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Scaled Langevin equation to describe the $1/f^{\alpha}$ spectrum

Junji Koyama
Geophysical Institute, Tohoku University, Aramaki-aoba, Sendai 980, Japan

Hiroaki Hara
Department of Engineering Science, Tohoku University, Aramaki-aoba, Sendai 980, Japan
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Based on an ideal system under external random forces, the dynamical process of random activation is studied. The time evolution of the system is described by the Langevin equation, and a scaling rule is introduced to generalize the system. The generalized system predicts the fractional power spectrum $1/f^{\alpha}$ from a white spectrum to a Lorentzian. The exponent α is a function of the fractal dimension of the scaling rule. It is found that the fractal dimensions of 2, 1, and about 0.47 specify the particular mode of the generalized system, where the total power of the fractional power spectrum is minimum. The values indicate a Lorentz spectrum, a $1/f$ spectrum, and a power spectrum of $1/f^{1.53}$ type, respectively. The system of the minimum total power in this study is equivalent to one in the minimum potential energy, where the system is in the steady state. Therefore the random-activated system in the steady state gives a $1/f$ spectrum, and a $1/f^{\alpha}$ spectrum is considered to represent the fluctuation of the complex system from the steady state.

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I. INTRODUCTION

Fluctuation phenomena with the $1/f$ spectrum have been observed in a wide variety of dissimilar physical systems. Noise experiments in electronic devices reveal fractional power spectra of $1/f^{\alpha}$ type [1]. Here α is a constant in the order of unity and is dependent on the experimental conditions. Noise current in semiconductors and turbulent flow field are also described by the fractional power spectra of $1/f^{3/2}$ and $1/f^{5/3}$ types [2,3].

Various theories have been proposed to explain such the spectral behavior [4] and its relationship to the long-tail behavior of the complex system [5]. Most of the theories start with an *a priori* assumption on the autocorrelation function of a particular process and/or the probability function of the random events. There has been little discussed on what conditions and material properties are necessary for the functions and what the physical significance of the functional forms is. Consequently, the difficulties are always encountered in generalizing the physical origin of the diversity of the systems. This leads us to accept that the physical origin of the $1/f$ fluctuations cannot be universal.

Recently disordered systems have been studied [5] in order to find the common stochastic property of the system. And it is suggested that the time-scale invariance is the essential property to the common temporal behavior of the complex system. In spite of these circumstances, it is further tempting to search for a universality in the fundamental equations underlying the $1/f$ fluctuations, since the phenomenon is so ubiquitous.

Our motivation to investigate the $1/f^{\alpha}$ fluctuation phenomena is rooted in the recent analysis of disastrous earthquakes [6]. Figure 1 shows a schematic diagram of an earthquake source: The size of an earthquake source is characterized by fault length L and width W, where the fracture induces the dislocation on the surface. Stress drop $\Delta\sigma_0$ (or force) acts on the surface of $L \times W$. Since the earthquake source process is essentially a transient phenomenon, the fracturing starts at one end of the source area and propagates to the other end in a finite velocity v. This is the macroscopic description of the earthquake source process.

The microscopic description of the earthquake source is also important: Strong ground motion is large-amplitude ground shaking near the earthquake source, which causes the earthquake disasters. The ground oscil-

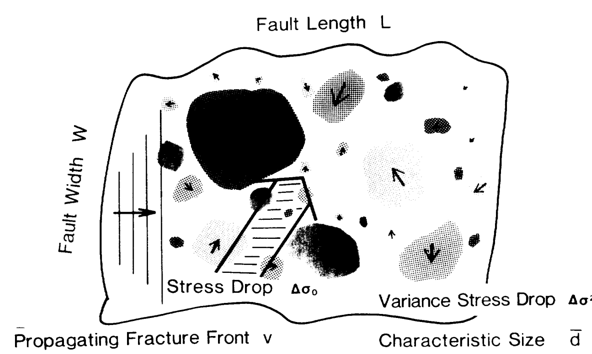

FIG. 1. Schematic illustration of the complex earthquake source as a representative of the complex system. Fault length L, width W, and stress drop $\Delta\sigma_0$ (or force) are the macroscopic parameters to describe the deterministic part of the earthquake source process. Fracture propagates on the surface with $L \times W$ with a velocity v. Characteristic size of small-scale heterogeneities $\bar{d}$ and variance stress drop $\Delta\sigma^2$ are the microscopic parameters to represent the stochastic part of the complex source process.

