Casimir effect in absorbing media

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(Received 3 February 1992)

The Casimir force between two dielectric slabs of finite thickness is calculated for the case of dissipative media. The medium is modeled as a continuous field of quantum harmonic oscillators interacting with the heat bath. The electromagnetic field inside and outside the cavity formed by the plates is found for the ground state of the coupled system, and its pressure is calculated. Two terms in the expression for the Casimir force are distinguished. One is the electromagnetic vacuum pressure, which is the only contribution in the case of lossless media. The other arises from the Langevin forces that appear together with the damping as the result of the interaction of atoms with the heat bath. It is shown that both contributions are necessary in order to arrive at a finite result.

PACS number(s): 12.20.−m, 42.50.−p, 03.65.−w

I. INTRODUCTION

One of the problems that has received considerable recent attention is that of the quantization of the electromagnetic field in the presence of a dielectric medium [1—11]. The interest in this problem is related to the development of cavity QED [12] and to experiments in cavities with non-perfectly-reflecting walls. Theories have been developed that allow one to calculate various quantum electrodynamical effects like the Casimir effect [5], spontaneous emission in a cavity [6], level shift and spontaneous emission in a dielectric microsphere [7], and the spectrum of squeezing in a leaky cavity [8]. The modification of the spontaneous emission has been also studied for an atom located in the vicinity of a dielectric wall [9], inside a dielectric slab [10], and in a bulk dielectric [11,13]. Another large group of papers has arisen from the need to describe the propagation of light through a medium [14,15], in particular the effect of the dielectric medium on the quantum statistical properties of the field. For example, nonlinear effects like the production of squeezed states have been examined [15].

The difficulty with the quantization of the electromagnetic field in dielectrics stems from the fact that the medium may be in general nonlinear, nonhomogeneous, dispersive and dissipative, and it is not easy to handle all of these properties together. The question of quantization in dispersive and lossy media was recently treated by Huttner and Barnett, who generalized the quantum microscopic approach [16] first for dispersive [2] and then for dissipative media [17]. However, so far they have considered homogeneous media only. On the other hand, the standard quantum macroscopic theory for nonhomogeneous media [1] does not take dispersion into account and refers to a medium described by a constant refractive index. Although useful for many applications, this approach, as well as all other approaches neglecting losses, is generally incorrect. It is well known that the dielectric function $\epsilon(\omega)$ must satisfy the Kramers-Kronig relations, otherwise causality would be violated. According to Kramers-Kronig relations, the imaginary part of a realistic, frequency-dependent dielectric function must not vanish, and that implies the dissipation of radiation energy. Therefore, a complete theory would have to include not only the field and atoms, but also a system that absorbs energy, usually called a heat bath or a reservoir.

The aim of the present paper is to recalculate the Casimir force between two dielectric plates, taking into account that the medium is not lossless. Reviews of the Casimir effect may be found in [18]–[21]. For previous approaches to the Casimir effect in dielectrics see [5] and [22]–[28]. In Sec. II of the present paper the method and the result obtained in [5] for the case of the nonabsorbing medium described by a constant refractive index is reviewed. Section III introduces the model of the medium composed of atoms–damped quantum oscillators, and the equations of motion for the atoms coupled with the electromagnetic field. It is shown in Sec. IV that the field and therefore the Casimir force have two contributions, which arise from the electromagnetic vacuum and from the Langevin forces. These are derived in Secs. V and VI. The resulting Casimir force for the case of semi-infinite slabs is studied in Sec. VII. Section VIII gives the results for the general case of finite thickness of the plates, and the conclusions.

II. CASIMIR FORCE BETWEEN THE PLATES: THE METHOD AND THE RESULTS FOR LOSSLESS MEDIA

In a previous paper [5] we calculated the Casimir force between two dielectric, non absorbing plates. As in the case of perfectly conducting slabs, for which the effect was first predicted by Casimir [29], this force can be viewed as the result of the electromagnetic vacuum pressure. The first step to find the force was the quantization of the electromagnetic field in the presence of dielectrics. The medium was assumed to be characterized by a frequency-independent refractive index. Thus it could be eliminated from the problem, its presence marked only by the modification of the modes of the quantized field. The physical vacuum—the ground state obtained in the

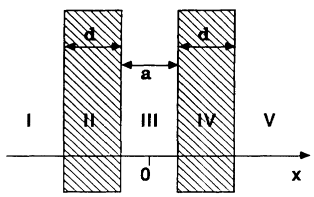

FIG. 1. Configuration of the dielectric plates.

above procedure of quantization— has an important property: the quantum averages of the components of the energy-momentum tensor of the electromagnetic field are not homogeneous in space. In other words, the presence of boundaries leads to the polarization of the vacuum. In particular, the difference between the T_{xx} components of the Maxwell stress tensor outside and inside the cavity results in the force between the plates.

In [5] the effect was calculated for the configuration presented in Fig. 1, in its one-dimensional version (which means that only the modes with wave vectors perpendicular to the plate surface were taken into account). The derived expression for the Casimir force has the form

$$F_C=\int_0^\infty \frac{dk}{2\pi}\hbar\omega_k 2\left[1-\frac{1-|r|^4}{|1-r^2e^{2ika}|^2}\right], \tag{1}$$

with r the reflection coefficient for a single plate:

$$r=\frac{\frac{n-1}{n+1}(e^{2iknd}-1)}{1-\left[\frac{n-1}{n+1}\right]^2 e^{2iknd}} .$$

It is important to note that in the above formula the refractive index n is constant. In the next sections we will generalize the method for the case of absorbing medium, characterized by the complex refractive index depending on frequency.

