Iterative supersymmetric construction of phase-equivalent potentials

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Potentials providing the same phase shifts as a given potential, but possibly different bound spectra, are constructed by successive supersymmetric transformations. Three situations are considered: suppression of several lowest bound states, addition of a number of bound states below the ground state, and no modification to the bound spectrum. Compact formulas involving physical or nonphysical solutions of the initial Schrödinger equation are established for the phase-equivalent potentials as well as for their bound or free wave functions. Such expressions, referring only to the initial problem, allow a comparison with other methods. The unchanged-spectrum case is shown to be a combination of the other two; it leads to a well-known result of inverse scattering. A general technique of classification of potentials arising from supersymmetry transformations is proposed. The method is illustrated by the Coulomb potential example, for which elementary analytical forms exist for the phase-equivalent potentials.

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I. INTRODUCTION

The interest manifested for supersymmetry [1–4] has given a new impetus to well-known methods in quantum mechanics. Indeed, a supersymmetry transformation does not differ basically from a Darboux transformation of the Schrödinger equation [5,6] or from the factorization method [7]. Whatever the motivation, this renewed interest has led to a wealth of papers on applications of supersymmetry.

The inverse-scattering problem [8,4] and in particular the construction of phase-equivalent potentials [9] are among the most interesting fields of application for supersymmetry. The construction of phase-equivalent potentials without modification of the bound spectrum is well known [8]. On the contrary, methods employed for adding or removing one or several bound states often modify the scattering properties of the potentials [5,4]. In Ref. [10], an iterative procedure is proposed for removing a number of bound states from the spectrum of a given Hamiltonian *without* modifying the phase shifts. In this case, the Levinson theorem and its generalizations [11] impose that the singularity of phase-equivalent potentials at the origin changes since the number of bound states varies. This property helped clarify the relations between different nucleus-nucleus potentials [10,12]. The two-step procedure of Ref. [10] is simplified in Ref. [13] and extended to phase-equivalent potentials obtained by adding a new ground state to the spectrum. More recently, the general problem of using supersymmetry for constructing iteratively phase-equivalent potentials without modifying the spectrum has been reconsidered (Ref. [14], and references therein). However, phase-equivalent potentials appearing in these methods depend on wave functions of several potentials. Relating them with other approaches is difficult. A canonical presentation, based on a reference Hamiltonian, is necessary.

The aim of the present paper is to provide compact formulas for phase-equivalent potentials and their associated wave functions obtained from supersymmetry transformations. Three different cases will be considered: suppression of several lowest bound states, addition of a number of bound states, and no modification to the spectrum. The latter case allows a comparison with literature results [8]. Potentials are constructed by iterating the three procedures proposed in Ref. [13] and by eventually expressing them only as a function of solutions of the initial Schrödinger equation. Our purpose is to clarify the contents of the supersymmetric method and to allow an easy comparison with other approaches to the inverse-scattering problem.

We also explore paths towards more general transformations of the spectrum. Indeed, only the lowest bound states are affected in the formulas that we derive. In fact, the principle of the method can be extended to more general modifications of the spectrum, such as removing or adding a single bound state anywhere in the spectrum.

In order to avoid ambiguities with the notion of phase equivalence, we focus here on the radial Schrödinger problem but the presented techniques could be applied as well to the one-dimensional problem on the full line. Indeed, in the radial problem, phase equivalence is uniquely defined with the phase shifts. On the line, different reflection and transmission amplitudes may lead to identical reflection and transmission probabilities [15,16]. Several types of phase equivalence therefore exist.

In Sec. II the principle of supersymmetric transformations [3] is recalled and notations are introduced. In Sec. III the basic phase-equivalent transformations [13] are reviewed and their properties are extended to nonphysical solutions of the Schrödinger equation. Suppression and addition of several bound states are, respectively, studied in Secs. IV and V. The suppression of bound states and

their reintroduction provide potentials with an unchanged spectrum (Sec. VI). Section VII is devoted to an attempt of general classification based on an associativity property of supersymmetric transformations. The Coulomb potential provides analytical examples of phase-equivalent potentials (Sec. VIII). Concluding remarks are presented in Sec. IX.

II. SUPERSYMMETRY TRANSFORMATIONS

We consider a radial potential $V_0(r)$, vanishing at infinity, which includes possible Coulomb and centrifugal terms. This potential is allowed to be singular at the origin in the following way:

$$V_0(r)\simeq n(n+1)r^{-2}, \quad r\to 0 \tag{1}$$

where n is a non-negative integer which may or may not be equal to the orbital momentum l of a given partial wave. Usually, the potential is considered as nonsingular for $n=l$ and as singular otherwise. The subscript 0 in V_0 refers to the initial potential, i.e., the potential to which transformations are applied.

After elimination of the first-order derivative in the radial part of the Laplacian, the Schrödinger equation for partial wave l reads

$$H_0\psi_0=\left[-\frac{d^2}{dr^2}+V_0(r)\right]\psi_0=E\psi_0 , \tag{2}$$

where units are chosen in such a way that $\hbar^2/2m=1$. Considered as a differential equation, (2) possesses two linearly independent real solutions. Because of the nature (1) of the singularity at the origin, these solutions may behave as r^{n+1} or as r^{-n} at small-r values. At infinity, negative-energy solutions either increase or decrease exponentially. The study of phase equivalence presented in the next sections will require detailed information on the different types of solutions to (2).

The notation ψ_0 will be employed to represent arbitrary (bound or free) *physical* solutions of (2), i.e., *bounded* solutions satisfying

$$\psi_0\sim r^{n+1}, \quad r\to 0 . \tag{3}$$

The increasing bound-state energies of H_0 are denoted as $E^{(0)},E^{(1)},E^{(2)},\ldots$ and the corresponding wave functions as $\psi_0^{(0)},\psi_0^{(1)},\psi_0^{(2)},\ldots$. In the following, we always assume that a sufficient number of bound states exists for the properties that we establish and that bound states are normalized to unity. Positive-energy wave functions are denoted without superscript.

At physical negative energies $E^{(i)}$, solutions of (2) which are linearly independent of $\psi_0^{(i)}$ behave as r^{-n} near the origin and increase exponentially at infinity. At other negative energies E, two linearly independent solutions can be defined as follows. The solution denoted as χ_0 is bounded at the origin, i.e., behaves as r^{n+1}. At nonphysical negative energies, χ_0 increases exponentially at infinity. At physical—negative or positive—energies, χ_0 does not differ from a physical state ψ_0. The solution denoted as f_0 is bounded at infinity. At nonphysical negative energies, f_0 behaves as r^{-n} near the origin and decreases exponentially at infinity. At physical negative energies, f_0 becomes identical to ψ_0.

With the method employed to prove the orthogonality of physical states, one easily obtains, for $E>E^{(i)}$,

$$\int_0^\infty \psi_0^{(i)}\chi_0 dr=0 \tag{4}$$

and, for any E,

$$\int_0^\infty \psi_0^{(i)}f_0 dr=(E^{(i)}-E)^{-1}[W(f_0,\psi_0^{(i)})]_{r=0} , \tag{5}$$

where $W(f,g)=fg'-f'g$ is the Wronskian of f and g.

Now, let us briefly recall the principle of supersymmetry transformations [3]. The operator H_0 is factorized into two mutually adjoint bounded operators

$$H_0=A_0^+A_0^-+\mathcal{E}_0 , \tag{6}$$

where $\mathcal{E}_0$ is a real parameter called the factorization energy. The first-order differential operators A_0^+ and A_0^- are given by

$$A_0^+=(A_0^-)^\dagger=\frac{d}{dr}+\frac{d}{dr}\ln\varphi_0 , \tag{7}$$

where φ_0 is an arbitrary solution of $H_0\varphi_0=\mathcal{E}_0\varphi_0$. In practice, boundedness of A_0^+ and A_0^- requires that φ_0 be nodeless. Therefore, factorization energies must be smaller than the ground-state energy $E^{(0)}$ of H_0. The supersymmetric partner H_1 of H_0 is defined as

$$H_1=A_0^-A_0^++\mathcal{E}_0 \tag{8}$$

and corresponds to the potential

$$V_1=V_0-2\frac{d^2}{dr^2}\ln\varphi_0 . \tag{9}$$

The spectrum of H_1 may differ from the spectrum of H_0 by its ground state according to the choice of φ_0, as detailed below.

