Proof of a generalized Robinson's theorem on beam damping

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A Robinson-type theorem concerning beam damping is proved, based directly on Liouville's theorem. The theorem so obtained is somewhat general: The radiating particles need not be relativistic, and they need not form a beam, the emitted radiation may be arbitrary (not necessarily electromagnetic), provided only that the emission process is short enough and not too violent. The proof makes no reference to transfer matrices, nor to properties of the guiding forces (which need not be electromagnetic), except that they are derivable from a Hamiltonian. Then in general the sum of the instantaneous damping coefficients is $\Sigma=\sum_{i=1}^{3}\alpha_i=\frac{1}{2}\sum_{i=1}^{3}\partial\dot{k}_i/\partial p_i=\frac{1}{2}\bar{\nabla}_p\bar{\dot{k}}$, where $\dot{k}$ is the time rate of momentum given up by a particle of momentum $\bar{p}$ and $i=1,2,3$ denotes the three components. When $\bar{\dot{k}}$ is parallel to $\bar{p}$, and $\dot{k}=\sum_{n=M}^{N}\dot{k}(n)$, where $\dot{k}(n)\sim p^n$, then $\Sigma=\sum_n(2+n)\dot{k}(n)/2p$. If, further, the emitting and emitted particles are both ultrarelativistic, and the sum of powers reduces to a single second-order term (as during photon emission by ultrarelativistic electrons in storage rings), then $\Sigma=4\dot{\varepsilon}/2E$, where $\dot{\varepsilon}$ is the power radiated out by particles of energy E. Taking the usual time average for quasi-harmonically oscillating beams gives $\langle\Sigma\rangle_T=4\dot{\varepsilon}/2E$: Robinson's original result. Special cases are discussed, including certain ones where the sum of the damping partition numbers $\neq 4$.

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A well-known theorem due to Robinson [1], concerning the radiation damping of relativistic electron beams, states that the sum of the damping partition numbers equals 4. The original proof of this theorem is based on the fact that the electrons to be damped form a beam, that they are ultrarelativistic, and that the guiding fields are produced by static magnets of rf cavities. The proof further makes use of the properties of beam transfer matrices and assumes that the radiation emitted by the electrons is electromagnetic in nature; i.e., photons are being radiated.

It turns out that a theorem of this type can be proven in an alternative way, considering directly the sum of the damping constants, and basing the argument on Liouville's theorem. The result so obtained is more general than the original one: The guiding forces may be arbitrary (need not be caused by static magnets or rf fields, indeed, they need not be electromagnetic in origin); the particles radiated away need not be photons, but may be arbitrary; the radiating particles need not be electrons, may be nonrelativistic, and they do not have to form a beam, provided that certain conditions hold. Indeed, a Robinson-type theorem of increasingly restrictive form can be proven, if enough of the conditions listed below hold.

(i) The guiding forces can be derived from a Hamiltonian.

(ii) The decay process is short enough, and not too violent:

(a) To evaluate a change in the phase-space volume of the radiating particles during the time δt that a radiated particle is emitted, it suffices to calculate to first order; higher-order terms can be neglected.

(b) During emission the momentum of a radiating particle changes, but any variation in position of the radiating particles, as a result of this change, can be neglected.

(c) The fractional change in the momentum of the emitting particles is small during the time δt.

(iii) The momentum $\delta\bar{k}$ given up by the radiating particles during the emission of (a) radiated particle(s) is parallel to the direction of $\bar{p}$ during the emission process.

(iv) δk is proportional to a polynomial of p.

(v) The radiating particles are relativistic, so that their energy E satisfies $p\approx E/c$, and δk is related to the energy, $\delta\varepsilon$, radiated out by $\delta k\approx\delta\varepsilon/c$.

When the polynomial reduces to a single second-order term, the original form of Robinson's theorem emerges.