III. SPECIFICATION OF THE MODEL OF THE ABSORBING MEDIUM COUPLED TO THE ELECTROMAGNETIC FIELD

A. Atoms plus the heat-bath subsystem

In order to take into account the absorption inside the material, we will assume that the medium is composed of the atoms modeled as quantum harmonic oscillators [30] interacting with the heat bath. Their positions and momenta are described by the operators $\mathbf{r}(\mathbf{x})$ and $\mathbf{p}(\mathbf{x})$ that fulfill the commutation relations

$$[r_i(\mathbf{x}),p_j(\mathbf{x}')]=\frac{i\hbar}{\eta}\delta(\mathbf{x}-\mathbf{x}')\delta_{ij} , \tag{2}$$

where η is the number density of atoms. The reservoir representing the system that absorbs the energy (for example, phonons) may be described with help of an infinite set of quantum harmonic oscillators [17,31] or effectively in terms of a damping constant γ and the Langevin force $\mathbf{F}(\mathbf{x},t)$. In the latter case, the interaction of the atoms with the heat bath leads to the equation of motion

$$m\ddot{\mathbf{r}}(\mathbf{x},t)+m\omega_0^2\mathbf{r}(\mathbf{x},t)+m\gamma\dot{\mathbf{r}}(\mathbf{x},t)=\mathbf{F}(\mathbf{x},t) .$$

The appearance of the Langevin force obeying the commutation relation

$$[F_i(\mathbf{x},t),F_j(\mathbf{x}',t')]=\frac{2m\gamma}{\eta}i\hbar\delta_{ij}\delta(\mathbf{x}-\mathbf{x}')\frac{\partial}{\partial t}\delta(t-t')$$

guarantees the conservation of the commutator (2) (see Appendix A). The heat bath is assumed to have zero temperature when we have

$$\begin{aligned}&\langle \mathbf{F}(\mathbf{x},t)\rangle=0 ,\\ &\langle F_i(\mathbf{x},t)F_j(\mathbf{x}',t')\rangle=\frac{\gamma m\hbar}{\eta\pi}\delta_{ij}\delta(\mathbf{x}-\mathbf{x}')\\ &\qquad\times\int_0^\infty d\omega\,\omega e^{-i\omega(t-t')} .\end{aligned} \tag{3}$$

This type of model of the interaction of oscillators with the heat bath has been discussed in [32] together with other models. See also [33]. Note the difference from the most commonly used model in which the rotating-wave approximation (RWA) is made in the interaction of the atomic oscillators with the heat bath [33,34]. There one has

$$\begin{aligned}&\dot a(t)=-i\omega_0 a(t)-\gamma a(t)+F(t) ,\\ &[F(t),F^\dagger(t')]=2\gamma\delta(t-t') ,\\ &\langle F(t)F^\dagger(t')\rangle=2\gamma\delta(t-t') .\end{aligned}$$

The important difference between both models lies in the spectral properties of the noise. While in the model with RWA the Langevin forces may be regarded as white noise, it is no longer true in the present case; from (3) it follows that

$$\begin{aligned}\langle F_i(\mathbf{x}_1,\omega_1)F_j(\mathbf{x}_2,\omega_2)\rangle=&\frac{4\pi m\gamma\hbar}{\eta}\delta_{ij}\delta(\mathbf{x}_1-\mathbf{x}_2)\\ &\times\int_0^\infty d\omega\,\omega\,\delta(\omega-\omega_1)\\ &\times\delta(\omega+\omega_2)\end{aligned} \tag{4}$$

and

$$\begin{aligned}\langle[F_i(\mathbf{x}_1,\omega_1),F_j^\dagger(\mathbf{x}_2,\omega_2)]\rangle=&\frac{4\pi m\gamma\hbar}{\eta}\delta_{ij}\delta(\mathbf{x}_1-\mathbf{x}_2)\\ &\times\omega_1\delta(\omega_1-\omega_2) ,\end{aligned}$$

so the frequency spectrum is not flat.

B. Equations of motion for the atoms and the electromagnetic field

We consider the configuration of the dielectric plates as in Fig. 1. The coupled system, the electromagnetic field plus atoms interacting with the heat bath, is described by the set of equations

$$\begin{aligned}&\dot{\mathbf{r}}(\mathbf{x},t)=\frac{\mathbf{p}(\mathbf{x},t)-(e/c)\mathbf{A}(\mathbf{x},t)}{m} ,\\ &\dot{\mathbf{p}}(\mathbf{x},t)=-m\omega_0^2\mathbf{r}(\mathbf{x},t)-\gamma m\dot{\mathbf{r}}(\mathbf{x},t)+\mathbf{F}(\mathbf{x},t) ,\\ &\Delta\mathbf{A}(\mathbf{x},t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}(\mathbf{x},t)=-\frac{4\pi}{c}\frac{\partial\mathbf{P}_\perp(\mathbf{x},t)}{\partial t} ,\end{aligned} \tag{5}$$

where the damping constant γ and Langevin forces $\mathbf{F}(\mathbf{x},t)$ are related through the fluctuation-dissipation relation (3). The operator $\mathbf{P}_\perp(\mathbf{x},t)$ represents the transverse part of the polarization $\mathbf{P}(\mathbf{x},t)$ of the medium:

$$\mathbf{P}(\mathbf{x},t)=\eta e\mathbf{r}(\mathbf{x},t) .$$

The electric and magnetic fields may be calculated from the vector potential $\mathbf{A}(\mathbf{x},t)$ by the usual relations:

$$\mathbf{E}_\perp(\mathbf{x},t)=-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) , \quad \mathbf{B}(\mathbf{x},t)=\nabla\times\mathbf{A}(\mathbf{x},t) .$$