lates so violently that the phase of the motion is incoherent [6]. Figure 2 shows an example of strong ground-motion accelerograms, which registered the maximum ground acceleration about 30% of the gravity G. The excitation of strong ground motion is strictly related to the random fracturing of small-scale heterogeneous areas where the stress drops are not constant but fluctuating. Variance stress drop $\Delta\sigma^2$ and size $\bar{d}$ of the heterogeneous areas are the parameters to describe the microscopic part of the complex earthquake source process [7]. L is the order of several hundred kilometers and $\bar{d}$ is several hundred meters for a particular disastrous earthquake.

Fracturing one heterogeneous area radiates one pulse of oscillation. Fracturing many of the heterogeneities in a random manner generates random pulses with incoherent phases. Then, the complex earthquake source could be observed as a random pulse time series (Fig. 2). These understandings lead us to consider that the complex earthquake source process could be simulated by a stochastic process of the fracture of small-scale heterogeneous areas activated randomly.

In this study we develop a theory to describe a dynamical process of random activation. The model process is capable of generating $1/f$ and $1/f^{\alpha}$ spectra. This study is not aimed at obtaining the $1/f^{\alpha}$ spectrum by assuming *a priori* the autocorrelation function and the stochastic property of the process. The time evolution of the random activation is described by the Langevin equation of motion, and the complexity of the process is taken into account by a scaling rule to the set of the Langevin equation. The scaling parameter is the only restriction to characterize the whole system, and we seek for a physical condition to determine the scaling parameter of the system and the exponent α of the spectrum.

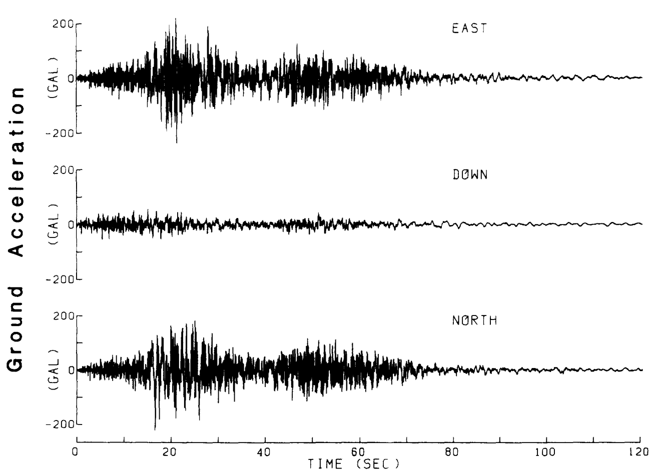

FIG. 2. Strong motion acceleration at Akita (Port Harbor Research Institute), Japan for the 1983 Japan Sea earthquake of May 26. Three components of ground acceleration, east-west, down-up, and north-south components, are shown. The ordinate of 200 gal is 20% of the gravity G. Note that the phases of the motion are not continuous like sine waves but change time to time abruptly.

II. SCALING RULE FOR THE LANGEVIN EQUATION

Generally speaking, the dynamics of random systems is so complicated that the systems are hard to be characterized only by their macroscopic parameters. The elementary process of the system is even unknown in some cases, nor is the dynamic temporal evolution known for the elementary processes. In this study we consider the dynamics of a complex system as the system response by random activation. And we model the complexity of the system by the self-similarity of component subsystems.