To obtain the result, choose three orthogonal coordinate axes [2] x_1, x_2, and x_3. Denote by ΔV_1 an infinitesimal phase-space element associated with the x_1 direction, etc.,

$$\Delta V_i=\Delta x_i\Delta p_i \quad (i=1,2,3)\ , \tag{1}$$

where p_i is the ith momentum component. A six-dimensional phase-space element can then be written as

$$\Delta V=\Delta V_1\Delta V_2\Delta V_3\ . \tag{2}$$

The fractional variation of ΔV can be expressed as

$$\frac{\delta\Delta V}{\Delta V}=[(\delta\Delta V_1)\Delta V_2\Delta V_3+\Delta V_1(\delta\Delta V_2)\Delta V_3+\Delta V_1\Delta V_2\delta\Delta V_3]/\Delta V_1\cdot\Delta V_2\cdot\Delta V_3=\sum_{i=1}^{3}\frac{\delta\Delta V_i}{\Delta V_i}\ . \tag{3}$$

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Consider the variation of ΔV during the time interval δt. Assume that during this time the ensemble of "original" objects, whose phase-space points are located within ΔV, are acted upon by two kinds of forces: first, by external guiding forces; second, by forces acting as a result of the original objects emitting certain particles. These emitted particles will be referred to as "radiation." The two kinds of forces will cause variations in ΔV, denoted by $\delta_1 \Delta V$ and $\delta_2 \Delta V$, respectively. Condition (iia) guarantees that the total variation is simply the sum of these two:

$$\delta \Delta V = \delta_1 \Delta V + \delta_2 \Delta V \ . \tag{4}$$

According to condition (i), the guiding forces are derivable from a Hamiltonian. Therefore, by Liouville's theorem $\delta_1 \Delta V = 0$. Consequently, one only needs to deal with $\delta_2 \Delta V$, and henceforth the subscript 2 will be omitted from the δ.

Write the variation of ΔV_1 as

$$\delta \Delta V_1 = (\delta \Delta x_1) \Delta p_1 + \Delta x_1 \delta \Delta p_1 \ ,$$

etc., so that

$$\frac{\delta \Delta V}{\Delta V} = \sum_{i=1}^{3} \left[\frac{\delta \Delta x_i}{\Delta x_i} + \frac{\delta \Delta p_i}{\Delta p_i} \right] \ . \tag{5}$$

It follows from condition (iib) that, during δt, any variation in position caused by the emission process is negligible; therefore so is the variation of any difference of positions: $\delta \Delta x_i = 0$. Consequently, now

$$\frac{\delta \Delta V}{V} = \sum_{i=1}^{3} \frac{\delta \Delta p_i}{\Delta p_i} = \frac{\delta(\Delta p_1 \cdot \Delta p_2 \cdot \Delta p_3)}{\Delta p_1 \cdot \Delta p_2 \cdot \Delta p_3} \ . \tag{6}$$

The right-hand side of Eq. (6) is the fractional volume change of a three-dimensional element in (p_1, p_2, p_3) space. To evaluate it, first consider a space element in any three-dimensional Euclidean space, denote its volume by U, and its surface by S. When the space element is distorted, each point on its surface may be moved by a displacement vector $\bar{d}$, where $\bar{d}$ in general depends on the surface point under consideration. As a result of these displacements, U changes by δU, where clearly δU is the surface integral over S of the normal component (pointing outward) of $\bar{d}$. By the divergence theorem this, in turn, equals the integral of $\bar{\nabla} \cdot \bar{d}$ over the volume U, enclosed by S:

$$\delta U = \int_S d\bar{\sigma} \cdot \bar{d} = \int_U du \ \bar{\nabla} \bar{d} \ . \tag{7}$$

When the volume element is small enough so that the fractional variation of $\bar{\nabla} \cdot \bar{d}$ across the volume is negligible, then the right-hand side of the above equation is $U \cdot \bar{\nabla} \bar{d}$, thus

$$\frac{\delta U}{U} = \bar{\nabla} \bar{d} \ . \tag{8}$$

Applying this result to Eq. (6), we are dealing with a three-dimensional Euclidean momentum space, in which a vector has components p_i $(i=1,2,3)$, therefore the divergence operator has components $\partial / \partial p_i$ $(i=1,2,3)$. We are considering an infinitesimal momentum-space element that suffers distortion as a result of particles being radiated, which may cause any point $\bar{p}$ on the surface of the space element to move by $\delta \bar{p}$, depending on the value of $\bar{p}$. If the particle with momentum $\bar{p}$ gives up a net momentum $\delta \bar{k}$ during the time interval δt as a result of radiation, then $\delta \bar{p} = -\delta \bar{k}$. Therefore the fractional variation of the volume ΔV is