IV. TWO CONTRIBUTIONS TO THE CASIMIR EFFECT

As in the case of a non absorbing medium, we will calculate the Casimir force as a difference between the T_{xx} components of the Maxwell stress tensor on the left and right sides of the plate in the ground state of the coupled system (physical vacuum). Our calculations for finding the electromagnetic field in the physical vacuum will follow in the spirit of Compagno, Passante, and Persico [35], where they found the density of the magnetic and electric field around the neutral source in the dressed ground state. We assume that at $t=0$ the field is in the vacuum ("free vacuum") state. It evolves according to the equations of motion and after $t\to\infty$ is damped to the ground-state ("physical vacuum") solution.

We will present calculations for the one-dimensional configuration that is restricted to the modes with the wave vector perpendicular to the plate surface. This will allow us to compare the present results with those obtained in our previous paper for lossless media. The Casimir force is then given by the difference of the mean energy density $e(x)=\langle(1/8\pi)[\mathbf{E}^2(x)+\mathbf{B}^2(x)]\rangle$ in the regions I and III.

$$F_C=e^{\mathrm{I}}(x=-a/2-d)-e^{\mathrm{III}}(x=-a/2)=e^{\mathrm{I}}-e^{\mathrm{III}} .$$

In order to find the electromagnetic field, we take the Laplace transform of Eqs. (5) and eliminate the atomic variables. As the result, we obtain the following equation for the field:

$$c^2\frac{\partial^2}{\partial x^2}\mathbf{A}(x,s)-s^2\left[1+\frac{4\pi e^2\eta}{m}\frac{1}{s^2+\omega_0^2+\gamma s}\right]\mathbf{A}(x,s)$$
$$=s\,\mathbf{A}(x,0)+\dot{\mathbf{A}}(x,0)-\frac{4\pi ec\eta}{m}\frac{s\mathbf{F}(x,s)}{s^2+\omega_0^2+\gamma s} . \quad (6)$$

This equation is modified in comparison with the equation for the free field by the presence of the complex refractive index $n(s)=1+(4\pi e^2\eta/m)[1/(s^2+\omega_0^2+\gamma s)]$, and by the presence of the inhomogeneous part containing the Langevin forces. In the above equation we have omitted the terms containing the positions and momenta of the oscillators at $t=0$ because they do not contribute to the field in the vicinity of the plates for $t\to\infty$. We are looking for the solution of Eq. (6) for $t\to\infty$, with the initial condition

$$\mathbf{A}(x,0)=\sum_\lambda\int_{-\infty}^{\infty}dk\,c\left[\frac{\hbar}{\omega_k}\right]^{1/2}\times\mathbf{e}_\lambda[b_{k\lambda}(0)e^{ikx}+b_{k\lambda}^\dagger(0)e^{-ikx}] ,$$

$$\dot{\mathbf{A}}(x,0)=\sum_\lambda\int_{-\infty}^{\infty}dk\,c\left[\frac{\hbar}{\omega_k}\right]^{1/2}\times\mathbf{e}_\lambda[-i\omega_k b_{k\lambda}(0)e^{ikx}+i\omega_k b_{k\lambda}^\dagger(0)e^{-ikx}] ,$$

$$\omega_k=c|k| ,$$

$$\langle b_{k\lambda}(0)\rangle=\langle b_{k\lambda}^\dagger(0)\rangle=\langle b_{k'\lambda'}^\dagger(0)b_{k\lambda}(0)\rangle=0 ,$$

$$\langle b_{k\lambda}(0)b_{k'\lambda'}^\dagger(0)\rangle=\delta(k-k')\delta_{\lambda\lambda'} .$$

This solution has the form

$$\mathbf{A}(x,t)=\mathbf{A}_V(x,t)+\mathbf{A}_L(x,t) ,$$

where we have distinguished two contributions: $\mathbf{A}_L(x,t)$, which depends linearly on the Langevin forces, and $\mathbf{A}_V(x,t)$, formed by the modified initial vacuum modes. Similarly, the Casimir force will be composed of two parts: $F_C=F_V+F_L$, where F_V and F_L will be called, respectively, the contribution from the vacuum and the contribution from the Langevin forces. (There is no cross term because of the property $\langle\mathbf{F}(x,t)\rangle=0$.)

V. CONTRIBUTION FROM THE VACUUM

The solution for $\mathbf{A}_V(x,t)$ for $t\to\infty$ has the form

$$\mathbf{A}_V(x,t)=\sum_\lambda\left[\int_0^{\infty}+\int_{-\infty}^{0}\right]dk\,c\left[\frac{\hbar}{\omega_k}\right]^{1/2}\times\mathbf{e}_\lambda[b_{k\lambda}(0)e^{-i\omega_k t}f_k(x)+b_{k\lambda}^\dagger(0)e^{i\omega_k t}f_k^*(x)] .$$

In the above expressions, the first integral comprises the waves going from left to right, and the second one the waves going from right to left. The mode functions $f_k(x)$ are described by the same formula as for the case of a nonabsorbing medium [5]; the only difference is that, instead of the constant refractive index, we now have

$$n^2=n^2(\omega_k)=1+\frac{\omega_p^2}{\omega_0^2-\omega_k^2-i\gamma\omega_k} , \quad (7)$$

TABLE I. Mode functions $f_k(x)$.

