omega] , \quad (5.16)$$

$$\Lambda^{\mu}{}_{\nu}[\omega]=\delta^{\mu}{}_{\nu}+i\epsilon^{\mu\rho}\, {}^{A}h_{\rho}{}^{\alpha} X_{\alpha\beta}[\omega]\, {}^{A}h_{\nu}{}^{\beta} , \quad (5.17)$$

where $\Lambda^{\mu}{}_{\nu}$ is the generalized transfer function.

We study now the limiting case where reservoir A is weakly coupled to the system and we show that the expressions of the output field fluctuations take a very simple form in this case. More precisely, we assume that the coupling of the system to reservoir A is negligible in comparison with the coupling to reservoir B:

$${}^{A}h_{\mu}{}^{\alpha}\ll{}^{B}h_{\mu}{}^{\alpha} . \quad (5.18)$$

Then, the relaxation coefficients (4.5) are determined by the coupling of the system with reservoir B since Eq. (4.6) becomes

$${}^{R}C^{\gamma\delta\,\mathrm{in}}={}^{B}h_{\mu}{}^{\gamma}\, {}^{B}h_{\nu}{}^{\delta}\, {}^{B}C^{\mu\nu\,\mathrm{in}}[\omega^{\mu}] . \quad (5.19)$$

Using the property demonstrated in Sec. IV, one obtains the correlation functions describing the proper fluctuations as

$$\langle\hat{S}_{\alpha}[\omega]\hat{S}_{\beta}[\omega']\rangle\approx\langle S_{\alpha}[\omega]S_{\beta}[\omega']\rangle=2\pi\delta(\omega+\omega')\, {}^{S}C_{\alpha\beta}[\omega] . \quad (5.20)$$

Then, the quantum response operators $X_{\alpha\beta}[\omega]$ may be replaced by the classical susceptibility functions ${}^{S}\chi_{\alpha\beta}[\omega]$:

$$A^{\mu\,\mathrm{out}}[\omega]=\lambda^{\mu}{}_{\nu}[\omega] A^{\nu\,\mathrm{in}}[\omega]+i\epsilon^{\mu\nu}\, {}^{A}h_{\nu}{}^{\alpha}\hat{S}_{\alpha}[\omega] , \quad (5.21)$$

$$\Lambda^{\mu}{}_{\nu}[\omega]=\delta^{\mu}{}_{\nu}+i\epsilon^{\mu\rho}\, {}^{A}h_{\rho}{}^{\alpha}\, {}^{S}\chi_{\alpha\beta}[\omega]\, {}^{A}h^{\nu\beta} . \quad (5.22)$$

As a matter of fact, the correlation function deduced from this expression is the same as the correct correlation function when the terms containing products of two response operators $X_{\alpha\beta}[\omega]$ are neglected (they are much smaller than the terms associated with proper fluctuations). The terms containing only one response operator are correctly given by the susceptibility functions and describe the transformation of the input field by the nonlinear system (absorption, dispersion, or parametric transformation).

The situation studied here of a weakly coupled reservoir corresponds to a large number of experimental situations. Its range of applicability is completely different from the semiclassical limit although the quantum fluctuations can be understood with the simple concept of classical susceptibility functions.

This approach has been used for studying the field emitted by a collection of atoms (which do not behave as semiclassical systems) placed inside a cavity [24,25]. Assuming that the atomic relaxation is determined essentially by the side modes, the atomic dipoles can be written as a sum of a linear response to the field in the cavity mode and of proper atomic fluctuations (in fact the linear response to the input field in the side modes).

VI. SUMMARY

In this last section, we summarize the results obtained in the paper. This summary can conveniently be used as a starting point for the application of the generalized linear input-output theory to specific problems.

The mean values of the system observables obey the following relaxation equations:

$$d_t\langle S_{\alpha}(t)\rangle=[-iL_{\alpha}{}^{\beta}-K_{\alpha}{}^{\beta}]\langle S_{\beta}(t)\rangle , \quad (6.1)$$

where the coefficients $L_{\alpha}{}^{\beta}$ are related to the system eigenfrequencies and the relaxation coefficients $K_{\alpha}{}^{\beta}$ are given by the expressions (4.5). These equations are obtained in a perturbative relaxation theory by assuming that the noise spectra characterizing the reservoir fluctuations are flat on each frequency band taken separately.

But it has not been assumed that the input reservoir fields are in the vacuum state.