$$\frac{\delta \Delta V}{\Delta V} = - \sum_{i=1}^{3} \frac{\partial \delta k_i}{\partial p_i} \equiv - \bar{\nabla}_p \delta \bar{k} \ . \tag{9}$$

This result can be recast, dividing by the time interval δt, and denoting time derivatives by dots: $\delta \Delta V / \delta t \approx \Delta \dot{V}$ and $\delta k_i / \delta t \approx \dot{k}_i$,

$$\Sigma \equiv \sum_{i=1}^{3} \alpha_i = - \frac{1}{2} \sum_{i=1}^{3} \frac{\Delta \dot{V}_i}{\Delta V_i} = - \frac{1}{2} \frac{\Delta \dot{V}}{\Delta V} = \frac{1}{2} \bar{\nabla}_p \dot{\bar{k}} \ . \tag{10}$$

Here $\Delta \dot{V}$ and $\dot{k}$ are the time rate of phase-space volume change and of momentum given up by a radiating particle, respectively, and the $\alpha_i \equiv \frac{1}{2} \Delta \dot{V}_i / \Delta V_i$ are the three damping coefficients. The factor $\frac{1}{2}$ is included to conform to customary notation.

In these equations one may assign two different meanings to $\delta \bar{k}$. First, one can denote by $\delta \bar{k}$ the net momentum actually given up during the time interval δt. In that case the α_i in Eq. (10) characterize the phase-space volume change that actually occurs during δt. Alternatively, one can denote by $\delta \bar{k}$ the *expected* net momentum given up during δt, in which case the calculated α_i characterize the *expected* phase-space volume change during δt. By "expected" here we mean the average taken over many events with identical initial conditions occurring during δt. For storage ring related applications, where the momenta of emitted photons fluctuate according to the laws of quantum mechanics, the latter interpretation is more useful, and in the following that will be the meaning assigned to δk.

Under special circumstances the above result can be made more specific.

According to condition (iic), any change in the momentum of the radiating particle during δt is relatively small, so that both the direction and magnitude of $\bar{p}$ are well defined during δt. Now, if condition (iii) holds, i.e., if $\dot{\bar{k}}$ always points along $\bar{p}$, the net momentum given up by the emitting particle as a result of emission is always parallel to the momentum of the emitting particle, then it is advantageous to describe $\dot{\bar{k}}$ in terms of its spherical coordinates $\dot{k}_p$, $\dot{k}_\Theta$, and $\dot{k}_\varphi$. By assumption, only the radial component $\dot{k}_p = \dot{k}$ is nonzero, and then

$$\bar{\nabla}_p \dot{\bar{k}} = \frac{1}{p^2} \frac{\partial (p^2 \dot{k})}{\partial p} = 2 \frac{\dot{k}}{p} + \frac{\partial \dot{k}}{\partial p} \ . \tag{11}$$

This case is often realized, for example, when the average net momentum carried away by the radiation is zero, as seen from the rest frame of the emitting particle [3]. It holds when ultrarelativistic electrons emit photons, in that case $\delta \bar{k}$ is simply the momentum carried away by the radiation. If, further, condition (iv) also holds, so that $\dot{k}$ is expressible as $\dot{k} = \sum_{n=M}^{N} \dot{k}(n)$ where $\dot{k}(n) = a_n p^n$ and a_n is independent of p (the n is not necessarily integer, nor necessarily positive), then

$$\overline{\nabla}_p\overline{k}=\sum_{n=M}^{N}(2+n)\frac{\dot{k}(n)}{p}\ . \tag{12}$$