As discussed in Refs. [10] and [13], (9) shows that the singularity of V_1 differs from the singularity of V_0 displayed in (1). The wave functions of H_1 are related to those of H_0 by

$$\begin{aligned}\psi_1&=(E-\mathcal{E}_0)^{-1/2}A_0^-\psi_0\\&=(E-\mathcal{E}_0)^{-1/2}\varphi_0^{-1}W(\psi_0,\varphi_0) .\end{aligned} \tag{10}$$

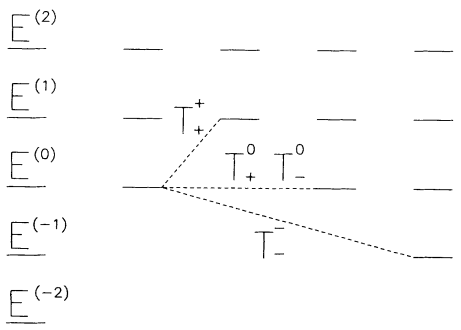

FIG. 1. Schematic representation of the supersymmetry transformations. The convention for the energy scale is also displayed.

TABLE I. Types of supersymmetric transformations.

	$\mathcal{E}_0$	φ_0	Action on the spectrum	Phase-shift modification
T_+^+ (T_1)	$E^{(0)}$	$\psi_0^{(0)}$	Suppresses ground state	$+\tan^{-1}(\lvert E/E^{(0)}\rvert^{1/2})$
T_-^- (T_2)	$\mathcal{E}_0<E^{(0)}$	$\left[\int_0^\infty \varphi_0^{-2}dr=1\right]$	Adds new ground state ($n>0$)	$-\tan^{-1}(\lvert E/\mathcal{E}_0\rvert^{1/2})$
T_-^0 (T_3)	$\mathcal{E}_0<E^{(0)}$	χ_0	None	$-\tan^{-1}(\lvert E/\mathcal{E}_0\rvert^{1/2})$
T_+^0 (T_4)	$\mathcal{E}_0<E^{(0)}$	f_0	None ($n>0$)	$+\tan^{-1}(\lvert E/\mathcal{E}_0\rvert^{1/2})$

This expression can also be written as

$$\psi_1=(E-\mathcal{E}_0)^{1/}\int_{r_0}^{'}\varphi_0^{2}\tilde{\varphi}_0^{-1}\psi_0 dt \tag{11}$$

if r_0 is a value for h the whic Wronskian $W(\psi_0,\varphi_0)$ vanishes. The study of the asymptotic behavior of ψ_1 at a positive energy E shows that the phase shifts provided by V_0 and V_1 differ by a simple monotonic function of E, i.e., $\pm\tan^{-1}(|E/\mathcal{E}_0|^{1/2})$, where the sign depends on the choice of φ_0.

Different types of Hamiltonians H_1 are found according to the choice of the factorization energy $\mathcal{E}_0$ and of the type of solution φ_0 [3]. The four possible transformations are summarized in Table I. We take advantage of the opportunity to replace Sukumar's original notations T_1,T_2,T_3,T_4 by the more mnemonic notations T_+^+,T_-^-,T_-^0,T_+^0. Before explaining them, let us define more precisely our convention about the notations of the bound-state energies. Since we shall be dealing with Hamiltonians which have the same bound states, we shall denote all the bound-state energies of those Hamiltonians by the single notation $E^{(i)}$, where i is an integer which may be negative. Of course, the bound spectrum of H_0 does not contain states with negative i values but such states may appear during the transformation process. The definition and main properties [13] of the four transformations T_+^+,T_-^-,T_-^0,T_+^0 are recalled in Table I and their action is depicted schematically in Fig. 1. The superscript in the notation indicates whether the ground-state energy is raised (+), lowered (−), or left unchanged (0). The subscript represents the sign of the phase-shift modification, i.e., whether the phase shift is increased (+) or decreased (−) (note that the signs in the last column of Table I in Ref. [13] have been inadvertently reversed). The advantage of the notation is to summarize the action of the transformation both on the spectrum and on the phase shifts. Notice that the transformation depends on a parameter, the factorization energy, which is hidden in the present notations. Finally, successive transformations are written from right to left, in an operatorlike manner.

III. PHASE-EQUIVALENT POTENTIALS DERIVED FROM A PAIR OF SUPERSYMMETRY TRANSFORMATIONS

Performing two successive supersymmetry transformations allows one to construct potentials V_2 which are phase equivalent to V_0, i.e., which provide the same phase shifts at all energies [10]. The second step transforms H_1 and V_1 into H_2 and V_2 in a way which depends on the choice of the factorization energy $\mathcal{E}_1$ and on a solution φ_1 of the differential equation $H_1\varphi_1=\mathcal{E}_1\varphi_1$. The phase shifts of V_0 and V_1 differ by $\pm\tan^{-1}(|E/\mathcal{E}_0|^{1/2})$ and those of V_1 and V_2 by $\pm\tan^{-1}(|E/\mathcal{E}_1|^{1/2})$. Therefore, phase equivalence immediately imposes $\mathcal{E}_1=\mathcal{E}_0$. In addition, a transformation of the T_- type must be combined with a transformation of the T_+ type and vice versa. At first sight, eight combinations seem to be possible. However, two of them ($T_+^+T_-^0$ and $T_+^0T_-^-$, to be read from right to left) are forbidden by the condition $\mathcal{E}_1=\mathcal{E}_0$ and three others ($T_+^+T_-^-$, $T_+^0T_-^0$, and $T_-^0T_+^0$) are trivial in the sense that $V_2=V_0$. The only three interesting cases are therefore [13]

$$T_-^0T_+^+,\ T_-^-T_+^0,\ T_-^-T_+^+\ . \tag{12}$$

As indicated by the superscripts, the first pair of transformations suppresses the ground state $E^{(0)}$ (the ground state "rises" from $E^{(0)}$ to $E^{(1)}$), the second pair adds a new ground state at energy $E^{(-1)}<E^{(0)}$, and the third pair leaves the spectrum unchanged. The first and third pairs require $\mathcal{E}_0=E^{(0)}$ while the second pair is valid for any $\mathcal{E}_0=E^{(-1)}<E^{(0)}$. Now we summarize in turn the main properties of the three couples of transformations. Additional information can be found in Ref. [13].

The combined transformation $T_-^0T_+^+$ provides a phase-equivalent potential whose bound spectrum does not contain the ground state of V_0. This potential is given by

$$V_2=V_0-2\frac{d^2}{dr^2}\ln\int_0^r\psi_0^{(0)2}dt\ . \tag{13}$$

Expression (13) clearly shows that V_2 and V_0 only differ by a short-range term. The eigenstates of H_2 also differ from those of H_0 by short-range expressions

$$\psi_2=\psi_0-\psi_0^{(0)}\left[\int_0^r\psi_0^{(0)2}dt\right]^{-1}\int_0^r\psi_0^{(0)}\psi_0 dt\ . \tag{14}$$

The wave function ψ_2 is normalized to unity if ψ_0 is square-integrable and normalized to unity. Another form of (14) arises from the orthogonality of ψ_0 with $\psi_0^{(0)}$: the last integral may be replaced by $-\int_r^\infty\psi_0^{(0)}\psi_0\,dt$.

Let us recall that Eq. (14) is a consequence of the Wronskian expression (10) involving the unbound function, regular at the origin, $\chi_1=[\psi_0^{(0)}]^{-1}\int_0^r\psi_0^{(0)2}dt$, and ψ_1 given by (11), with $\varphi_0=\psi_0^{(0)}$ and $r_0=0$. Such a simple derivation is obviously not restricted to physical states, provided that convergence of the integrals is ensured.