$k>0$	$k<0$	
$e^{ikx}+R_k e^{-ikx}$	$T_{-k}e^{ikx}$	$x\in(-\infty,-a/2-d)$
$C_k e^{ikx}+D_k e^{-ikx}$	$C_{-k}e^{ikx}+D_{-k}e^{-ikx}$	$x\in(-a/2,+a/2)$
$T_k e^{ikx}$	$e^{ikx}+R_{-k}e^{-ikx}$	$x\in(+a/2+d,+\infty)$

where ω_p is the plasma frequency given by

$$\omega_p^2=\frac{4\pi e^2\eta}{m} .$$

The explicit expressions for the mode functions outside and between the plates that are necessary to calculate the Casimir force are given in Table I. The coefficients R_k, C_k, D_k, and T_k are

$$R_k=r\left[1+\frac{t^2e^{2ika}}{1-r^2e^{2ika}}\right]e^{-ik(a+2d)} ,$$

$$C_k=te^{-ikd}/(1-r^2e^{2ika}) ,$$

$$D_k=rte^{ik(a-d)}/(1-r^2e^{2ika}) ,$$

$$T_k=t^2e^{-2ikd}/(1-r^2e^{2ika}) ,$$

$$R_{-k}=R(-k) , \quad C_{-k}=C(-k) ,$$

$$D_{-k}=D(-k) , \quad T_{-k}=T(-k) ,$$

where r and t are the reflection and transmission coefficients for a single plate:

$$r=\frac{\frac{n-1}{n+1}(e^{2iknd}-1)}{1-\left[\frac{n-1}{n+1}\right]^2e^{2iknd}} ,$$

$$t=\frac{\frac{4n}{(n+1)^2}e^{iknd}}{1-\left[\frac{n-1}{n+1}\right]^2e^{2iknd}} . \quad (8)$$

Note that, due to the dissipation, the relations $|R_k|^2+|T_k|^2=1$ and $|r|^2+|t|^2=1$ are no longer valid.

The contribution from the vacuum to the energy density outside and inside the cavity is

$$e_V^{\rm I}=\int_0^\infty\frac{dk}{2\pi}\hbar\omega_k(1+|R_k|^2)+\int_{-\infty}^0\frac{dk}{2\pi}\hbar\omega_k|T_{-k}|^2$$

$$=\int_0^\infty\frac{dk}{2\pi}\hbar\omega_k(1+|R_k|^2+|T_k|^2) ,$$

$$e_V^{\rm III}=\int_0^\infty\frac{dk}{2\pi}\hbar\omega_k(|C_k|^2+|D_k|^2)$$

$$+\int_{-\infty}^0\frac{dk}{2\pi}\hbar\omega_k(|C_{-k}|^2+|D_{-k}|^2) \quad (9)$$

$$=\int_0^\infty\frac{dk}{2\pi}\hbar\omega_k 2(|C_k|^2+|D_k|^2) .$$

The resulting contribution to the Casimir force takes the form

$$F_V=\int_0^\infty\frac{dk}{2\pi}\hbar\omega_k(1+|R_k|^2+|T_k|^2-2|C_k|^2-2|D_k|^2) .$$

For a lossless medium this expression represents the total Casimir force. This is no longer the case when dissipation is present. It is easy to demonstrate this in the limit of infinitely thick plates ($d\to\infty$). We have then $|T_k|^2\to 0$, $|C_k|^2\to 0$, $|D_k|^2\to 0$ (the incoming vacuum is absorbed in the first plate), and we are left with the infinite value

$$F_V^{d\to\infty}=\int_0^\infty\frac{dk}{2\pi}\hbar\omega_k(1+|R_k|^2) .$$

In Sec. VII we show that the addition of the contribution from the Langevin forces will lead to a finite Casimir force.

VI. CONTRIBUTION FROM THE LANGEVIN FORCES

The long-time solution $\mathbf{A}_L(x,t)=\int_{-\infty}^{\infty}(d\omega/2\pi)\mathbf{A}_L(x,k)e^{-i\omega t}$, $k=\omega/c$ has a rather complicated structure, but with the use of a proper notation the formulas may be contracted to a form in which the interpretation of each term is easy. The field in the regions I and III is of primary interest and is given by