When, in addition to the above assumptions, the emitting particles are ultrarelativistic, and have energy E, then $p\approx E/c$. If, also, the emitted radiation consists of electromagnetic radiation, with energy $\delta\varepsilon$, then the radiation is emitted overwhelmingly in the forward direction, so that $\delta k\approx\delta\varepsilon/c$. Then condition (v) is satisfied, and one can write $\dot{k}(n)/p\approx\dot{\varepsilon}(n)/E$, and

$$\overline{\nabla}_p\overline{k}=\sum_{n=M}^{N}(2+n)\frac{\dot{\varepsilon}(n)}{E}\ . \tag{13}$$

One recognizes $\dot{\varepsilon}=\sum_{n=M}^{N}\dot{\varepsilon}(n)$ as the power radiated out by a particle of energy E, during δt.

If under these circumstances the electromagnetic radiation is emitted by electrons accelerated by transverse classical electomagnetic guide fields, then the sum of powers reduces to a single term: $\dot{k}=\dot{k}(2)=a_2p^2$, and

$$\Sigma=\tfrac{1}{2}\overline{\nabla}_p\overline{k}=4\frac{\dot{\varepsilon}}{2E}\ . \tag{14}$$

Except for a time averaging, this is the result originally derived [1] by Robinson; that averaging, usually performed in connection with storage rings, will be discussed before Eq. (21).

VALUES OF THE INDIVIDUAL α_i

The results obtained so far have to do with the sum $\sum_i\alpha_i$ and not with any individual α_i. The reason is that Liouville's theorem concerns itself with the total (six-dimensional) phase-space volume ΔV, rather than the individual ΔV_i. To be able to make statements about any particular ΔV_i one would have to consider not only changes in the volume of the chosen phase-space element, but also know its changes in shape which, in turn, depend on the detailed properties of the guide forces. When additional information about these properties is available, then one may be able to make statements about certain individual α_i. Such is the case when the external guide forces preserve separately the volume of a two-dimensional phase-space element, say ΔV_2 (and then also the remaining four-dimensional volume $\Delta V_1\Delta V_3$). This happens, e.g., when ultrarelativistic electrons circulating in a storage ring emit photons, while being guided by a static transverse magnetic field, if the design orbit lies in the $[\hat{x}_1\hat{x}_3]$ plane and the magnetic field is symmetric around that plane. Then an argument analogous to the one given earlier shows that the time rate of change $\Delta\dot{V}_2$ caused by the emission process satisfies

$$\alpha_2=-\frac{1}{2}\frac{\Delta\dot{V}_2}{\Delta V_2}=\frac{1}{2}\frac{\partial\dot{k}_2}{\partial p_2}\ . \tag{15}$$

If, further, $\dot{k}$ is parallel to p then $\dot{k}_2=p_2\dot{k}/p$, and

$$\frac{\partial\dot{k}_2}{\partial p_2}=\frac{\dot{k}}{p}+p_2\frac{\partial(\dot{k}/p)}{\partial p_2}\ . \tag{16}$$

If $\dot{k}=\sum_{n=M}^{N}\dot{k}(n)$, where $\dot{k}(n)=a_np^n$, and a_n does not depend on p (the n do not necessarily have integer values, and may be negative),

$$\frac{\partial\dot{k}_2}{\partial p_2}=\frac{\dot{k}}{p}+\left[\frac{p_2}{p}\right]^2\sum_{n=M}^{N}(n-1)\frac{\dot{k}(n)}{p}\ , \tag{17}$$

so that whenever p_2/p is small enough, only the first term on the right-hand side of the above equation survives, and

$$\alpha_2\approx\frac{1}{2}\frac{\dot{k}_2}{p_2}\ . \tag{18}$$

For ultrarelativistic electrons radiating photons while circulating in storage rings, p_2 is sufficiently smaller than p_3 (if $\hat{x}_2$ and $\hat{x}_3$ are chosen along the vertical and the beam direction, respectively), so that if the above listed conditions hold, then Eq. (18) is valid. One may then also write $\dot{k}_2/p_2\approx\dot{\varepsilon}/E$. Under these conditions Eq. (14) must hold too, therefore, one also has

