PHYSICAL REVIEW A VOLUME 46, NUMBER 4 15 AUGUST 1992

Directed alternating lattices and the site-to-bond ratio for animals and trees

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(Received 16 December 1991)

We present here a brief summary of results on so-called "exotic" directed lattices having nonregular periodicity. Such lattices, e.g., the Archimedean nets, are characterized by different site types and a spread of coordination numbers. The evidence adduced here shows that the exponent structure for the growth of animals and trees on such lattices is of the form predicted, with θ_0 for trees equivalent to θ for unrestricted animals, independent of the periodicity property. This supports the general form $\theta_c=\theta_0-c$, for cycles $c=0$. We quote values for the growth parameter (or inverse critical fugacity) λ for bond trees and animal growth on selected directed lattices. The convergence of such series is well known to be subject to the influence of subdominant singularities in addition to θ, and we report on results obtained using the second-log-derivative scheme for the lattices of interest. Recent results from percolation studies for the alternating nets [H. J. Ruskin, Phys. Lett. A **162**, 215 (1992)] have proved particularly encouraging for the Archimedean lattices, with $\bar{z}=2.498+-0.010$, but uncertainties for the honeycomb were found to be large. Investigation of the site-to-bond ratios for animals and trees on the honeycomb gives somewhat smoother series behavior, which, though subject to confluence effects, supports a much lower value of the effective coordination number. We quote $\bar{z}=1.450+-0.050$.

PACS number(s): 05.50.+q

INTRODUCTION

The statistics of finite cluster growth on regular lattices has proved a useful study in terms of the understanding of various biological growth processes ([1] and references cited therein) for dilute polymer problems [2], and more recently in the formulation and understanding of various lattice gauge theories [3]. For isotropic lattices, trees and full animals have been shown to have the same critical exponents in all dimensions [4,5] and their hierarchical structure has been explored [6,7]. It was early noted [8] that on general types of lattice all site trees are neighbor-avoiding walks, and hence no proper zero-cycle configurations are possible. This phenomenon is observed for all lattices which are bond-to-site transformations of other nets, with additional numbers of loops dependent on the valence partition. The establishment of a similar hierarchical structure for directed animals [9] has meant that the exploitation of recent developments with respect to site-to-bond conversion for directed graphs [10,11] has provided the means of direct investigation of the cyclomatic number partition of directed bond configurations. Utilizing the information from site-to-bond conversion, we explore here the singularity structure for directed animals and trees for the Archimedean (3,3,3,4,4) and honeycomb nets (bond), which fall into the class of "exotic" lattices. These lattices are characterized by alternating periodicity with coordination number, z, having more than one value which is dependent on the site type. The connectivity features of the different site types thus provide different origins for cluster growth [illustrated in Fig. 1(b) for the Archimedean]. Results for the alternating nets are contrasted with those of the Kagomé lattice, which, despite regular periodicity (single-valued z), shares several similar properties [Fig. 1(a), with site types distinguished by the number of incoming and outgoing bonds on the hexagon].

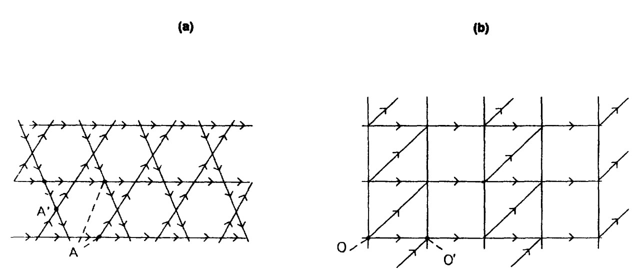

FIG. 1. (a) Directed Kagomé lattice showing different site types. (b) Directed Archimedean lattice showing different site types (different origins for cluster growth).

EXPONENT STRUCTURE AND SITE-TO-BOND RATIOS

The critical exponent structure which we investigate for these exotic lattices assumes a generating function of the form

$$A_c(x)=1+\sum_{n=1} A_{n,c}x^n\ , \tag{1}$$

with

$$A_{n,c}\sim n^{-\theta_c}\lambda_c^n=\sum_c A_n\lambda_0^n n^{-\theta_0+c} \tag{2}$$

and

$$\theta_c = \theta_0 - c \ , \tag{3}$$

where n is the size of the site (bond) cluster, λ is the growth parameter or inverse critical fugacity, and $A_{n,c}$ is the total number of directed animals with size n having c cycles (or loops). λ can also be regarded as tree multiplicity, where $c=0$ for trees. Further, the presence of subdominant singularities is known to influence the convergence of animal series, so that we may extend Eq. (1) above [3,12] to write for animals of loop size c

$$A_{n,c} \sim \lambda_0^n n^{-\theta_c}(1+Bn^{-w}+\cdots) \ . \tag{4}$$

Thus by applying the second log-derivative scheme, we may obtain information on the effect of the confluences.