$$\mathbf{A}_L^{\rm I}(x,k)=\mathbf{W}_1(k)e^{-ikx} ,$$

$$\mathbf{A}_L^{\rm III}(x,k)=\mathbf{W}_2(k)e^{ikx}+\mathbf{W}_3(k)e^{-ikx} ,$$

where $\mathbf{W}_1(k)$, $\mathbf{W}_2(k)$, and $\mathbf{W}_3(k)$ are

$$\mathbf{W}_1(k)=W(k)e^{iknd}e^{-ik(a+d)}$$

$$\times[\mathbf{K}(1+rr_ne^{2ika})+\mathbf{L}(r_n+re^{2ika})$$

$$+\mathbf{M}te^{ik(a-nd)}+\mathbf{N}tr_ne^{ik(a+nd)}] ,$$

$$\mathbf{W}_2(k)=W(k)[\mathbf{K}r_ne^{2iknd}+\mathbf{L}+\mathbf{M}re^{ika}$$

$$+\mathbf{N}rr_ne^{ik(a+2nd)}] ,$$

$$\mathbf{W}_3(k)=W(k)[\mathbf{K}rr_ne^{ik(a+2nd)}+\mathbf{L}re^{ika}$$

$$+\mathbf{M}+\mathbf{N}r_ne^{2iknd}] ,$$

$$\mathbf{K}=e^{ikna/2}\int_{-a/2-d}^{-a/2}\mathbf{G}(x,\omega)e^{iknx}dx ,$$

$$\mathbf{L}=e^{-ikna/2}\int_{-a/2-d}^{-a/2}\mathbf{G}(x,\omega)e^{-iknx}dx ,$$

$$\mathbf{M}=e^{-ikna/2}\int_{a/2}^{a/2+d}\mathbf{G}(x,\omega)e^{iknx}dx ,$$

$$\mathbf{N}=e^{ikna/2}\int_{a/2}^{a/2+d}\mathbf{G}(x,\omega)e^{-iknx}dx ,$$

$$\mathbf{G}(x,\omega)=\frac{4\pi e\eta}{m}\frac{\mathbf{F}(x,\omega)}{\omega_0^2=\omega^2-i\gamma\omega} .$$

$$W(k)=\frac{we^{ika/2}}{2n(1-r^2e^{2ika})} ,$$

$$w=\frac{2n/(n+1)}{1-r_n^2e^{2iknd}} ,$$

$$r_n=\frac{n-1}{n+1} .$$

r and t are given by (8) and $n^2=n^2(\omega)$ by (7).

The above formulas together with the expression (4) for the correlation of the Langevin forces are sufficient to obtain the mean energy density of the electromagnetic field caused by the Langevin forces. To this end we will have to evaluate integrals of the type

$$\left\langle \int_{-\infty}^{\infty} d\omega' C_1(k') \int_{-a/2-d}^{a/2} dx' f_1(k',x') \times \mathbf{F}(x',\omega') e^{-ik'x} e^{-i\omega' t} \int_{-\infty}^{\infty} d\omega'' C_2(k'') \times \int_{-a/2-d}^{a/2} dx'' f_2(k'',x'') \mathbf{F}(x'',\omega'') e^{-ik''x} e^{-i\omega'' t} \right\rangle ,$$

which give, when (4) is used,

$$\text{const} \times \int_0^\infty d\omega\, \omega\, C_1(k) C_2(-k) \times \int_{-a/2-d}^{-a/2} dx' f_1(k,x') f_2(-k,x') .$$

For the region outside the plates we get

$$e_L^{\mathrm{I}} = \frac{1}{2\pi c} \int_0^\infty dk\, \hbar\omega_k^2 \frac{\omega_p^2 \gamma \omega_k |w|^2 e^{ik(n-n^*)d}}{|\omega_0^2 - \omega_k^2 - i\gamma\omega_k|^2 |n|^2 |1 - r^2 e^{2ika}|^2}$$
$$\times \Bigg[|1 + r r_n e^{2ika}|^2 \int_{-d}^{0} e^{ik(n-n^*)x} dx + |r_n + r e^{2ika}|^2 \int_{-d}^{0} e^{-ik(n-n^*)x} dx$$
$$+ \left[(1 + r r_n e^{2ika})(r_n^* + r^* e^{-2ika}) \int_{-d}^{0} e^{ik(n+n^*)x} dx + \text{c.c.} \right]$$
$$+ |t e^{-iknd}|^2 \int_0^d e^{ik(n-n^*)x} dx + |t r_n e^{iknd}|^2 \int_0^d e^{-ik(n-n^*)x} dx$$
$$+ \left[|t|^2 r_n^* e^{-ik(n+n^*)d} \int_0^d e^{ik(n+n^*)x} dx + \text{c.c.} \right] \Bigg] .$$

Further, we will use the following notation:

$$n = n_1 + i n_2 , \quad n_1 = \mathrm{Re}(n) , \quad n_2 = \mathrm{Im}(n) , \quad e^{2ikn_1 d} = e^{iz_1} , \quad e^{-2kn_2 d} = e^{-z_2} . \tag{10}$$

We will also take advantage of the property

$$2n_1(\omega) n_2(\omega) = \frac{\gamma \omega \omega_p^2}{|\omega_0^2 - \omega^2 - i\gamma\omega|^2} .$$

The contribution from the Langevin forces to the energy density outside the cavity now takes the form

$$e_L^{\mathrm{I}} = \int_0^\infty \frac{dk}{2\pi} \hbar\omega_k \frac{|w|^2 e^{-z_2}}{|n|^2 |1 - r^2 e^{2ika}|^2}$$
$$\times \{ n_1 (e^{z_2} - 1) |1 + r r_n e^{2ika}|^2 + n_1 (1 - e^{-z_2}) |r_n + r e^{2ika}|^2 [i n_2 (e^{-iz_1} - 1)(1 + r r_n e^{2ika})(r_n^* + r^* e^{-2ika}) + \text{c.c.}]$$
$$+ n_1 |t|^2 (e^{z_2} - 1) + n_1 |t r_n|^2 (1 - e^{-z_2}) + [i n_2 |t|^2 r_n^* (e^{-iz_1} - 1) + \text{c.c.}] \} .$$

For the region between the plates we get in a similar way

$$e_L^{\mathrm{III}} = \int_0^\infty \frac{dk}{2\pi} \hbar\omega_k \frac{2(1 + |r|^2)|w|^2}{|n|^2 |1 - r^2 e^{2ika}|^2} \times \{ n_1 |r_n|^2 (1 - e^{-z_2}) e^{-z_2} + n_1 (1 - e^{-z_2}) + [i n_2 (e^{-z_2} - e^{2iknd}) r_n + \text{c.c.}] \} . \tag{11}$$

The contribution from the Langevin forces to the Casimir effect is $F_L = e_L^{\mathrm{I}} - e_L^{\mathrm{III}}$.