$$\alpha_1+\alpha_3\approx3\frac{\dot{\varepsilon}}{2E}\ . \tag{19}$$

EXTENSION TO FINITE TIMES AND PHASE-SPACE SEGMENTS

So far we considered only an infinitesimal phase-space element, and a short time interval δt. In this way we obtained information about the instantaneous values of $\sum_i\alpha_i$ (and perhaps certain individual α_i) referring to the chosen element. By following the evolution of the phase-space element while various external guide forces act on it, one can chart the time evaluation of the quantities of interest, and one can also define their average value calculated over some macroscopic time interval T. The time average of T of any quantity x will be denoted by $\langle x\rangle_T$. This procedure is particularly useful when the particles under consideration follow closed orbits located within a finite configuration volume, e.g., electrons circulating in a storage ring. Different groups of electrons, of course, follow different orbits around the ring, are subject to different guide forces, and experience different damping along the way. However, after a sufficiently long time T_0, all groups of electrons have followed arbitrarily closely to any possible orbit, and experienced guide forces arbitrarily close to all possible values. If one chooses $T>T_0$, but T not too long, i.e., still short enough so that during T the fractional change in phase-space volumes is still small, and so is the net fractional change in the beam momentum p, then $\langle\Delta\dot{V}/\Delta V\rangle_T\approx\langle\Delta\dot{V}\rangle_T/\Delta V$ and $\langle\partial\dot{\overline{k}}_i/\partial p_i\rangle_T\approx\langle\partial\dot{\overline{k}}_i\rangle_T/\partial p_i$, so that the sum rule for the time-averaged damping coefficients $\langle\alpha_i\rangle_T$ is

$$\sum_{i=1}^{3}\langle\alpha_i\rangle_T=\overline{\nabla}_p\langle\partial\dot{\overline{k}}_i\rangle_T\ . \tag{20}$$

Similarly, Eqs. (9)–(19) all hold when, in these equations, δk_i, $\delta\Delta V$, $\Delta\dot{V}_i$, $\dot{k}_i$, $\dot{\varepsilon}$, α_i, and Σ are replaced by their respective time-averaged values. These average values are now the same for all circulating electrons; they do not depend on where the phase-space volume to be studied is located.

When considering an infinitesimal phase-space element, a calculation to the lowest order in small quantities yields exact results. The same calculation will give good accuracy, even for finite phase-space segments, provided that its dimensions are small enough. The six-dimensional phase-space segment, occupied by an electron bunch circulating in a storage ring, usually satisfies this condition. One can then describe the volume change of this entire finite segment using the formulas given above. One defines the surface of this segment by some means [4], and observes its change as a function of time. Usually these surfaces are ellipses in each of the two-dimensional phase spaces, and any point on the circumference of an ellipse moves along that surface as a result of interactions with the guide forces. As discussed earlier, photon emission leaves the ellipse diameter parallel to x_i unchanged, but alters the diameter parallel to p_i. This process is fastest when the emitting point is at the end points of the largest diameter along the p_i direction (denoted by $2\sigma_i' p$), slower otherwise, smallest when the point is where the tangent to the ellipse is parallel to the p_i axis. The finite phase-space volume V_i, associated with the ith direction is now the area of the ellipse A_i. The A_i is proportional to $(\sigma_i')^2$. The change in σ_i' at any moment is, in turn, proportional to the change in that infinitesimal phase volume ΔV_i which moves along the perimeter of the ellipse, therefore

$$\frac{\delta V_i}{V_i}=\frac{\delta A_i}{A_i}=2\frac{\delta\sigma_i'}{\sigma_i'}=2\frac{\delta\Delta V_i}{\Delta V_i} .$$

The factor 2 appearing above is canceled by the effect of time averaging. Since any phase-space point moves along the ellipse harmonically, and since the change of $(2\sigma_i p)^2$ is proportional to the square of p_i, time averaging in this special case is particularly simple: equivalent of taking the average of $\cos^2$, which gives a factor $\frac{1}{2}$. Denoting by $\underline{\alpha}_i$ the damping coefficients related to the macroscopic phase-space volumes V_i, one obtains from Eq. (14),

$$\langle\underline{\Sigma}\rangle_T=\sum_{i=1}^{3}\langle\underline{\alpha}_i\rangle_T=4\dot{\varepsilon}/2E . \tag{21}$$

This is Robinson's original result [1].