The investigation of a critical site-to-bond ratio for directed animals examines the form [10] for size-dependent growth by means of the bond perimeter distribution, i.e.,

$$\lambda(a) = (a+1)^{a+1}/a^a - k(a) \ , \quad k(a) > 0 \text{ for } a > a_c \ , \tag{5}$$

where

$$a = t/b = \tilde{z} - 1 + (\tilde{z}/b) - (\tilde{z}c/b) \ , \tag{6}$$

with $\tilde{z}$ the "effective" coordination number, t the perimeter, b the number of bonds, and a_c the critical ratio. In a previous paper, Ruskin [13] has considered percolation thresholds for selected alternating bond lattices, and has obtained estimates of $\tilde{z}$ for these cases. For the Archimedean lattice, results were good, with $\tilde{z} \sim 2.5$, in agreement with the conjectured exact value based on averaging the number of sites of different types. Estimates were less satisfactory for the honeycomb lattice, given the characteristic oscillation of sign for the percolation series. However, there was some evidence to show that $\tilde{z}$ could be as low as 1.5, which would support a conjecture based on similar configurational considerations. Furthermore, Eqs. (5) and (6) above clearly indicate that an investigation of trees (cycle or loop size $c=0$) and animals with $c>0$ should provide an alternative estimate of $\tilde{z}$. The hierarchical structure of exponents for these fixed-cycle clusters is expected to be of the form given in Eq. (2), with some support for this assumption provided by earlier studies [2,11,14]. Here we investigate series for both trees and animals for selected directed nets with regular and alternating periodicity.

RESULTS

Series data on the total number of directed animals are given by Ruskin [14] for the Archimedean (3,3,3,4,4) to order 14 (15) for the bond (site), and for the Kagomé bond and site to order 22. The honeycomb bond animals are given in Table I, and the bond trees for Archimedean, Kagomé, and honeycomb lattices are given in Table II. Relevant site-to-bond ratios, denoted $\langle t \rangle / b$ and $\langle s \rangle / b$,

TABLE I. Honeycomb bond animals.

1	2
2	3
3	6
4	11
5	22
6	44
7	89
8	182
9	374
10	775
11	1611
12	3363
13	7042
14	14 790
15	31 139
16	65 697
17	138 869
18	294 029
19	623 492
20	1323 898
21	2814 533
22	5990 156
23	12 761 699
24	27 213 259
25	58 079 355
26	124 051 993
27	265 155 766
28	567 141 107
29	1213 817 958
30	2599 382 044

TABLE II. Bond trees.

	Honeycomb	Kagomé	Archimedean
1	2	4	5
2	3	10	16
3	6	27	56
4	11	76	206
5	22	221	781
6	43	657	3019
7	86	1982	11 831
8	173	6039	46 837
9	350	18 543	186 876
10	712	57 305	750 267
11	1452	178 049	3027 441
12	2975	555 650	12 267 584
13	6108	1740 394	49 885 928
14	12 577	5468 171	203 473 454
15	25 944	17 226 909	
16	53 626	54 399 777	
17	111 020	172 142 129	
18	230 189	545 724 610	
19	477 902	1732 885 298	
20	993 373	5510 656 445	
21	2067 084	17 547 367 745	
22	4305 588	55 942 528 625	
23	8976 406		
24	18 729 822		
25	39 111 032		
26	81 728 463		
27	170 896 956		
28	357 570 769		
29	748 580 760		

TABLE III. Bond averages.