Here, we extend (14) to the unbound functions f_2 and χ_2. For a nonphysical energy E different from $E^{(0)}$, the solutions regular at infinity become

$$f_2=f_0+\psi_0^{(0)}\left[\int_0^r\psi_0^{(0)2}dt\right]^{-1}\int_r^\infty\psi_0^{(0)}f_0dt \tag{15}$$

while for the newly nonphysical energy $E^{(0)}$, the function $f_2^{(0)}$ is given by

$$f_2^{(0)}=\psi_0^{(0)}\left[\int_0^r\psi_0^{(0)2}dt\right]^{-1}. \tag{16}$$

The corresponding expressions for the solution regular at the origin read

$$\chi_2=\chi_0-\psi_0^{(0)}\left[\int_0^r\psi_0^{(0)2}dt\right]^{-1}\int_0^r\psi_0^{(0)}\chi_0dt \tag{17}$$

and

$$\chi_2^{(0)}=f_2^{(0)}\int_0^r f_2^{(0)-2}dt\ . \tag{18}$$

The differences in the integration limits are imposed by the value r_0 at which the Wronskian of f_0 or χ_0 with $\psi_0^{(0)}$ vanishes. Obviously, (14) is a particular case of (15) or (17) since ψ_0 is a common particular case of f_0 and χ_0. Relations (15) and (16) will be useful later on when several bound states are removed (Sec. IV).

The pair $T^-T_+^0$ adds a bound state at energy $E^{(-1)}$ below the ground state without modifying the phase shifts. The potential V_2 is then

$$V_2=V_0-2\frac{d^2}{dr^2}\ln\left[\alpha_0+\int_r^\infty f_0^{(-1)2}dt\right], \tag{19}$$

where $f_0^{(-1)}$ is an exponentially decreasing irregular solution of (2) at energy $E^{(-1)}$. The potential depends on two parameters $E^{(-1)}$ and α_0 which are arbitrary but for the conditions $E^{(-1)}<E^{(0)}$ and $\alpha_0>0$.

The corresponding physical wave functions are given for $E>E^{(-1)}$ by

$$\psi_2=\psi_0-f_0^{(-1)}\left[\alpha_0+\int_r^\infty f_0^{(-1)2}dt\right]^{-1}\int_r^\infty f_0^{(-1)}\psi_0dt \tag{20}$$

and for the new ground state by

$$\psi_2^{(-1)}=\alpha_0^{1/2}f_0^{(-1)}\left[\alpha_0+\int_r^\infty f_0^{(-1)2}dt\right]^{-1}. \tag{21}$$

The wave function $\psi_2^{(-1)}$ is normalized to unity, as well as the other ψ_2 for negative energies. Again, we extend (20) to nonphysical states as

$$f_2=f_0-f_0^{(-1)}\left[\alpha_0+\int_r^\infty f_0^{(-1)2}dt\right]^{-1}\int_r^\infty f_0^{(-1)}f_0dt \tag{22}$$

and

$$\chi_2=\chi_0-f_0^{(-1)}\left[\alpha_0+\int_r^\infty f_0^{(-1)2}dt\right]^{-1}\int_r^\infty f_0^{(-1)}\chi_0dt\ . \tag{23}$$

Equation (23) assumes that the nonphysical solution χ_0 is calculated at an energy E higher than $E^{(-1)}$; with an equation analog to (5) a slightly more complicated expression without any restrictive condition can readily be found. Again, (20) is a particular case of (22) and (23). Notice that at energy $E^{(-1)}$, $f_0=f_0^{(-1)}$ introduced in (22) provides (up to a multiplicative factor) the physical wave function $\psi_2^{(-1)}$.

The couple $T^-T_+^+$ of transformations leads to a potential V_2, distinct from V_0, but with the same bound spectrum and phase shifts. Its expression reads

$$V_2=V_0-2\frac{d^2}{dr^2}\ln\left[(1-\alpha_0)^{-1}-\int_0^r\psi_0^{(0)2}dt\right], \tag{24}$$

where the parameter α_0 must be positive. If α_0 is larger than unity, the argument of the logarithm in (24) is negative but this does not cause any trouble since $(\ln u)'$ is only a convenient notation for u'/u. Notice that the parameter α_0 replaces the parameter α appearing in Eq. (19) of Ref. [13] for reasons which will appear in Sec. VI. Both parameters are related by $\alpha=\alpha_0(1-\alpha_0)^{-1}$. The cases $\alpha_0<1$ and $\alpha_0>1$ lead, respectively, to positive and negative values of the old parameter α. The limit $\alpha_0=1$ leads to the trivial transformation $V_2=V_0$. The limit $\alpha_0\to\infty$ in (24) leads to (13), i.e., to a suppression of the ground state. The physical wave functions become for $E>E^{(0)}$

$$\psi_2=\psi_0-\psi_0^{(0)}\left[(1-\alpha_0)^{-1}-\int_0^r\psi_0^{(0)2}dt\right]^{-1}\int_r^\infty\psi_0^{(0)}\psi_0dt \tag{25}$$

and are normalized if ψ_0 is normalized. The ground-state wave function reads

$$\psi_2^{(0)}=\alpha_0^{1/2}(1-\alpha_0)^{-1}\psi_0^{(0)}\left[(1-\alpha_0)^{-1}-\int_0^r\psi_0^{(0)2}dt\right]^{-1}. \tag{26}$$

Analog expressions for f_2 and χ_2 are given at nonphysical energies by

$$f_2=f_0-\psi_0^{(0)}\left[(1-\alpha_0)^{-1}-\int_0^{(r)}\psi_0^{(0)2}dt\right]^{-1}\int_r^\infty\psi_0^{(0)}f_0dt \tag{27}$$

and

$$\chi_2=\chi_0+\psi_0^{(0)}\left[(1-\alpha_0)^{-1}-\int_0^r\psi_0^{(0)2}dt\right]^{-1}\int_0^r\psi_0^{(0)}\chi_0dt\ . \tag{28}$$

With (4), (28) can be rewritten in a form very similar to (25) or (27), if $E>E^{(0)}$. Equation (25) is a particular case of (27) and (28).

IV. SUPPRESSION OF THE m LOWEST BOUND STATES

The combined transformation $T_0^-T_+^+$ suppresses the ground state of H_0. A similar transformation (with $E^{(1)}$ as factorization energy) suppresses the ground state of H_2, which corresponds to the first excited state of H_0. The potential V_4 appearing in H_4 is phase equivalent to V_0. After m steps, one obtains a potential V_{2m} equivalent to V_0, but without its m lowest bound states [10]. In the following, we establish a compact formula providing V_{2m} directly from V_0 and from the wave functions of its m lowest bound states.

First we define the integrals

$$\Psi_{2k}^{(i,j)}(r)=\int_0^r\psi_{2k}^{(i)}\psi_{2k}^{(j)}dt\ , \tag{29}$$

where $\psi_{2k}^{(i)}$ is the wave function of the bound state at energy $E^{(i)}$ of the Hamiltonian H_{2k} $(i \geq k)$. Choosing the m lowest bound states of H_{2k} $(i=k$ to $k+m-1)$ allows us to define an $m \times m$ symmetric matrix $\Psi_{2k}^{(m)}$, depending on the coordinate r. This matrix vanishes for $r=0$ and tends to the unit matrix when r tends to infinity.