It may happen that the variables $\bar{x}$ and $\bar{p}$ do not satisfy conditions (ii), but certain related quantities $\bar{x}'$ and $\bar{p}'$ do. In that case the theorem predicts the fractional change in $(\bar{x}',\bar{p}')$ phase space. An example is given at the end of the next section.

EXAMPLES

Here we list examples for which Eq. (14) does *not* hold. First consider a circulating beam of relativistic ions of energy E and momentum $\bar{p}$. Each ion contains at least one electron bound to the nucleus. Let the first excited state of the ions have an energy ε_1 above the ground state. The transition between the ground state and the first excited state has a line shape as shown in Fig. 1. Damping of this ion beam can be induced by irradiation with a beam of photons of appropriate energy ε_p [5]. To be effective, ε_p should be chosen such that as seen from the rest frame of the circulating ions, the photon energy $\varepsilon_p'\approx\varepsilon_1$. In that case, if the ions are originally in the first excited state, the photon beam will induce stimulated emission. As seen from the laboratory, the emitted photon will carry away an expected momentum $\delta\bar{k}$ essentially parallel to $\bar{p}$, the momentum of the emitting ion. To achieve significant damping in this manner requires high-intensity photon beams. However, the same result can be accomplished without energy investment, if one replaces the photons by an undulating transverse static magnetic field [6] of appropriate wavelength λ_u. To be effective, the λ_u has to be chosen so that as seen from the rest frame of the circulating ions, it has the wavelength of photons with energy ε_1. Again $\delta\bar{k}$ is expected to be essentially parallel to p. Since the emitting particles are ultrarelativistic, and the emitted particles are photons, $\bar{k}/p\approx\varepsilon/E$. The intensity of stimulated photon emission is proportional to the square of the transverse field intensity as seen in the ion's rest frame, i.e., to E^2. If this were the whole story one could now substitute $n=2$ in Eq. (13) and obtain Eq. (14). But the emission is also proportional to the cross section, which varies as shown in Fig. 1. When the energies of all the ions lie in the interval $(E\pm\Delta E/2)$, so that in the ionic rest frame the magnetic undulator field is perceived as a (virtual) photon with energy ε_p' lying in region 1 in the figure, then the cross section varies as $(\varepsilon_p')^n$, and from Eq. (13) one obtains $\Sigma=(4+n)\dot{\varepsilon}_s/2E$, where $\dot{\varepsilon}_s$ is the power radiated out due to stimulated emission. If, on the other hand, the ε_p' lies in region 2, the result is $\Sigma=(4-m)\dot{\varepsilon}_s/2E$: for $m=1$, 2, 3 one has damping, for $m=4$ there is no damping, and for $m>4$ the Σ is negative, the ion beam is *anti*damped.

So far we neglected spontaneous emission, which would be there, even in the absence of an external field. The spontaneously emitted power $\dot{\varepsilon}_{sp}$ is essentially a

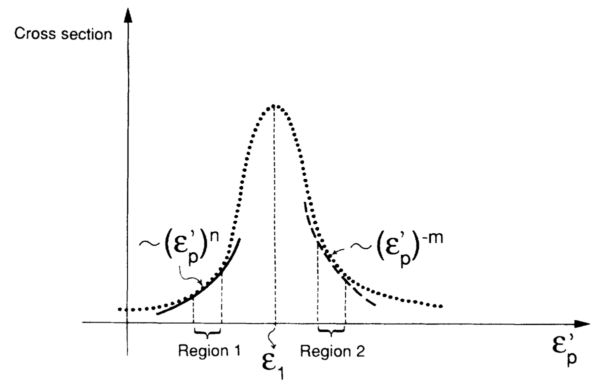

FIG. 1. The photon excitation cross section as a function of the photon energy ε_p. The maximum is located at $\varepsilon_p=\varepsilon_1$. The solid section of the line shape (in region 1) varies as the nth power with energy, the dashed section (in region 2) varies with inverse mth power.