VII. CASIMIR FORCE ($F_L + F_V$) FOR THE CASE $d \to \infty$

The appearance of the factor $e^{-2kn_2 d}$ and the fact that t is proportional to $e^{-kn_2 d}$ mean that the above complicated formulas simplify significantly when we take the limit $d \to \infty$, that is, for the case of semi-infinite slabs. As this is also the case most often considered by other authors, it seems to be a good starting point for further considerations.

As previously said, the contribution from the vacuum to the Casimir force is infinite and so is the contribution from the Langevin forces. Using the formulas from Secs. V and VI it is easy to check that

$$F_V(d \to \infty) = e_V^{\mathrm{I}}(d \to \infty) = \int_0^\infty \frac{dk}{2\pi} \hbar\omega_k (1 + |r_n|^2) ,$$

$$e_V^{\mathrm{III}}(d \to \infty) = 0 ,$$

$$e_L^{\mathrm{I}}(d \to \infty) \lim_{d \to \infty} \int_0^\infty \frac{dk}{2\pi} \hbar\omega_k \frac{|w|^2 n_1}{|n|^2 |1 - r^2 e^{2ika}|^2} \times |1 + r r_n e^{2ika}|^2$$
$$= \int_0^\infty \frac{dk}{2\pi} \hbar\omega_k \left| \frac{2}{n+1} \right|^2 n_1$$
$$= \int_0^\infty \frac{dk}{2\pi} \hbar\omega_k (1 - |r_n|^2) ,$$

$$e_L^{\mathrm{III}}(d \to \infty) = \int_0^\infty \frac{dk}{2\pi} \hbar\omega_k \frac{2(1 - |r_n|^4)}{|1 - r_n^2 e^{2ika}|^2} .$$

Two last equalities follow from the identity $|n+1|^2-|n-1|^2=4n_1$.

The total energy density of the field in the region I is equal to

$$e^{\mathrm{I}}=e_V^{\mathrm{I}}+e_L^{\mathrm{I}}=\int_0^\infty \frac{dk}{2\pi}\hbar\omega_k 2 \ , \tag{12}$$

which is exactly the energy density of the free vacuum. We should have expected this result, as in the one-dimensional case the mean energy density does not depend on the position in the region with the constant refractive index, and infinitely far from the plates it should be the same as in the free vacuum.

Although both contributions to the Casimir effect are infinite, the sum (F_L+F_V) gives the finite result

$$F_C=F_V+F_L=(e_V^{\mathrm{I}}-e_V^{\mathrm{III}})+(e_L^{\mathrm{I}}-e_L^{\mathrm{III}})$$
$$=\int_0^\infty \frac{dk}{2\pi}\hbar\omega_k 2\left[1-\frac{1-|r_n|^4}{|1-r_n^2e^{2ika}|^2}\right] . \tag{13}$$

Comparing it with (1), one sees that eventually we have exactly the same formula that was valid for the case without absorption in the medium.

The expression (13) has the advantage that its physical origin is clearly seen. Before applying it, it is useful to transform it further. In Appendix B we show that (13) is equivalent to

$$F_C=\frac{2\hbar}{\pi c}\int_0^\infty ds\, s\frac{r_n^2(is)e^{-(2sa/c)}}{1-r_n^2(is)e^{-(2sa/c)}} \ . \tag{14}$$

Having written it in this form, it is easy to check that our result is in agreement with the expression derived by Lifshitz, who was the first to calculate the force between two semi-infinite dielectric slabs [22].

From (14) it follows that for large distances $(a\geq c/\omega_0)$ between the plates the main contribution to the Casimir force comes from small frequencies where n is constant and we may make the approximation

$$F_C\approx\frac{2\hbar}{\pi c}\int_0^\infty ds\, s\frac{r_n^2(0)e^{-(2sa/c)}}{1-r_n^2(0)e^{-(2sa/c)}}$$
$$=\frac{\hbar c}{2\pi a^2}r_n^2(0)\int_0^\infty d\xi\frac{\xi e^{-\xi}}{1-r_n^2(0)e^{-\xi}} \ .$$

The last integral can be evaluated by use of the special function Φ [36], which satisfies the equations

$$\Phi(z,s,v)=\sum_{m=0}^{\infty}(v+m)^{-s}z^m=\frac{1}{\Gamma(s)}\int_0^\infty dt\frac{t^{s-1}e^{-vt}}{1-ze^{-t}} \ .$$

The expression for the casimir force then takes the form

$$F_C\approx\frac{\hbar c}{2\pi a^2}r_n^2(0)\Phi[r_n^2(0),2,1]$$
$$=\frac{\hbar c}{2\pi a^2}r_n^2(0)\sum_{m=0}^{\infty}\frac{[r_n^2(0)]^m}{(1+m)^2}$$
$$=\frac{\hbar c}{2\pi a^2}\sum_{m=1}^{\infty}\frac{r_n^{2m}}{m^2} \ . \tag{15}$$

The sum appearing in this formula is known as Euler's dilogarithm [36]. This result for the Casimir force was also obtained by another method in [5].