Then, we prove by recurrence that the potential V_{2m} is given by

$$V_{2m} = V_0 - 2\frac{d^2}{dr^2}\ln \det \Psi_0^{(m)} \tag{30}$$

and the wave functions of H_{2m} by

$$\psi_{2m} = (\det \Psi_0^{(m)})^{-1} \begin{vmatrix} \Psi_0^{(0,0)} & \cdots & \Psi_0^{(0,m-1)} & \psi_0^{(0)} \\ \vdots & & \vdots & \vdots \\ \Psi_0^{(m-1,0)} & \cdots & \Psi_0^{(m-1,m-1)} & \psi_0^{(m-1)} \\ \int_0^r \psi_0^{(0)} \psi_0 dt & \cdots & \int_0^r \psi_0^{(m-1)} \psi_0 dt & \psi_0 \end{vmatrix} . \tag{31}$$

For $m=1$, (30) and (31) reduce to (13) and (14). Assuming that (30) is valid for removing $m-1$ states, one may suppress the $m-1$ lowest bound states of H_2 and write

$$V_{2m} = V_2 - 2\frac{d^2}{dr^2}\ln \det \Psi_2^{(m-1)} . \tag{32}$$

Now, we express the elements of $\Psi_2^{(m-1)}$ as a function of the elements of $\Psi_0^{(m)}$. With (29) and (14), one easily shows with a simple integration that

$$\Psi_2^{(i,j)} = \Psi_0^{(i,j)} - [\Psi_0^{(0,0)}]^{-1} \Psi_0^{(i,0)} \Psi_0^{(0,j)} . \tag{33}$$

The determinant property (A2) leads to

$$\det \Psi_2^{(m-1)} = [\Psi_0^{(0,0)}]^{-1} \det \Psi_0^{(m)} \tag{34}$$

from which (30) follows with (13). In the same way, assuming (31) to be valid for removing $m-1$ states, we employ it to express ψ_{2m} as a function of the different ψ_2. Equation (14) can be rewritten as

$$\psi_2 = \psi_0 - [\Psi_0^{(0,0)}]^{-1} \psi_0^{(0)} \int_0^r \psi_0^{(0)} \psi_0 dt \tag{35}$$

and a simple integration leads to

$$\int_0^r \psi_2^{(i)} \psi_2 dt = \int_0^r \psi_0^{(i)} \psi_0 dt - [\Psi_0^{(0,0)}]^{-1} \Psi_0^{(i,0)} \int_0^r \psi_0^{(0)} \psi_0 dt . \tag{36}$$

Equations (33), (35), and (36) show that all the elements of the determinant appearing in the expression of ψ_{2m} share the same form (A1) so that (31) results from (A2).

The wave functions ψ_{2m} corresponding to negative energies are normalized to unity. Equation (31) shows that ψ_{2m} vanishes if ψ_0 is replaced by any $\psi_0^{(i)}$ $(i<m)$: the m lowest bound states of H_0 disappear in the spectrum of H_{2m}. This property clarifies the fact that ψ_{2m} can be written as a function of determinants.

Near the origin V_{2m} behaves as

$$V_{2m} \simeq (n+2m)(n+2m+1) r^{-2}, \quad r \to 0 \tag{37}$$

as shown by the iterative process [10]. This behavior cannot easily be proved from (30). In agreement with (37), ψ_{2m} can be shown by iteration to behave as r^{n+2m+1} near the origin. The asymptotic properties of V_{2m} and ψ_{2m} are easily derived from (30) and (31). Indeed,

$$\det \Psi_0^{(m)} \simeq 1 - \sum_{i=0}^{m-1} \int_r^\infty \psi_0^{(i)2} dt, \quad r \to \infty \tag{38}$$

leads to

$$V_{2m} - V_0 \simeq -2\frac{d}{dr}\psi_0^{(m-1)2}, \quad r \to \infty \tag{39}$$

since $E^{(m-1)}$ is the highest energy among the $E^{(i)}$. Since $\Psi_0^{(m)}$ tends to the unit matrix and the integrals $\int_0^r \psi_0^{(i)} \psi_0 dt$ tend to zero, (31) clearly shows that ψ_{2m} and ψ_0 have the same asymptotic behavior and provide therefore the same phase shifts.

V. ADDITION OF m BOUND STATES BELOW THE GROUND STATE

By successive applications of combined transformations $T^- T_+^0$ with appropriate factorization energies, we add bound states at energies $E^{(-1)} < E^{(0)}$, $E^{(-2)} < E^{(-1)}, \ldots, E^{(-m)} < E^{(-m+1)}$. After m steps, a potential V_{2m} is obtained which is phase equivalent to V_0 provided that m is not larger than $\frac{1}{2}n$ [13].

The addition of bound states necessitates exponentially decreasing solutions $f_{2k}^{(-i)}$ $(i>k)$ at energy $E^{(-i)}$, of the differential equation involving V_{2k}. Let us introduce the integrals

$$F_{2k}^{(i,j)}(r) = \alpha_i \delta_{ij} + \int_r^\infty f_{2k}^{(-i)} f_{2k}^{(-j)} dt , \tag{40}$$

where α_i is a positive parameter. With the functions $f_{2k}^{(-k-1)}$ to $f_{2k}^{(-k-m)}$, we define the $m \times m$ symmetric matrix $\mathbf{F}_{2k}^{(m)}$ whose general element is given by (40). This matrix is in general not bounded when r tends towards 0. At infinity, it becomes a constant diagonal matrix with the α_i as diagonal elements.

A compact form of the potential V_{2m} is given by

$$V_{2m} = V_0 - 2\frac{d^2}{dr^2}\ln \det \mathbf{F}_0^{(m)} . \tag{41}$$

The corresponding wave functions read for $E \neq E^{(-i)}$

$$\psi_{2m}=(\det\mathbf{F}_0^{(m)})^{-1}\begin{vmatrix} F_0^{(1,1)} & \cdots & F_0^{(1,m)} & f_0^{(-1)} \\ \vdots & & \vdots & \vdots \\ F_0^{(m,1)} & \cdots & F_0^{(m,m)} & f_0^{(-m)} \\ \int_r^\infty f_0^{(-1)}\psi_0 dt & \cdots & \int_r^\infty f_0^{(-m)}\psi_0 dt & \psi_0 \end{vmatrix} \tag{42}$$

and for $E=E^{(-i)}$

$$\psi_{2m}^{(-i)}=\alpha_i^{1/2}(\det\mathbf{F}_0^{(m)})^{-1}\begin{vmatrix} F_0^{(1,1)} & \cdots & F_0^{(1,i-1)} & F_0^{(1,i+1)} & \cdots & F_0^{(1,m)} & f_0^{(-1)} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ F_0^{(m,1)} & \cdots & F_0^{(m,i-1)} & F_0^{(m,i+1)} & \cdots & F_0^{(m,m)} & f_0^{(-m)} \end{vmatrix} . \tag{43}$$

In the $m\times m$ determinant appearing in (43), the ith column of matrix $\mathbf{F}_0^{(m)}$ is missing. The wave functions (43) are normalized to unity, as well as the wave functions (42) corresponding to a negative energy.

The proofs of (41) and (42) parallel the proofs encountered in Sec. IV. Relations (41)–(43) reduce, respectively, to (19)–(21) for $m=1$. Let us assume that they are valid for the addition of $m-1$ states. Then, we proceed in two steps: a state is added at energy $E^{(-1)}$ below $E^{(0)}$ and then the $m-1$ other states are added at energies $E^{(-2)}$ to $E^{(-m)}$. The potential V_{2m} is given by

$$V_{2m}=V_2-2\frac{d^2}{dr^2}\ln\det\mathbf{F}_2^{(m-1)} . \tag{44}$$

Again, a simple integration provides with (22)

$$F_2^{(i,j)}=F_0^{(i,j)}-[F_0^{(1,1)}]^{-1}F_0^{(i,1)}F_0^{(1,j)} . \tag{45}$$

With (A2), (45) leads to the relation

$$\det\mathbf{F}_2^{(m-1)}=[F_0^{(1,1)}]^{-1}\det\mathbf{F}_0^{(m)} . \tag{46}$$

Employing (19) then completes the proof of (41). The proof of (42) follows the same pattern: ψ_{2m} is expressed as a function of ψ_2. Then (45), (20), (22), and

$$\int_r^\infty f_2^{(-i)}\psi_2 dt=\int_r^\infty f_0^{(-i)}\psi_0 dt -[F_0^{(1,1)}]^{-1}F_0^{(i,1)}\int_r^\infty f_0^{(-1)}\psi_0 dt \tag{47}$$

show that all the elements appearing in the determinant take the form (A1) so that (A2) is applicable.