What we consider here is the dynamical process of a complex system. The system is composed of elements, and the elements are grouped into clusters. Each cluster represents a set of elements in which respective stochastic behavior is governed by the same equation. This means that the state of the whole system is expressed as a sum of "local states" of clusters: Let the state of the system be $X(t)$ and the local states $X_n(t)$, then we have

$$X(t)=\sum_{n=0} X_n(t)\ . \tag{1}$$

Suppose that the state of the system $X(t)$ is described by

$$\dot{X}(t)+\bar{\gamma}X(t)=n(t)\ , \tag{2}$$

where the dot stands for the time derivative, $\bar{\gamma}$ is a positive constant, and $n(t)$ represents random forces or random noises.

If the whole system is expressed by a single cluster $X_0(t)$, (2) is reduced to an equation

$$\dot{X}_0(t)+\bar{\gamma}X_0(t)=n_0(t)\ . \tag{3}$$

The random force is assumed by zero mean

$$\langle n_0(t)\rangle=0$$

and the autocorrelation function of

$$\langle n_0(t+\tau)n_0(t)\rangle=\sigma^2\delta(\tau)\ , \tag{4}$$

where σ^2 is a constant depending upon the frequency of random activation, and δ is the Dirac delta function. The convention is adopted to denote the mean value by angular brackets $\langle\ \rangle$.

Clearly, (2) is oversimplified to generally represent the complex system. However, the primary interest of this study is to describe the stochastic behavior of the complex system, and not to derive a model appropriate to the actual details. So far we understand (2). (2) is the Langevin equation to describe the Brownian motion. It also represents the rate change of membrane potential of a neuron [8] and/or the response of a random activation of a viscoelastic spring [9] when $X(t)$ is considered as velocity and $n(t)$ is the random force.

Subscript 0 indicates a generator of the system and N the number of the clusters. The generator is composed of elements activated by $n_0(t)$ with a characteristic decay of $\bar{\gamma}$. $\langle X_0(t)\rangle$ becomes zero as $t\rightarrow\infty$, because of (2). This is similarly understood as that the temporal summation of excitatory and inhibitory potentials [10] of the synapse goes back toward to the resting level. The autocorrelation function $C_0(\tau)$ of $X_0(t)$ is calculated as

$$C_0(\tau)=\langle X_0(t+\tau)X_0(t)\rangle = \frac{\sigma^2}{2\bar{\gamma}}\exp(-\bar{\gamma}|\tau|) \tag{5}$$

under the initial condition of $X_0(-\infty)=0$.

We introduce parameters a and b to consider the scaling rule for local states of X_{i-1} and X_i, and for n_{i-1} and n_i $(i=1,2,\ldots,N)$,

$$\sqrt{a/b}\,X_{i-1}(bt)=X_i(t)\ , \tag{6}$$

$$\sqrt{ab}\,n_{i-1}(bt)=n_i(t)\ , \tag{7}$$

where a and b are positive real here, and subscript i indicates the ith level of the scaling. We have from (3) and (6)

$$\begin{aligned}&\dot{X}_1(t)+b\bar{\gamma}X_1(t)=n_1(t)\ ,\\&\dot{X}_2(t)+b^2\bar{\gamma}X_2(t)=n_2(t)\ ,\end{aligned} \tag{8}$$

and so on. The autocorrelation of the noises is expressed similarly

$$\langle n_i(t+\tau)n_i(t)\rangle = a\langle n_{i-1}(t+\tau)n_{i-1}(t)\rangle \quad (i=1,2,\ldots,N)\ . \tag{9}$$

The autocorrelation function of each cluster is then scaled as

$$C_i(\tau)=\frac{a}{b}C_{i-1}(b\tau)\ . \tag{10}$$

Each level of the scaling specifies a cluster response with b (>1) times more rapid decay than that of a younger scaling level. The responses are triggered a times as often in a random manner, which measures the activation rate. Different activation rates for different clusters would be a manifestation of many threshold levels for the activation of elements. There we introduce the scaling region.