VIII. CASIMIR FORCE FOR THE FINITE PLATE THICKNESS d; CONCLUSIONS

In this section we will use the results of the previous sections to derive the expression for the Casimir force that is valid for any thickness of the plates.

According to (9), the contribution from the vacuum to the field between the plates is

$$e_V^{\mathrm{III}}=\int_0^\infty \frac{dk}{2\pi}\hbar\omega_k\frac{2(1+|r|^2)|t|^2}{|1-r^2e^{2ika}|^2} \ .$$

The contribution from the Langevin forces is given by (11), which turns out (see Appendix C) to be equivalent to

$$e_L^{\mathrm{III}}=\int_0^\infty \frac{dk}{2\pi}\hbar\omega_k\frac{2(1+|r|^2)}{|1-r^2e^{2ika}|^2}(1-|r|^2-|t|^2) \ . \tag{16}$$

Taking into account that for the region outside the cavity the relation (12) is valid, we get

$$F_C=(e_V^{\mathrm{I}}+e_L^{\mathrm{I}})-(e_V^{\mathrm{III}}+e_L^{\mathrm{III}})$$
$$=\int_0^\infty \frac{dk}{2\pi}\hbar\omega_k 2\left[1-\frac{1-|r|^4}{|1-r^2e^{2ika}|^2}\right] . \tag{17}$$

At this point we have proved that the expression (1) obtained for the case of a constant refractive index holds also for a refractive index depending on frequency, in particular for the realistic case when the absorption is present. The difference is that while in the case of a constant refractive index we had to introduce the cutoff function decreasing to zero for large frequencies in order to get a finite result, now this cutoff is achieved in a natural way by $r(\omega_k)\to 0$ for $\omega_k\to\infty$. Moreover, the interpretation of the effect is different in the two cases. A refractive index that is real and does not depend on frequency means that there are no damping and no Langevin forces, and the whole effect comes from the vacuum modified by the presence of boundaries. In the real situation of a dispersive medium, the refractive index is complex and there are losses. According to the quantum theory, damping is accompanied by the Langevin forces. The pressure of the field radiated by atoms due to the Langevin forces adds to the pressure exerted by the modified vacuum.

The expressions (15) and (17) correspond to one-dimensional calculations with only $\mathbf{k}$ vectors normal to the surface taken into account. To get the result valid for a real situation, an integral over $\mathbf{k}_{\parallel}$ should be added (leading to a^{-4} in place of a^{-2} dependence of the force on a separation) and evanescent modes should be included. But our main idea—the necessity of taking into account both contributions to the Casimir force— remains valid.

ACKNOWLEDGMENTS

I am grateful to J. Mostowski for stimulating and supervising this work. I also acknowledge helpful discussions with K. Rzążewski, M. Lewenstein, and T.

Radożycki. Last, but not least, I thank P. Knight and G. Barton for stimulating comments, and C. Eberlein for a careful reading of the manuscript.

APPENDIX A

The quantum evolution of a damped harmonic oscillator is described by the equations

$$\dot{r}=\frac{p}{m}\ ,$$
$$\dot{p}=-m\omega_0^2 r-\gamma m\dot{r}+F\ . \tag{A1}$$

We wish to prove that the commutation relation

$$[F(t),F(t')]=2m\gamma i\hbar\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}i\omega e^{i\omega(t-t')} \tag{A2}$$

guarantees the conservation of the commutator for the position and momentum of the oscillator. The solution of Eqs. (A1) is

$$r(t)=\frac{p(0)}{m(z-z^*)}\left[e^{zt}-e^{z^*t}\right]-\frac{r(0)}{z-z^*}(z^*e^{zt}-ze^{z^*t})+\frac{1}{m(z-z^*)}\int_0^t dt'F(t')(e^{z(t-t')}-e^{z^*(t-t')})\ ,$$

$$p(t)=\frac{p(0)}{z-z^*}(ze^{zt}-z^*e^{z^*t})-\frac{m\omega_0^2 r(0)}{z-z^*}(e^{zt}-e^{z^*t})+\frac{1}{z-z^*}\int_0^t dt'F(t')(ze^{z(t-t')}-z^*e^{z^*(t-t')})\ ,$$

where

$$z=-\frac{\gamma}{2}-i\left[\omega_0^2-\left[\frac{\gamma}{2}\right]^2\right]^{1/2}\ .$$

Then we get

$$[r(t),p(t)]=-[p(0),r(0)]\frac{\omega_0^2}{(z-z^*)^2}(e^{zt}-e^{z^*t})^2-[r(0),p(0)]\frac{1}{(z-z^*)^2}(z^*e^{zt}-ze^{z^*t})(ze^{zt}-z^*e^{z^*t})$$
$$+\frac{1}{m(z-z^*)^2}\int_0^t dt'\int_0^t dt''[F(t'),F(t'')](e^{z(t-t')}-e^{z^*(t-t')})(ze^{z(t-t'')}-z^*e^{z^*(t-t'')})\ . \tag{A3}$$