The relaxation equations are solved in the frequency space in terms of Green functions obtained as the inverse of the evolution matrix:

$$\Theta(t)\langle S_\alpha(t)\rangle = G_\alpha{}^\beta(t)\langle S_\beta(0)\rangle \ , \tag{6.2}$$

$$G_\alpha{}^\gamma[\omega][-i\omega\delta_\gamma{}^\beta + iL_\gamma{}^\beta + K_\gamma{}^\beta] = \delta_\alpha{}^\beta \ . \tag{6.3}$$

The correlation functions of the system observables are evaluated by using the quantum regression theorem:

$$\Theta(t)\,{}^S C_{\alpha\beta}(t) = G_\alpha{}^\gamma(t)\,\bar{S}_{\gamma\beta} \ , \tag{6.4}$$

$${}^S C_{\alpha\beta}[\omega] = G_\alpha{}^\gamma[\omega]\,\bar{S}_{\gamma\beta} + G_\beta{}^\gamma[-\omega]\,\bar{S}_{\alpha\gamma} \ . \tag{6.5}$$

One then deduces the retarded susceptibility functions:

$${}^S\chi_{\alpha\beta}[\omega] = iG_\alpha{}^\gamma[\omega](\bar{S}_{\gamma\beta} - \bar{S}_{\beta\gamma}) \tag{6.6}$$

which describe the linear response of the mean values $\langle \delta S_\alpha \rangle$ of the system observables to classical modulations $\mathcal{A}^\mu$ of the input reservoir fields:

$$\langle \delta S_\alpha[\omega]\rangle = {}^S\chi_{\alpha\beta}[\omega]\,\mathcal{R}^\beta[\omega] \ , \tag{6.7}$$

$$\mathcal{R}^\beta = h_\mu{}^\beta \mathcal{A}^\mu \ . \tag{6.8}$$

The quantum fluctuations of the system observables are given by generalized linear-response formula in terms of the quantum fluctuations $A^{\mu\,\mathrm{in}}$ of the input reservoir fields:

$$S_\alpha[\omega] = X_{\alpha\beta}[\omega]\,R^{\beta\,\mathrm{in}}[\omega] \ , \tag{6.9}$$

$$R^{\beta\,\mathrm{in}} = h_\mu{}^\beta\,A^{\mu\,\mathrm{in}} \ . \tag{6.10}$$

The response operators

$$X_{\alpha\beta}[\omega] = iG_\alpha{}^\gamma[\omega](S_{\gamma\beta} - S_{\beta\gamma}) \tag{6.11}$$

have their mean values equal to the susceptibility functions:

$${}^S\chi_{\alpha\beta}[\omega] = \langle X_{\alpha\beta}[\omega]\rangle \ . \tag{6.12}$$

The dynamical terms $G_\alpha{}^\gamma[\omega]$, which are determined by the relaxation equations and contain all the frequency dependence of the response operators, play the same role in the expressions of the susceptibility functions and of the response operators. The commutators $(S_{\gamma\beta} - S_{\beta\gamma})$ are related to the structure of the nonlinear system and do not depend upon the dynamics. They have to be treated as noncommuting system operators.

For example, one can infer the system correlation functions from the generalized linear response formula:

$$\begin{aligned}\langle S_\alpha[\omega]S_\beta[\omega']\rangle &= \langle X_{\alpha\gamma}[\omega]X_{\beta\delta}[\omega']\rangle \\ &\quad\times\langle R^{\gamma\,\mathrm{in}}[\omega]R^{\delta\,\mathrm{in}}[\omega']\rangle \ .\end{aligned} \tag{6.13}$$

The fluctuations derived from these equations are in perfect agreement with the correlation functions computed previously. A comparison of these equations with the case of a linear scatterer (see Sec. II) suggests that the response operators $X_{\alpha\beta}[\omega]$ may be considered as noncommuting quantum impedances.

There exists an operatorial expression for the output field in terms of the input ones and of the system operators:

$$A^{\mu\,\mathrm{out}}[\omega] = A^{\mu\,\mathrm{in}}[\omega] + i\epsilon^{\mu\nu}h_\nu{}^\alpha S_\alpha[\omega] \ . \tag{6.14}$$

Using the generalized linear-response formula, it is then simple to obtain input-output relations for the reservoir fields. The output fluctuations can be written in terms of noncommuting transfer functions which generalize the frequency-dependent reflectivities describing the scattering upon a harmonic scatterer (see Sec. II):

$$A^{\mu\,\mathrm{out}}[\omega] = \Lambda^\mu{}_\nu[\omega]\,A^{\nu\,\mathrm{in}}[\omega] \ , \tag{6.15}$$

$$\Lambda^\mu{}_\nu[\omega] = \delta^\mu{}_\nu + i\epsilon^{\mu\rho}h_\rho{}^\alpha X_{\alpha\beta}[\omega]\,h_\nu{}^\beta \ . \tag{6.16}$$

These quantum transfer functions preserve the commutation relations of the reservoir fields for any nonlinear system, which can be considered as an *a posteriori* justification of the method.