Lorentz invariant [7], so that now $n=0$, and this process contributes a term $\Sigma_{sp}=2\dot{\varepsilon}_{sp}/2E$.

Even if spontaneous processes are neglected, one may have to take into account several terms in Eq. (13). That happens, for example, whenever in the energy region of interest, the cross section is approximated not by a single power of ε'_p but a sum of them. Indeed, the wider the energy region of interest, the more likely it is that a single term approximation will not suffice [8].

As a second example, consider an ensemble of relativistic electrons, whose rest mass, momentum, and energy will be denoted by m, $\bar{p}$, and E, respectively. The electrons initially move in arbitrary directions in a static magnetic field B_2, which is parallel to the $\hat{x}_2$ direction, and oscillates harmonically as a function of x_3. The amplitude of B_2 is sufficiently small, so that changes in the electron momentum are small. Again, $\bar{\dot{k}}$ is parallel to $\bar{p}$, and its magnitude is $\dot{\varepsilon}/c$. The radiated power is proportional to the square of the force acting on the electron, i.e., to $(p\sin\Theta)^2$, where Θ is the angle between $\bar{p}$ and the $\hat{2}$ axis. Therefore, now $n=2$ in Eq. (13), and $\Sigma=4\dot{\varepsilon}/2E$. Since $\dot{\varepsilon}$ varies as $\sin^2\Theta$, the Σ is a function of the direction in which the phase-space element under consideration is moving.

When the electrons are nonrelativistic, then a similar argument holds, except now $\dot{k}=\dot{\varepsilon}(m/p)$. That leads to $\dot{k}\sim p$, so that $n=1$, and $\Sigma=3\dot{k}/2p$. One can express this in terms of the electron kinetic energy, $K=p^2/2m$, as $\Sigma=\frac{3}{4}\dot{\varepsilon}/K$.

Finally, consider the example of a classical electron oscillating linearly and harmonically in a potential well with periodic τ. During the typical photon emission time δt, the electron oscillates many times, and neither $\delta\bar{p}$ nor $\delta\bar{x}$ is negligible. However, one may select one moment during each oscillation period when $\bar{x}=0$ and $\bar{p}$ points in one chosen direction (as opposed to the other) and has the value $\bar{p}=\bar{p}_0$. Looking only at these moments (separated by integer multiples of τ), one can describe the state of the electron by the variables $\bar{x}_0=0$ and $\bar{p}_0$. As a result of photon emission, the change $\delta\bar{p}_0$ during δt will be small compared to $\bar{p}_0$, and $\delta\bar{x}_0=0$, so that these variables satisfy conditions (iib) and (iic). As in the previous example, $\dot{\varepsilon}\sim p^2$, and $\dot{k}\sim p$. Therefore the fractional change of the phase-space volume filled by an ensemble of such oscillating electrons will be determined by $\langle\underline{\Sigma}\rangle_T=3\dot{k}/2p$.

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[1] K. W. Robinson, Phys. Rev. **111**, 373 (1958).

[2] These may, but need not refer to the horizontal, vertical, and longitudinal directions, respectively.

[3] This condition holds, in particular, during the emission of classical dipole electromagnetic radiation.

[4] For example, one may choose some criterion related to the phase-space density. That density can often be considered to be Gaussian.

[5] Paul J. Channell, J. Appl. Phys. **52**, 3792 (1981).

[6] Paul L. Csonka, Proc. SPIE **582**, 298 (1985).

[7] The energy, emitted essentially in the forward direction, is proportional to γ, while the rate of emission varies as γ^{-1}.

[8] In addition to photon emission, the undulator field can also induce absorption. Since the photons are now *absorbed from* the forward direction, the momentum change, $\delta\dot{k}$ is in the same direction as when a photon is *emitted* forward. Therefore this process contributes to damping. Its contribution can be significant, if the magnetic field is strong enough to deexcite a significant fraction of the ions passing through it. We do not discuss this process here. Nor the question of how the ions came to be excited in the first place, or how that excitation may have affected the phase-space volume.