Addition of the individual responses of such clusters is considered to form a kind of parallel network of elements for the random activation. This would be the same as spatial summation [10] for many inputs at different locations. Figure 3 draws a sketch of the random responses activated in this context. The autocorrelation function of the whole system composed of all the clusters is expressed by the sum of (10) as

$$C(\tau)=\frac{\sigma^2}{2\bar{\gamma}}\sum_i\left[\frac{a}{b}\right]^i\exp(-b^i\bar{\gamma}|\tau|)\ . \tag{11}$$

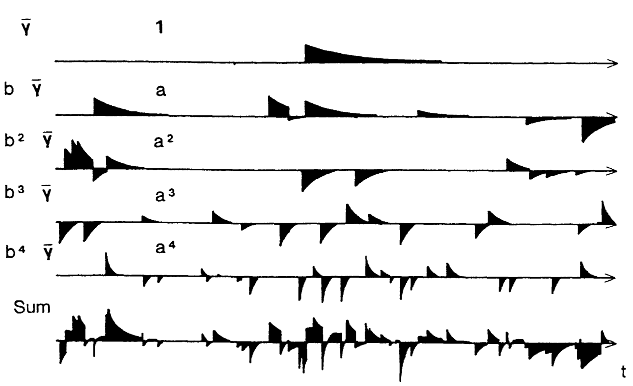

FIG. 3. Response of the generalized system by random activation. The amplitude and unit of time in the abscissa are arbitrary. A self-similar set of clusters is illustrated in rows. Each cluster is composed of random-activated pulses with the same amplitude decay rate but with random heights.

III. FRACTIONAL POWER SPECTRUM OF THE COMPLEX SYSTEM

When $a/b>1$, $(a/b)^i$ within the summation in the right-hand side of (11) diverges whereas the exponential function converges. We could apply the steepest descent approximation [11] for (11) considering large $|\tau|$,

$$\begin{aligned}&C(\tau)\simeq A_\xi|\tau|^{-(\xi-1)}\ ,\\&A_\xi=\left[\frac{2\pi}{\xi-1}\right]^{1/2}\frac{\exp\{(\xi-1)[\ln(\xi-1)-1]\}}{\ln(b)}\end{aligned} \tag{12}$$

where ξ is the fractal dimension of the random activation defined by the scaling parameters a and b as

$$\xi=\frac{\ln(a)}{\ln(b)}\ . \tag{13}$$

Here $\bar{\gamma}=1$ and $\sigma^2=2$ have been assumed without loss of generality. Note that $\xi>1$ in this case. b (>1) then specifies the amplitude decay of responses as found in Fig. 3. a $(>b>1)$ and b are left to be a free parameter here, which will be determined later by a physical condition.

We should notice that the long-tail behavior of the complex system [12] is generally obtained in (12). The behavior could be commonly found in the complex systems and disordered materials and has been studied extensively. However, we could study more about the property beyond the general trend and relative behavior, since (12) is the solution of the Langevin equation of motion without any *a priori* assumption of the probability density and/or the autocorrelation function of the system.

The power spectrum of the system is calculated from (12)

$$\begin{aligned}P_s(\omega)&=\int_{-\infty}^{\infty}C(\tau)\exp(-i\omega\tau)d\tau\\&=2A_\xi\Gamma(2-\xi)\cos\left[\frac{\pi}{2}(2-\xi)\right]|\omega|^{\xi-2}\end{aligned}$$

$$(2-\xi\neq\text{integer})\ , \tag{14}$$

where Γ is the Γ function, and ω is the angular frequency. It is $\omega=2\pi f$ and we presumed upon $\bar{\gamma}$ to be the normalization factor. This describes the fractional power spectrum from a white spectrum, when ξ tends to 2, to $1/f$ spectrum, when ξ tends to 1.

The power spectrum in (14) is not always true for the entire frequency range because of the assumption in deriving (12). The power spectrum would be white in very low frequencies $\omega\ll1$. This is because the spectrum in the low-frequency limit represents that of random impulses and not of random pulses with finite-time durations. Whereas in the extremely high frequencies, $\omega > b^N$ (N being the maximum scaling level), the spectrum would be of $1/\omega^2$ type, since each $C_i(\tau)$ in (5) and (10) is attributed to the corresponding Lorentz spectrum.