Proving (43) requires discussing two subcases. For $i\geq 2$, the proof is parallel to the other ones and makes use of (45), (22), and (A2). For $i=1$, the situation is different since $\psi_{2m}^{(-1)}$ is given as a function of the solutions of H_2 by (42) rather than by (43). However, the property

$$\int_r^\infty f_2^{(-i)}\psi_2^{(-1)}dt=\alpha_1^{1/2}[F_0^{(1,1)}]^{-1}F_0^{(i,1)} \tag{48}$$

combined with (45), (21), and (22) leads to a determinant which is equal (except for its sign) to the determinant appearing in (43), by elementary operations on its rows.

Near the origin, V_{2m} behaves as

$$V_{2m}\simeq(n-2m)(n-2m+1)r^{-2},\quad r\to 0 \tag{49}$$

Relation (49) is easily proved by iteration from Eq. (17) of Ref. [13]. Accordingly, ψ_m behaves as r^{n-2m+1} for $r\to 0$ and is physical only if

$$m\leq\tfrac{1}{2}n . \tag{50}$$

The asymptotic behavior of V_{2m} is given by

$$V_{2m}-V_0\simeq\frac{2}{\alpha_1}\frac{d}{dr}f_0^{(-1)2},\quad r\to\infty \tag{51}$$

because $|E^{(-1)}|$ is smaller than the other $|E^{(-i)}|$ and therefore $f_0^{(-1)}$ presents the slowest exponential decrease. From (42), we see that the asymptotic behavior of ψ_{2m} does not differ from the asymptotic behavior of ψ_0 in full agreement with (51).

VI. UNCHANGED SPECTRUM

The suppression of the m lowest bound states as in Sec. IV and their reintroduction as in Sec. V provides a potential which is equivalent to the initial one, and possesses the same spectrum. This potential differs from the original one by m parameters which appeared during the reintroduction of the bound states.

In order to obtain a compact formula for the equivalent potential, we combine results from the two preceding paragraphs. Let V_{2m} be the potential obtained after the suppression of m bound states and V_{4m} the final potential. From (41), we have

$$V_{4m}=V_{2m}-2\frac{d^2}{dr^2}\ln\det\mathbf{F}_{2m}^{(m)} , \tag{52}$$

where the elements of matrix $\mathbf{F}_{2m}^{(m)}$ are given by (40). We shall employ results from Sec. IV to transform (52) into an expression involving V_0 and its m first wave functions. For this aim, let us define $m\times m$ symmetric matrices $\mathbf{M}_{2k}^{(m)}$ for $k\leq m$. Their elements are

$$M_{2k}^{(i,j)}(r)=\begin{cases} F_{2k}^{(i,j)}(r) & (i,j=0,\ldots,k-1) \\ \int_r^\infty \psi_{2k}^{(i)}f_{2k}^{(j)}dt & (i=k,\ldots,m-1;j=0,\ldots,k-1) \\ (1-\alpha_i)^{-1}\delta_{ij}-\Psi_{2k}^{(i,j)}(r) & (i,j=k,\ldots,m-1) , \end{cases} \tag{53}$$

where $\Psi_{2k}^{(i,j)}$ is given by (29), $F_{2k}^{(i,j)}$ by a formula analogous to (40) but with functions $f_{2k}^{(i)}$ with non-negative superscripts i, and where the α_i are positive parameters. The functions appearing in the definition of matrix $\mathbf{M}_{2k}^{(m)}$ correspond to a potential V_{2k} for which the k lowest energies $E^{(0)}$ to $E^{(k-1)}$ are nonphysical and the $m-k$ energies $E^{(k)}$ to $E^{(m-1)}$ are physical. It is composed of four blocks: a $k\times k$ diagonal block corresponding to nonphysical functions, a $(m-k)\times(m-k)$ diagonal block formed with physical wave functions, and two mixed nondiagonal blocks. Two important particular cases of this matrix are obtained for $k=m$,

$$\mathbf{M}_{2m}^{(m)}=\mathbf{F}_{2m}^{(m)} \tag{54}$$

and for $k=0$

$$\mathbf{M}_{0}^{(m)}=(\mathbf{I}^{(m)}-\boldsymbol{\alpha}^{(m)})^{-1}-\boldsymbol{\Psi}_{0}^{(m)} \tag{55}$$

where $\mathbf{I}^{(m)}$ is the $m\times m$ unit matrix and $\boldsymbol{\alpha}^{(m)}$ is an arbitrary positive-definite constant diagonal matrix.

Equation (54) shows that (52) can be written as

$$V_{4m}=V_{2k+2}-2\frac{d^2}{dr^2}\ln\det\mathbf{M}_{2k+2}^{(m)}\ , \tag{56}$$

where $k=m-1$. We shall now prove that if (56) holds for some $k<m$, one also has

$$V_{4m}=V_{2k}-2\frac{d^2}{dr^2}\ln\det\mathbf{M}_{2k}^{(m)} \tag{57}$$

so that (57) is valid for any $k\le m$. Hence, (55) leads to the final result

$$V_{4m}=V_0-2\frac{d^2}{dr^2}\ln\det[(\mathbf{I}^{(m)}-\boldsymbol{\alpha}^{(m)})^{-1}-\boldsymbol{\Psi}_0^{(m)}]\ . \tag{58}$$

In order to establish (57) from (56), we employ the relation

$$M_{2k+2}^{(i,j)}=M_{2k}^{(i,j)}+[\Psi_{2k}^{(k,k)}]^{-1}M_{2k}^{(i,k)}M_{2k}^{(k,j)} \tag{59}$$

valid for all $i\neq k$ and $j\neq k$. The proof of (59) involves three sectors corresponding to the three kinds of elements in (53). Each proof is based on (14) and/or (15) and leads to a very simple integration as in (33) or (45). With the additional equation (16), one easily proves

$$M_{2k+2}^{(i,k)}=[\Psi_{2k}^{(k,k)}]^{-1}M_{2k}^{(i,k)} \tag{60}$$

for any $i\neq k$, and

$$M_{2k+2}^{(k,k)}=\alpha_k-1+[\Psi_{2k}^{(k,k)}]^{-1}\ . \tag{61}$$

By subtracting row k of $\det\mathbf{M}_{2k+2}^{(m)}$, multiplied by $M_{2k}^{(i,k)}$, from any row i, one obtains

$$\det\mathbf{M}_{2k+2}^{(m)}=(1-\alpha_k)[\Psi_{2k}^{(k,k)}]^{-1}\det\mathbf{M}_{2k}^{(m)}\ . \tag{62}$$

Equation (57) follows from (62) and (13) applied to V_{2k+2}.