	Kagomé	Honeycomb			
	Animals	Animals		Trees	
	$\langle s\rangle/b$	$\langle t\rangle/b$	$\langle s\rangle/b$	$\langle t\rangle/b$	$\langle s\rangle/b$
4	1.4634	0.8636		0.8636	
5	1.3579	0.7818		0.7818	
6	1.2870	0.7159	1.1629	0.7209	1.1667
7	1.2357	0.6774	1.1380	0.6844	1.1429
8	1.1966	0.6435	1.1188	0.6517	1.1250
9	1.1658	0.6197	1.1040	0.6298	1.1111
10	1.1410	0.5994	1.0919	0.6104	1.1000
11	1.1207	0.5829	1.0818	0.5955	1.0909
12	1.1037	0.5692	1.0735	0.5827	1.0833
13	1.0892	0.5576	1.0664	0.5720	1.0769
14	1.0768	0.5476	1.0603	0.5629	1.0714
15	1.0660	0.5389	1.0550	0.5549	1.0667
16	1.0565	0.5313	1.0503	0.5480	1.0625
17	1.0481	0.5245	1.0462	0.5418	1.0588
18	1.0406	0.5185	1.0425	0.5364	1.0556
19	1.0339	0.5132	1.0392	0.5316	1.0526
20	1.0278	0.5083	1.0362	0.5272	1.0500
21	1.0223	0.5039	1.0335	0.5232	1.0476
22	1.0173	0.4999	1.0310	0.5197	1.0455
23		0.4963	1.0288	0.5164	1.0435
24		0.4929	1.0267	0.5134	1.0417
25		0.4898	1.0248	0.5106	1.0400
26		0.4870	1.0230	0.5081	1.0385
27		0.4843	1.0214	0.5057	1.0370
28		0.4819	1.0199	0.5035	1.0357
29		0.4796	1.0184	0.5015	1.0345
Central limiting value	0.908	0.414	0.978	0.447	0.995

are given in [13] for the Archimedean bond lattice, and Table III gives the appropriate ratios for the Kagomé and honeycomb lattices, together with the central value obtained from asymptotic extrapolation. For the Kagomé lattice (with $\bar{z}=2$), $\langle s\rangle/b$ ratios are included for completeness but $\langle t\rangle/b$ averages are not given, since these follow directly [Eq. (6)] for $\bar{z}$ known and constant. For the honeycomb trees, $\langle s\rangle/b$ are counted according to the usual definition $s=b+1$, and, as can be seen from Table III, closely parallel the animal sequences, giving slightly higher central values. Estimates of the growth parameter (or tree multiplicity) for the bond trees for all lattices and for the honeycomb bond animals are summarized in Table IV, together with central values for the dominant critical exponent (θ_0 and θ, respectively). These are seen to be in good agreement with the form predicted ([9] and references cited therein), with exponent estimates for trees well within the error bounds quoted by Ruskin [14] for the unrestricted animals. (The animal estimates for Archimedean and Kagomé lattices are repro-

TABLE IV. Summary of growth parameter and dominant exponent estimates for bond trees. Animal estimates are included in comparison.

	λ	$\theta_0(\theta)$
Kagomé site animals	2.7000 ± 0.0025	0.505 ± 0.005
Kagomé bond animals	3.426 ± 0.006	0.515 ± 0.015
Kagomé bond trees	3.2660 ± 0.0005	0.500 ± 0.005
Archimedean site animals	3.416 ± 0.004	0.505 ± 0.010
Archimedean bond animals	4.477 ± 0.007	0.510 ± 0.020
Archimedean bond trees	4.2350 ± 0.0025	0.500 ± 0.005
Honeycomb bond animals	2.21875 ± 0.0005	0.500 ± 0.005
Honeycomb bond trees	2.1778 ± 0.0002	0.500 ± 0.001

TABLE V. Three point fits, A, B, and ϕ.