Although the integral formula itself in (14) is valid [13] even for $2<\xi<3$, we observe a negative power spectrum. This is because the Γ-function is evaluated as the finite part of hyperfunctions [14] excluding the singularities. Another approximation to evaluate the integral in (14) could be obtained by the Euler-Maclaurin formula as

$$P_e(\omega) \simeq \int_0^N \left[\frac{a}{b}\right]^x \frac{2b^x}{b^{2x}+\omega^2}dx - \sum_{n=0}^{N-1}\int_n^{n+1}\left\{\left[\frac{a}{b}\right]^x \frac{2b^x}{b^{2x}+\omega^2}\right\}' dx \ , \qquad (15)$$

where $'$ stands for the derivative. In the above, we could relax the restriction of the fractal dimension as to $\xi>0$.

The approximation (15) is rewritten taking a new variable $t=|\omega|b^x$ as

$$P_e(\omega) \simeq \frac{2}{\ln(b)}|\omega|^{\xi-2}\int_{1/|\omega|}^{b^N/|\omega|}\frac{t^{\xi-1}}{t^2+1}dt - \sum_{n=0}^{N-1}\int_n^{n+1}\left\{\left[\frac{a}{b}\right]^x \frac{2b^x}{b^{2x}+\omega^2}\right\}' dx \ . \qquad (16)$$

If our interest is confined to the fractional power spectrum in the scaling region of frequency $1 \ll \omega \ll b^N$, the first integral is approximated to the integrand from 0 to ∞. Then we could evaluate by the branch cut integral, when $0<\xi<1$, as

$$P_e(\omega) \simeq \frac{2\pi}{\ln(b)} \frac{\sin\left\{\frac{\pi}{2}(\xi-1)\right\}}{\sin\{\pi(\xi-1)\}}|\omega|^{\xi-2}+O\left[\frac{2}{1+\omega^2}\right] . \qquad (17)$$

This is identical to the spectrum in the limit of $\xi \to 0_+$, which we could evaluate by integrating each component of the Taylor series for the numerator in (16)

$$P_e(\omega) \simeq \frac{2}{\xi\ln(b)}|\omega|^{\xi-2}+O\left[\frac{2}{1+\omega^2}\right] . \qquad (18)$$

Similarly when $\xi \to 2_-$, we could derive the power spectrum of

$$P_e(\omega) \simeq \frac{2}{(2-\xi)\ln(b)}|\omega|^{\xi-2}+O\left[\frac{2}{1+\omega^2}\right] . \qquad (19)$$

This is identical to the power spectrum by the steepest descent method in (14).

In the case when $\xi>2$, the contribution of the cluster in the maximum scaling level prevails, giving an approximate spectrum of

$$P_e(\omega) \simeq \left[\frac{a}{b}\right]^N \frac{2b^N}{b^{2N}+\omega^2} . \qquad (20)$$

(20) is a Lorentz spectrum with a characteristic corner frequency of b^N. Consequently, these spectra show a fractional power decay from one over f to the Lorentz decay. Combining the result of (14), the generalized system considered in this study describes a fractional power spectrum $1/f^{\alpha}$ from a white to a Lorentzian. The exponent α of $1/f^{\alpha}$ spectrum is, therefore, a function of the fractal dimension of random activation as

$$\begin{aligned} &\alpha = 2-\xi \quad (0<\xi<2) \ , \\ &\alpha = 0 \quad (\text{or } 2) \quad (\xi>2) \ . \end{aligned} \qquad (21)$$

IV. TOTAL POWER OF THE FRACTIONAL POWER SPECTRUM

It is true that the total power of the spectrum from the generalized system monotonously increases as ξ. This is because the number of elements included within a unit time increases as ξ. However, our interest does not lie on the white spectrum in very low frequencies nor on the Lorentz spectrum in the extremely high frequencies. Two cutoff frequencies ω_1 and ω_2 are considered. These specify the scaling region of frequency mentioned in the preceding sections. Physically, the lower limit ω_1 gives the low frequency below which the power spectrum is always white, whereas the upper limit ω_2 corresponds to the very high frequency above which the power spectrum is a Lorentzian. These are said to be $\omega_1=1$ (normalized by $\bar{\gamma}$) and $\omega_2=b^N$ in the preceding section. In this scaling region, the fractional decay of the power spectrum is observed.