The wave functions of H_{4m} are given for $E>E^{(m-1)}$ by

$$\psi_{4m}=[\det\mathbf{M}_0^{(m)}]^{-1}\begin{vmatrix} M_0^{(0,0)} & \cdots & M_0^{(0,m-1)} & \psi_0^{(0)} \\ \vdots & & \vdots & \vdots \\ M_0^{(m-1,0)} & \cdots & M_0^{(m-1,m-1)} & \psi_0^{(m-1)} \\ \int_r^\infty \psi_0^{(0)}\psi_0 dt & \cdots & \int_r^\infty \psi_0^{(m-1)}\psi_0 dt & \psi_0 \end{vmatrix} \tag{63}$$

and for $E^{(i)}$ $(i<m)$ by

$$\psi_{4m}^{(i)}=\alpha_i^{1/2}(1-\alpha_i)^{-1}[\det\mathbf{M}_0^{(m)}]^{-1}\begin{vmatrix} M_0^{(0,0)} & \cdots & M_0^{(0,i-1)} & M_0^{(0,i+1)} & \cdots & M_0^{(0,m-1)} & \psi_0^{(0)} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ M_0^{(m-1,0)} & \cdots & M_0^{(m-1,i-1)} & M_0^{(m-1,i+1)} & \cdots & M_0^{(m-1,m-1)} & \psi_0^{(m-1)} \end{vmatrix}\ . \tag{64}$$

At negative energies, these wave functions are normalized to unity. These relations follow from (42) and (43) by a simple proof, completely analogous to the proof of (58), making use of relations such as

$$\int_r^\infty f_{2k+2}^{(i)}\psi_{2k+2}dt=\int_r^\infty f_{2k}^{(i)}\psi_{2k}dt-[\Psi_{2k}^{(k,k)}]^{-1}M_{2k}^{(i,k)}\times\int_0^r\psi_{2k}^{(k)}\psi_{2k}dt \tag{65}$$

for $i<k$.

The potential V_{4m} possesses the same behavior as V_0 near the origin, as shown by combining (37) and (49). For large distances, (58) indicates that

$$V_{4m}-V_0\simeq 2(1-\alpha_{m-1})\frac{d}{dr}\psi_0^{(m-1)2},\quad r\to\infty \tag{66}$$

because $|E^{(m-1)}|$ is smaller than the other $|E^{(i)}|$. Again V_{4m} and V_0 only differ by short-range terms.

Before closing this section, we would like to emphasize that (58) is a classical result of the inverse problem: see, e.g., Eq. (IV.2.14b) in Ref. [8]. Therefore, in the present case, the supersymmetric approach is only a rather simple way of deriving known results. Notice, however, that the wave functions corresponding to the phase-equivalent potentials are provided simultaneously without additional difficulties.

VII. DISCUSSION

Let us first observe that for $m=1$, the potential V_4 provided by (58) is identical to the potential V_2 displayed in (24). Simultaneously, the wave functions (63) and (64) reduce, respectively, to (25) and (26). This identity justifies the change of parameter between Sec. 3.3 of Ref. [13] and the present equations (24)–(26). In fact, we observe here that $T^-T^+_+$ is completely equivalent to $(T^-T^0_+)(T^0_-T^+_+)$. This property is the basis of a general classification of the transformations leading to phase-equivalent potentials.

Now, we complement the basic properties of supersymmetric transformations, explained in Sec. II, by observing that they satisfy an *associativity* property. Indeed, each modification of the potential introduces an additive term written as the second derivative of the logarithm of a function φ_0 [see (9)]. Successive transformations will involve the product of the different functions φ_0, which is associative. The same associativity holds for the transformation of wave functions since they involve a product of the successive differential operators A^-.

As discussed at the beginning of Sec. III, some couples of transformations are equivalent to an identity and can be written

$$T^+_+T^-_-=T^0_+T^0_-=T^0_-T^0_+=1 \tag{67}$$

provided that both transformations in each pair share the same factorization energy. With the associativity property and (67), the equivalence discussed above becomes obvious

$$(T^-T^0_+)(T^0_-T^+_+)=T^-(T^0_+T^0_-)T^+_+=T^-T^+_+ \ . \tag{68}$$

Note that while the right-hand side of (68) can always be decomposed into the left-hand side, the opposite is true only if the added bound state has the same energy as the removed one. Hence, in the search for the most general transformations involving the pairs $T^0_-T^+_+$, $T^-T^0_+$, and $T^-T^+_+$, the latter pair can be disregarded since it can be obtained by combining the other two.

Taking account of the trivial character of $T^+_+T^-_-$ and $T^0_-T^0_+$, the most general transformation obtainable from $T^0_-T^+_+$ and $T^-T^0_+$ can be written symbolically as

$$\prod_1^{m_-}(T^-T^0_+)\prod_1^{m_+}(T^0_-T^+_+) \ . \tag{69}$$

This corresponds to removing the m_+ lowest bound states followed by adding m_- new bound states. Sections IV–VI describe, respectively, the cases $m_-=0$, $m_+=0$, and $m_+=m_-$ (where the energies of the suppressed and added states are chosen equal). Moreover, the intermediate steps in the proof of the case $m_+=m_-$ provide a solution to the case $m_+>m_-$ (again with common energies for the states suppressed and reintroduced). Most probably, general formulas corresponding to the most arbitrary cases should be derivable in a rather similar way. With $m_-=m_++1$, the expression (69) allows one for example to add a bound state *anywhere* in the bound spectrum without modifying the other bound-state energies.

Notice that the transformations (69) are the most general ones for which the potential is phase equivalent at each intermediate step. Other cases where only the final potential is phase equivalent to V_0 can be thought of, such as

$$\prod_1^m T^-\prod_1^m T^+_+ \ . \tag{70}$$

This type of transformation has been proposed recently by Keung *et al.* [14]. These authors did not try to express the potential only with solutions of the initial problem. Expressing transformations such as (70) or more general ones as a function of the initial Hamiltonian is a difficult task.

VIII. AN EXAMPLE: THE COULOMB POTENTIAL

The attractive Coulomb potential provides an interesting application in which equivalent potentials with elementary analytical expressions can be derived. Though sometimes complicated, such expressions are easily obtained with symbolic computer calculations [17].

First, we consider the suppression of bound states. Amado [18] has already discussed the suppression of the lowest bound state of the Coulomb potential for an arbitrary l value. Let us start from the potential

$$V^l_0=-\frac{2}{r}+\frac{l(l+1)}{r^2} \tag{71}$$

of the lth partial wave. The $l=0$ potentials obtained by removing the ground state (V^0_2) and the two lowest s states (V^0_4) are displayed in Fig. 2. The analytic expression of V^0_2 reads

$$V^0_2=V^0_0+16re^{-2r}\frac{(r-1)+(1+r)e^{-2r}}{[1-(1+2r+2r^2)e^{-2r}]^2} \tag{72}$$

and is a particular case of Amado's Eq. (28). The potentials V^0_2 and V^0_4 are strongly repulsive at small distances because of their respective singular behaviors $6r^{-2}$ and $20r^{-2}$ [Eq. (37)]. They tend to V^0_0 beyond $r\approx 6$ for V^0_2

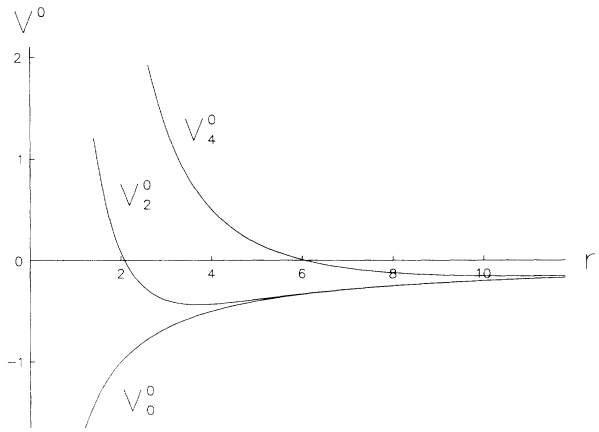

FIG. 2. Suppression of bound states of the $l=0$ Coulomb potential (V^0_0). Phase-equivalent potentials are obtained after removing the ground state (V^0_2) or the two lowest bound states (V^0_4).

and $r \approx 14$ for V_4^0 [see (39)]. Notice that they tend to V_0^0 from above in agreement with the sign displayed in (39) [see also (72)].