The calculation of the terms containing the commutator $[r(0),p(0)]$ is straightforward and gives

$$[r(t),p(t)]_0=i\hbar e^{-\gamma t}\ . \tag{A4}$$

This result shows that damping not accompanied by the Langevin forces would lead to the damping of the commutator of the position and momentum. The term in (A3) containing the Langevin forces, when expressed by (A2), takes the form

$$[r(t),p(t)]_F=i\hbar\frac{2\gamma z}{(z-z^*)^2}\int_0^t dt'\int_0^t dt''\left[-\frac{1}{2\pi i}\right]\int_{-\infty}^{\infty}d\omega\,\omega\, e^{i\omega(t'-t'')}(ze^{z(2t-t'-t'')}-ze^{z^*(t-t')}e^{z(t-t'')})-\text{c.c.}$$

Integrating over t' and t'' we get

$$[r(t),p(t)]_F=i\hbar\frac{2\gamma z}{(z-z^*)^2}\left[-\frac{1}{2\pi i}\right]\int_{-\infty}^{\infty}d\omega\,\omega\left[e^{2zt}\frac{e^{(i\omega-z)t}-1}{i\omega-z}\frac{e^{-(i\omega+z)t}-1}{-i\omega-z}-e^{-\gamma t}\frac{e^{(i\omega-z^*)t}-1}{i\omega-z^*}\frac{e^{-(i\omega+z)t}-1}{-i\omega-z}\right]-\text{c.c.}$$

The first integral is zero, and we are left with

$$[r(t),p(t)]_F=i\hbar\frac{2\gamma e^{-\gamma t}}{z-z^*}\frac{1}{2\pi i}\left[\int_{-\infty}^{\infty}d\omega\frac{\omega(e^{\gamma t}+1)}{(\omega+iz^*)(\omega-iz)}-\int_{-\infty}^{\infty}d\omega\frac{\omega(e^{(i\omega-z^*)t}+e^{-(i\omega+z)t})}{(\omega+iz^*)(\omega-iz)}\right]$$
$$=i\hbar(1-e^{-\gamma t})\ .$$

Adding together $[r(t),p(t)]_0$ and $[r(t),p(t)]_F$, we get the proper value of the commutator $[r(t),p(t)]$.

APPENDIX B

Here we show that

$$F_C=\int_0^{\infty}\frac{dk}{2\pi}\hbar\omega_k 2\left[1-\frac{1-|r_n|^4}{|1-r_n^2e^{2ika}|^2}\right]$$

may be rewritten as (14). First let us notice that the formula above is equivalent to

$$F_C=\frac{\hbar}{\pi c}\int_0^{\infty}d\omega\,\omega\frac{2|r_n|^4-2\,\text{Re}(r_n^2e^{2i(\omega/c)a})}{|1-r_n^2e^{2i(\omega/c)a}|^2}$$
$$=-\frac{2\hbar}{\pi c}\text{Re}\int_0^{\infty}d\omega\,\omega\frac{r_n^2e^{2i(\omega/c)a}}{1-r_n^2e^{2i(\omega/c)a}}\ .$$

Now we may replace the integral over the real axis by the integral over the positive imaginary axis (the integrand has no singularities on the upper half-plane). We then get

$$F_C=-\frac{2\hbar}{\pi c}\text{Re}\int_{i0}^{i\infty}d\omega\,\omega\frac{r_n^2e^{2i(\omega/c)a}}{1-r_n^2e^{2i(\omega/c)a}}\ .$$

Putting $\omega=is$ we immediately get (14).

APPENDIX C

Like most formulas from Sec. VI, the expression (11) looks rather complicated, but one requires only some algebra to show that the much simpler formula (16) is equivalent to it. In the transformations below we will use the notation introduced by (10) and the identity $1-|r_n|^2=4n_1/|n+1|^2$. Using the definition (8) we get

$$1-|r|^2=\frac{1}{|1-r_n^2e^{2iknd}|^2}[1-|r_n|^2-|r_n|^2(1-|r_n|^2)e^{-2z_2}-(r_n^2e^{2iknd}+\text{c.c.})+(|r_n|^2e^{2iknd}+\text{c.c.})]$$

$$=\frac{1}{|1-r_n^2e^{2iknd}|^2}\left[\frac{4n_1}{|n+1|^2}-|r_n|^2\frac{4n_1}{|n+1|^2}e^{-2z_2}+\left[r_ne^{2iknd}\frac{2(n^*-n)}{|n+1|^2}+\text{c.c.}\right]\right]$$

and

$$|t|^2=\frac{4/|n+1|^2}{|1-r_n^2e^{2iknd}|^2}\frac{4(n_1^2+n_2^2)}{|n+1|^2}e^{-z_2}$$

$$=\frac{4/|n+1|^2}{|1-r_n^2e^{2iknd}|^2}\left[n_1\frac{4n_1}{|n+1|^2}+\frac{2in_2(n^*-n)}{|n+1|^2}\right]e^{-z_2}$$

$$=\frac{4/|n+1|^2}{|1-r_n^2e^{2iknd}|^2}[n_1(1-|r_n|^2)-(in_2r_n+\text{c.c.})]e^{-z_2}\ .$$

Taking into account the form of the coefficient w we get

$$1-|r|^2-|t|^2=\frac{|w|^2}{|n|^2}\{n_1(1-|r_n|^2e^{-2z_2}-e^{-z_2}+|r_n|^2e^{-z_2})+[(in_2r_ne^{-z_2}-in_2r_ne^{2iknd})+\text{c.c}]\}\ ,$$

which is what was to be proved.

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