The correlations between system and reservoir fluctuations are obtained in a simple manner from the input fluctuations and from the susceptibility functions:

$$\begin{aligned}\langle S_\alpha[\omega]A^{\nu\,\mathrm{in}}[\omega']\rangle &= {}^S\chi_{\alpha\gamma}[\omega]\langle R^{\gamma\,\mathrm{in}}[\omega]A^{\nu\,\mathrm{in}}[\omega']\rangle \ , \\ \langle A^{\mu\,\mathrm{in}}[\omega]S_\beta[\omega']\rangle &= {}^S\chi_{\beta\delta}[\omega']\langle A^{\mu\,\mathrm{in}}[\omega]R^{\delta\,\mathrm{in}}[\omega']\rangle \ .\end{aligned} \tag{6.17}$$

In the particular case where the input field is in the vacuum state, the normally ordered correlation functions characterizing the output field fluctuations (to be compared with the results of standard quantum-optical methods) may be written

$$\begin{aligned}\Delta C^{\mu\nu}[\omega] &= \epsilon^{\mu\rho}\epsilon^{\sigma\nu}h_\rho{}^\alpha h_\sigma{}^\beta D_\alpha{}^\gamma{}_\beta{}^\delta[\omega] \\ &\quad\times\{\bar{S}_{\delta\gamma}\Theta(\omega^\mu) - \bar{S}_{\gamma\delta}\Theta(-\omega^\mu)\} \ ,\end{aligned} \tag{6.18}$$

$$D_\alpha{}^\gamma{}_\beta{}^\delta[\omega] = G_\alpha{}^\gamma[\omega]\delta_\beta{}^\delta - \delta_\alpha{}^\gamma G_\beta{}^\delta[-\omega] \ . \tag{6.19}$$

In the limiting case where we are interested in some components weakly coupled to the system, the quantum fluctuations of the system and of the output fields can be written as a sum of two contributions:

$$S_\alpha[\omega] = {}^S\chi_{\alpha\beta}[\omega]\,h_\mu{}^\beta A^{\mu\,\mathrm{in}} + \hat{S}_\alpha[\omega] \ , \tag{6.20}$$

$$A^{\mu\,\mathrm{out}}[\omega] = \lambda^\mu{}_\nu[\omega]\,A^{\nu\,\mathrm{in}}[\omega] + i\epsilon^{\mu\nu}h_\nu{}^\alpha\hat{S}_\alpha[\omega] \ , \tag{6.21}$$

$$\lambda^\mu{}_\nu[\omega] = \delta^\mu{}_\nu + i\epsilon^{\mu\rho}h_\rho{}^\alpha\,{}^S\chi_{\alpha\beta}[\omega]\,h_\nu{}^\beta \ . \tag{6.22}$$

The first contribution is described by the classical susceptibility functions ${}^S\chi_{\alpha\beta}$ or the classical transfer functions $\lambda^\mu{}_\nu$. The second contribution corresponds to extra fluctuations coming from the other components which are uncorrelated with the previous ones. These extra fluctuations can be considered as a proper noise of the system described by the usual correlation functions [24,25]:

$$\langle \hat{S}_\alpha[\omega]\hat{S}_\beta[\omega']\rangle = 2\pi\delta(\omega+\omega')\,{}^S C_{\alpha\beta}[\omega] \ . \tag{6.23}$$

ACKNOWLEDGMENTS

Thanks are due to C. Cohen-Tannoudji, M. Devoret, D. Esteve, C. Fabre, E. Giacobino, P. Grangier, A. Heidmann, L. Hilico, and M. T. Jaekel for discussions. This work has been done in the frame of the EEC Contract No. Esprit BRA 3186. Laboratoire de l'Ecole Normale Supérieure et de l'Université Pierre et Marie Curie associé au Centre National de la Recherche Scientifique.

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[43] The operators S_α (S^α) are interpreted as elements of a column (row) matrix while expressions having the form $M_\alpha{}^\beta$ are interpreted as elements of a square matrix. However, one has to be careful with a tensorial form such as ${}^s\eta^{\alpha\beta}$.
[44] This theorem has been extensively used in resonance fluorescence theory [28]. It is valid in the situations studied in this paper because of the Markov approximation (white noise for each band taken separately [36]). It can indeed be derived from the quantum Langevin equations (26) by multiplying on the right by $S_\gamma(0)$, taking the average, and using the fact that system operators are not correlated with the input fields evaluated at a later time.
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