Adding bound states is more complicated since two free parameters appear at each addition and because of the restrictive condition (50). Indeed, a single bound state can be added—without modifying the phase shifts—only if the potential singularity is at least $6r^{-2}$, i.e., for $l \geq 2$ in the present case. Two bound states can be added only if $l \geq 4$, and so on. The functions $f_0^{(-i)}$ appearing in (19) or (41) are given by the Whittaker functions [19]

$$f_0^{(-i)}(r) = W_{\gamma_i^{-1}, l+1/2}(2\gamma_i r) = (2\gamma_i r)^{l+1} \exp(-\gamma_i r) \times U(l+1-\gamma_i^{-1}, 2l+2, 2\gamma_i r) , \quad (73)$$

where $E^{(-i)} = -\gamma_i^2$ and U is an hypergeometric function. Notice that while the normalization chosen for $f_0^{(-i)}$ is physically irrelevant, it affects the definition, and therefore the meaning, of parameter α_i. The expression (73) takes an especially simple form if γ_i is the inverse of an integer

$$\gamma_i = 1/n_i . \quad (74)$$

In this case, $f_0^{(-i)}$ can be expressed as a function of generalized Laguerre polynomials as

$$f_0^{(-i)}(r) = (-1)^{l+n_i} (l+n_i)! (2r/n_i)^{-l} \times \exp(-r/n_i) L_{l+n_i}^{(-2l-1)}(2r/n_i) . \quad (75)$$

Hence analytic expressions can be derived for the potentials, which only involve polynomial, exponential, and exponential integral functions. Because even the simplest case is rather complicated, we do not display it here.

In Fig. 3 we display potentials equivalent to V_0^2, with a bound state added at energy -1. The new ground state is related to a potential pocket whose location depends on

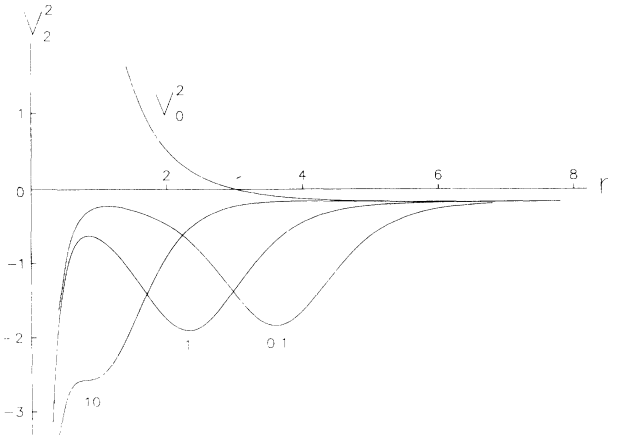

FIG. 3. Addition of a bound state at energy -1 to the $l=2$ Coulomb potential (V_0^2): phase-equivalent potentials V_2^2 for $\alpha_0 = 0.1, 1, 10$.

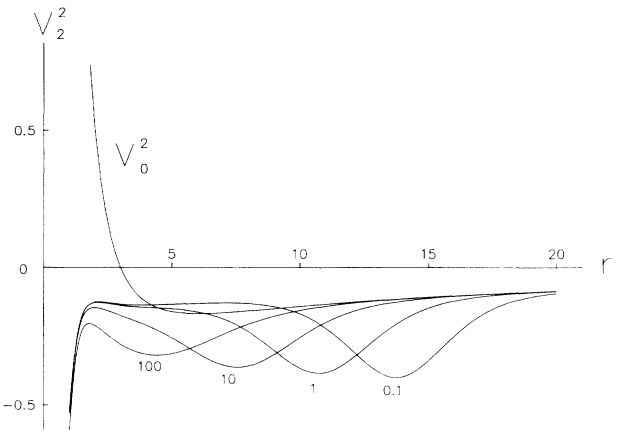

FIG. 4. Addition of a bound state at energy $-\frac{1}{4}$ to the $l=2$ Coulomb potential (V_0^2): phase-equivalent potentials V_2^2 for $\alpha_0 = 0.1, 1, 10, 100$.

α_0. For $\alpha_0 = 0.1$ and 1, this pocket appears between 2 and 4. For the larger parameter 10, the pocket starts fusing with the $-2/r$ singularity near the origin. The potentials V_2^2 tend to V_0^2 from below in agreement with (51). In Fig. 4, the bound state is added to the other energy satisfying (74), i.e., $-\frac{1}{4}$. The properties observed in Fig. 3 remain true for Fig. 4 but the potential pockets are broader and located at larger distances. The fact that the potentials V_2^2 tend more slowly towards V_0^2 for a smaller energy is also explained by (51). Two bound states are added to V_0^4 in Fig. 5, at the energies -1 and $-\frac{1}{4}$. The potential depends on two parameters α_0 and α_1. For the fixed value $\alpha_0 = 10$, the deeper potential pocket containing the state at energy -1 depends on the value of the corresponding parameter α_1. On the contrary, the shallow potential pocket associated with the bound state at energy $-\frac{1}{4}$ is rather insensitive to α_1. The sensitivity to

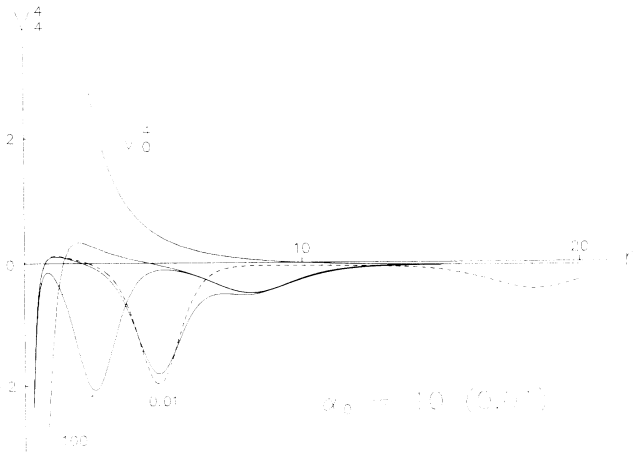

FIG. 5. Addition of two bound states at energies -1 and $-\frac{1}{4}$ to the $l=4$ Coulomb potential (V_0^4): phase-equivalent potentials V_4^4 for $\alpha_0 = 10$ and $\alpha_1 = 0.01, 1, 100$ (solid lines) and for $\alpha_0 = \alpha_1 = 0.01$ (dashed line).

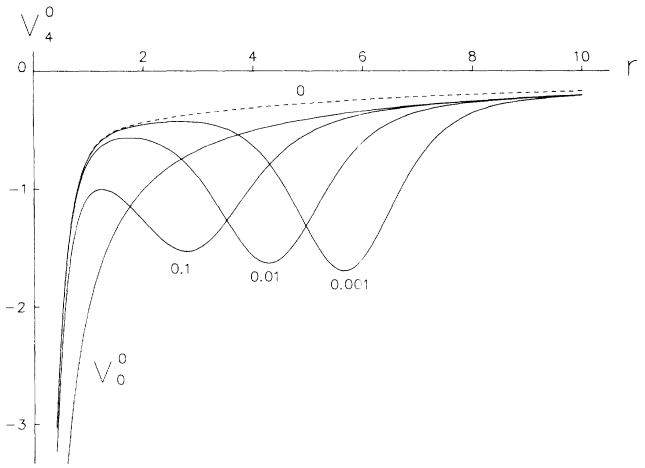

FIG. 6. One-parameter phase-equivalent potentials V_4^0 to the $l=0$ Coulomb potential V_0^0 for $\alpha_0=0.001, 0.01, 0.1$. For $\alpha_0=0$, the potential (dashed curve) is not equivalent to V_0^0.

α_0 (and the insensitivity of the other pocket) is illustrated by the dashed curve ($\alpha_0=0.01$).