Considering these two characteristic frequencies, we define a band-limited total power of the fractional power spectrum

$$T_p(\xi)=\frac{1}{\pi}\int_{\omega_1}^{\omega_2}P_{s,e}(\omega)d\omega \ . \qquad (22)$$

This is equivalent to the system energy in a unit time through Parseval's theorem. Since the dynamical process of the present system is stationary, one would obtain the band-limited total energy by multiplying a time duration of the system response to (22).

Figure 4 shows T_p as a function of the fractal dimension ξ. Although ξ $[=\ln(a)/\ln(b)]$ has been an arbitrary parameter specifying the dynamics of the system, we observe three values of ξ, 2, 1, and about 0.47, where the generalized system is specified by the total power (energy) minimum,

$$\frac{dT_p(\xi)}{d\xi}=0 \ . \qquad (23)$$

When $\xi>2$, many responses of the elements overlapped one another on the time axis, since $a>b^2$. Therefore, the system is characterized by random-and-dense phenomena. In this case the Lorentz spectrum is plausible, ξ becoming 2, where the system is specified by the local minimum in the band-limited total power. Figure 4

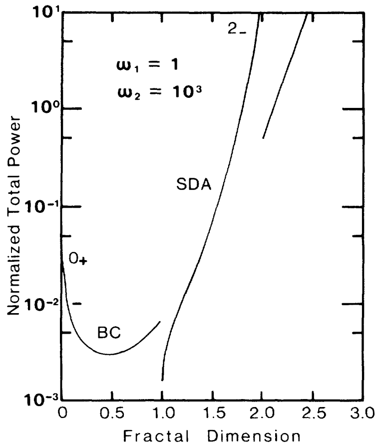

FIG. 4. Total power of the fractional power spectrum as a function of the fractal dimension of the scaling parameter. Maximum and minimum frequencies specify the scaling region of the system response, which is assumed 1000 and 1. The value of total power is normalized by the maximum frequency. SDA indicates the steepest descent approximation in the text, and BC indicates the branch-cut integral for the fractal dimension of $0<\xi<1$; $\xi\to 0_+$ and $\xi\to 2_-$ are understood.

also indicates that the $1/f$ spectrum is another mode, when ξ is 1. This is when the scaling parameters satisfy the condition of $a=b$. The other is the system specified by the power spectrum of $1/f^{1.53}$ type. Since $a<b$ in the last case, the random activation is sporadic compared to the characteristic response duration.

In a thermal equilibrium, the steady-state system is characterized by the minimum free energy. In contrast to this, the steady-state propagation of elastic fractures is specified by the minimum potential energy (elastic energy). And the power $T_p(\xi)$ in (22) corresponds to this elastic energy in the scaling region. Therefore, the above three modes indicate the system in the steady state.

V. CONCLUDING REMARKS

It is well known that the Langevin equation describes the stochastic behavior of the Brownian motion. The Brownian motion is a highly correlated stochastic process, and its derivative is the Gaussian random noise [15]. The Gaussian random noise is completely uncorrelated. There defined is a stochastic process with the temporal and spectral behavior between the Brownian motion and the Gaussian random noise. This has been studied as the fractional Brownian motion [15]. The fractional Brownian motion is formally obtained by a moving average of the Gaussian random noise weighted by the kernel of past time with the exponent of $H-\frac{1}{2}$. For $H=\frac{1}{2}$, the process is the classical Brownian motion. The interdependence of the fractional Brownian motion is featured by H. However, H is an empirical parameter $(0<H<1)$ and its physical significance is not obvious.