Now, we apply the techniques of Sec. VI in order to construct potentials, equivalent to V_0^0, which possess the usual hydrogenic spectrum. Equation (58), or its particular case (24), provides the potential

$$V_4^0 = V_0^0 + 16re^{-2r} \times \frac{\alpha_0(1-\alpha_0)^{-1}(1-r)+(1+r)e^{-2r}}{[\alpha_0(1-\alpha_0)^{-1}+(1+2r+2r^2)e^{-2r}]^2} . \tag{76}$$

Notice the similarity of (76) with (72). In fact, the potential in (72) is the limit of (76) for α_0 tending to infinity, as expected from a comparison of (58) and (30). Equation (76) is illustrated by Fig. 6 for $0<\alpha_0<1$ and by Fig. 7 for $\alpha_0>1$. The different potentials in Fig. 6 are successively

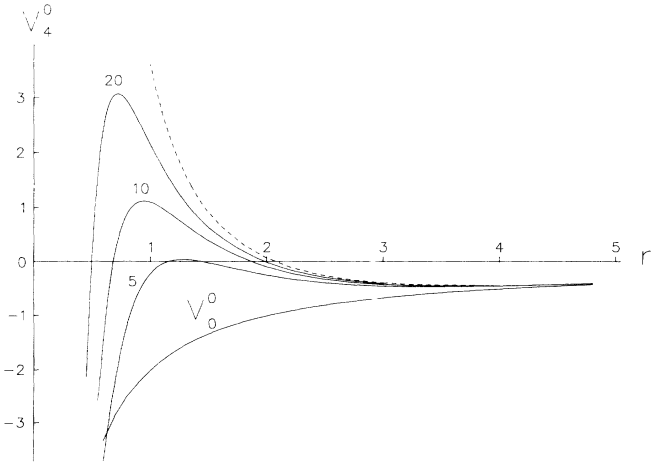

FIG. 7. One-parameter phase-equivalent potentials V_4^0 to the $l=0$ Coulomb potential V_0^0 for $\alpha_0=5, 10, 20$. The limit $\alpha_0\to\infty$ (dashed curve) corresponds to removing the ground state (see V_2^0 in Fig. 2).

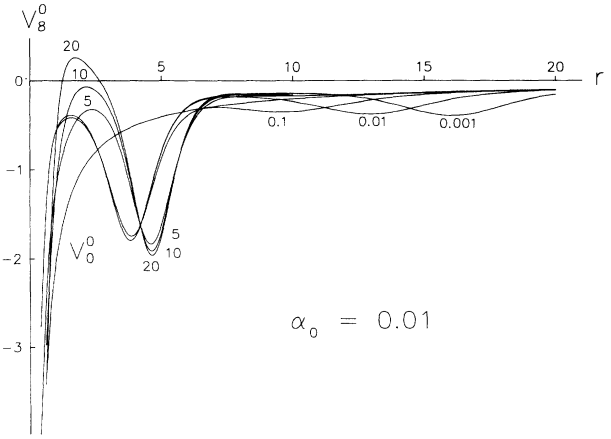

FIG. 8. Two-parameter phase-equivalent potentials V_8^0 to the $l=0$ Coulomb potential V_0^0 for $\alpha_0=0.01$ and $\alpha_0=0.001, 0.01, 0.1, 5, 10, 20$.

larger and smaller than V_0^0 and tend to it from below [see (66)]. A potential pocket moves towards large-r values when α_0 tends to zero. The limit $\alpha_0=0$ (dashed curve) is the envelope of the other curves but is *not* phase equivalent to V_0^0. For $\alpha_0>1$, the curves cross V_0^0 at small-r values, become larger, and tend to it from above. The envelope obtained when α_0 tends to infinity (dashed curve) corresponds to (72) as discussed above. Potentials V_8^0 involving two parameters α_0 and α_1 are presented in Figs. 8 and 9. In Fig. 8 the parameter α_0 is smaller than unity ($\alpha_0=0.01$) and different α_1 are chosen. All the curves display deep pockets around 3.8 ($\alpha_1<1$) or 5 ($\alpha_1>1$). These pockets correspond to the removal and reintroduction of the ground state. A shallow pocket associated with the first excited state appears at large distances for $\alpha_1<1$. If α_0 is modified, the location of the deeper pocket varies as shown by Fig. 9 ($\alpha_0=10$). Otherwise, the qualitative behavior of the different curves in Fig. 9 is very close to the one observed in Fig. 8. As in the case of addition of bound states, we see that each pa-

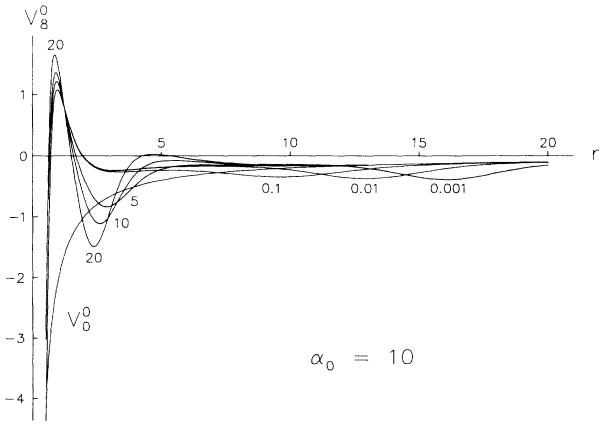

FIG. 9. Two-parameter phase-equivalent potentials V_8^0 to the $l=0$ Coulomb potential V_0^0 for $\alpha_0=10$ and $\alpha_1=0.001, 0.01, 0.1, 5, 10, 20$.

rameter controls essentially one of the qualitative properties of the potential. Phase-equivalent potentials to V_0^0 are displayed in Fig. 3 of Ref. [14] but they correspond to the scheme (70) and involve different parameter definitions.

IX. CONCLUSION

The supersymmetric factorization is an efficient tool for establishing potentials which are phase equivalent to a given potential. The present derivation only involves elementary properties of determinants and of second-order differential equations. For all types of spectrum modifications considered in the present paper, compact formulas involving determinants of simple integrals are obtained for the potentials. Furthermore, the wave functions are derived almost as easily as the potentials and also involve determinants. The reasons for a systematic apparition of determinants remain unclear except in the case of a suppression of bound states (Sec. IV). The compact formulas are not necessarily more efficient in numerical calculations than the original iterative procedure [10] but allow an easy comparison with other approaches. For example, potentials with an unchanged spectrum (Sec. VI) are already known from the traditional treatment of the inverse-scattering problem [8].

One of the three cases considered here (unchanged spectrum) is in fact a consequence of the other two. Each parameter appearing in a phase-equivalent potential with an unchanged spectrum is due to the reintroduction of a bound state after its suppression. Convenient notations for the four basic supersymmetry transformations [3] and an associativity property allow us to perform a general classification of supersymmetric transformations which leave the phase shifts unchanged at each step of the transformation. More general spectrum modifications should be derivable along the lines of the proofs encountered in the paper. Such extensions would provide transformations of the spectrum which are forbidden until now to the supersymmetric approach, such as removing or adding a bound state *above* the ground state.

More general combinations of transformations exist, in which only the final potential is phase equivalent to the initial one. The present notations easily allow us to write such transformations and to classify them. However, whether these more general combinations lead to new families of equivalent potentials, i.e., provide more freedom in adding or removing bound states, remains an open question.

ACKNOWLEDGMENT

We would like to thank M. Kruglanski for his help in employing the MATHEMATICA symbolic computer language.

APPENDIX

In this appendix we present the simple determinant property on which the proofs in Secs. IV and V are based. Consider an $m \times m$ matrix $\mathbf{A}^{(m)}$ with elements a_{ij} $(i,j=1,\ldots,m)$. With these elements and the assumption $a_{11} \neq 0$, we define the $(m-1)\times(m-1)$ matrix $\mathbf{B}^{(m-1)}$ whose elements are defined for $i,j=1,\ldots,m-1$ as

$$b_{ij}=(a_{11})^{-1}\begin{vmatrix} a_{11} & a_{1j+1} \\ a_{i+11} & a_{i+1j+1} \end{vmatrix} . \tag{A1}$$

In $\det \mathbf{A}^{(m)}$, we subtract from row i $(i>1)$ the product of the first row by the ratio a_{i1}/a_{11}. The $i>1$ elements of the first column vanish and one obtains the property

$$\det \mathbf{A}^{(m)} = a_{11} \det \mathbf{B}^{(m-1)} . \tag{A2}$$

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