The fractional Brownian motion [15] provides the power spectrum of the form $1/f^{2H-1}$ [16]. Since the present theory describes the stochastic behavior of the complex system based on the fundamental Langevin equation, the theory would give an insight into the fractional Brownian motion from the very basic equation. The primary result in this study is the description of the spectral behavior of $1/f^{\alpha}$, where $\alpha=2-\xi$ in (21). Then we formally obtain a relation between the fractal dimension of the scaling parameter and the above exponent H as

$$\xi=3-2H \ . \tag{24}$$

It is clear that the classical Brownian motion $H=\frac{1}{2}$ corresponds to $\xi=2$ in the present representation, where the trace of the Brownian motion covers the whole two-dimensional space [15]. For $\frac{1}{2}<H<1$, the process is characterized specifically by the long-tail behavior, where $1<\xi<2$ in the present case. An infinitely long-tail behavior of the system gives the $1/f$ spectrum where $H\to 1_-$ and $\xi\to 1_+$. All these reflect that the degree of the correlation of the fractional Brownian motion by H is represented by the fractal dimension ξ of the scaling parameter of the system. We also understand the geometrical property of the fractional Brownian motion in terms of H and ξ.

In the elastodynamic system, a uniform propagation of fracture is characterized by the minimum elastic energy (minimum potential energy). Seismic short waves are generated from the earthquake source by the random fracture of small-scale heterogeneous areas, whose wavelengths are closely related to the heterogeneity sizes. Such waves, namely strong ground motion, are composed of many pulses with incoherent phases. Therefore, the total energy of strong ground motion is said to be minimum, if the fracture sequence of the random heterogeneities is in the steady state. And this is a theoretical basis for the reason why we observe the fractional earthquake-source spectra in high frequencies [7].

It could be concluded from the above consideration that $1/f$ represents the spectrum of the complex system in the steady state and that $1/f^{\alpha}$ is the spectrum of the generalized system close to the steady state. The difference of the exponent coefficient α from 1 would measure the distance of the system from the steady state. This would be the reason why the $1/f$ spectrum is so ubiquitous in the varieties of the physical phenomena. This conclusion is derived from the minimum total power criterion and is not available until the complex system is described by the fundamental Langevin equation of motion.

It is also found in this study that there are two other modes specified by the minimum total power criterion beside one over f. One mode gives the Lorentz spectrum. The other is characterized by the random-but-sporadic phenomena with a power spectrum close to the $1/f^{1.53}$ type. This contrasts with the $1/f$ and the Lorentz modes. Since the system generally describes the sparse phenomena in the last case, this mode would be one of the representations to describe the intermittence in the turbulent flow fields. Considering the broad minima in Fig. 4, the spectral behavior of $1/f^{1.53}$ type would be consistent with the $\frac{5}{3}$ exponent energy spectrum of turbulent velocity fields [3] and may also be consistent with the $\frac{3}{2}$ exponent spectrum of semiconductor noises [2].

Finally, the present theory could be modified to describe a more complicated system in which the local clusters are characterized by the plural scaling factors of $b_1, b_2, \ldots, b_d$. The theory could be extended for the special case of $d=2$ in the same framework of this study, where the basic equations are expressed by complex scaling parameters. The system would provide us with a process predicted by the complex fractional Brownian motion [15].

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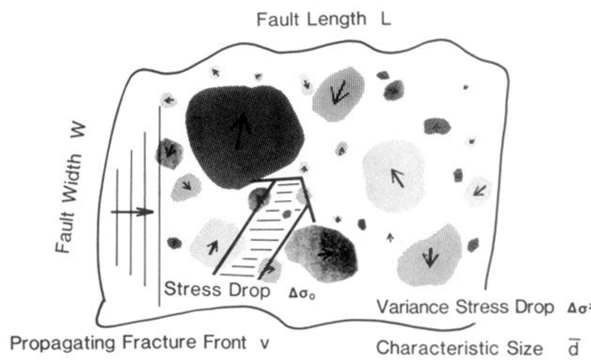

FIG. 1. Schematic illustration of the complex earthquake source as a representative of the complex system. Fault length L, width W, and stress drop $\Delta\sigma_0$ (or force) are the macroscopic parameters to describe the deterministic part of the earthquake source process. Fracture propagates on the surface with $L \times W$ with a velocity v. Characteristic size of small-scale heterogeneities $\bar{d}$ and variance stress drop $\Delta\sigma^2$ are the microscopic parameters to represent the stochastic part of the complex source process.