Lattice	Discrimination	A	B	ϕ
Honeycomb	$\langle t\rangle/b$	0.413 82	1.753 60	−0.975 03
		0.413 64	1.745 83	−0.972 90
		0.413 86	1.755 66	−0.975 54
		0.413 84	1.754 81	−0.975 31
		0.413 88	1.756 74	−0.975 81
		0.413 90	1.758 06	−0.976 14
		0.413 94	1.760 22	−0.976 68
	$\langle s\rangle/b$	0.976 51	1.032 19	−0.951 29
		0.976 52	1.032 73	−0.951 55
		0.976 64	1.037 44	−0.953 71
		0.976 68	1.039 24	−0.954 51
		0.976 70	1.040 23	−0.954 95
		0.976 75	1.042 52	−0.955 93
		0.976 79	1.044 85	−0.956 92
Kagomé	$\langle s\rangle/b$	0.906 36	2.095 18	−0.950 74
		0.907 29	2.115 48	−0.956 44
		0.907 51	2.121 10	−0.957 96
		0.907 40	2.118 03	−0.957 16
		0.907 33	2.115 85	−0.956 61
		0.907 41	2.118 34	−0.957 23
		0.907 56	2.123 85	−0.958 53
Archimedean (Sub 2)	$\langle s\rangle/b$	0.948 76	1.118 39	−0.968 37
		0.944 29	1.090 20	−0.941 89
		0.943 87	1.087 03	−0.939 18
		0.943 65	1.085 07	−0.937 65
		0.943 53	1.083 69	−0.936 66
		0.943 76	1.086 71	−0.938 69
		0.944 09	1.091 56	−0.941 73
	$\langle t\rangle/b$	1.349 03	2.879 45	−0.927 60
		1.355 45	2.916 47	−0.941 17
		1.357 15	2.929 45	−0.945 30
		1.358 74	2.944 67	−0.949 64
		1.359 93	2.958 38	−0.953 22
		1.360 31	2.963 52	−0.954 46
		1.360 32	2.963 81	−0.954 53
(Sub 1)	$\langle s\rangle/b$	0.941 49	1.033 67	−0.928 45
		0.943 15	1.043 17	−0.938 18
		0.943 60	1.046 61	−0.941 25
		0.944 16	1.051 79	−0.945 40
		0.944 63	1.057 18	−0.949 35
		0.944 82	1.059 85	−0.951 17
		0.944 86	1.060 36	−0.951 49
	$\langle t\rangle/b$	1.376 80	2.365 26	−1.003 13
		1.363 75	2.273 02	−0.963 11
		1.361 56	2.255 11	−0.955 89
		1.360 53	2.245 03	−0.952 16
		1.359 87	2.237 36	−0.949 52
		1.360 30	2.243 22	−0.951 40
		1.360 99	2.253 96	−0.954 64
Honeycomb trees	$\langle t\rangle/b$	0.444 50	1.666 40	−1.002 56
		0.445 44	1.718 62	−1.017 09
		0.445 05	1.691 32	−1.009 71
		0.445 20	1.700 10	−1.012 06
		0.445 16	1.697 85	−1.011 47
		0.445 09	1.692 99	−1.010 21
		0.445 14	1.696 62	−1.011 17

duced here for convenience.) As we might expect, the growth parameters for trees are lower for all lattices than those for the unrestricted animals. The asymptotic limit for central θ is subject to some curvature in all cases, although this is less marked for the longer series. Trees in all cases tend to underestimate the exponent value with respect to the exact result predicted by Dhar [2]. Biasing the sequences with respect to exact θ leads, however, to generally good agreement with respect to the limiting estimates of λ, and this is reflected in the uncertainties quoted. It would appear, therefore, on the evidence of the tree and animal data, that the property of nonregular periodicity does not modify the hierarchy of the exponent structure.

Given the known shortcomings of ratio procedures with respect to confluence singularities of the type typically found for animal and fixed-cycle series, we have also looked at the correction for the subdominant singularity as described in Eq. (4) together with the limiting behavior of the averages, $\langle t \rangle / b$. Results for the modified asymptotic form for the averages $A + Bb^{-\phi}$ (Eq. (6) and [15]) are given in Figs. 2(a) and 2(b) for ϕ, together with three point fits on A, B, and ϕ (Table V). Here, ϕ does not necessarily relate to the sum of $\theta_c + w$ [Eq. (4)] but depends on the amptitudes $A_{n,c}$. The numerical evidence appears good for a central estimate of $\phi = 1.0$ (honeycomb animals and trees), although the Archimedean results (based on shorter sequences) would suggest a lower

TABLE VI. Sequences for w (second log-derivative scheme) animals and trees.

	Site	Bond	Trees
Kagomé	−2.463 06	0.260 53	
	−0.897 78	−0.001 61	
	−0.619 62	−0.063 45	
	−0.809 01	0.253 09	
	−0.724 52	0.379 39	
	−0.459 78	0.401 19	
Honeycomb			39.321 43
			−38.046 87
			16.997 20
			−11.390 97
		8.218 17	6.830 82
		−6.316 87	−2.922 40
		2.568 86	1.473 62
		1.450 93	9.983 18
		2.491 42	0.084 13
		1.102 75	2.004 03
		0.805 47	−0.380 05
		1.781 98	1.494 64
		1.472 02	0.132 50
		1.431 00	1.294 18
		1.174 99	0.520 94
		1.416 27	0.965 67
Archimedean		−3.017 12	
Subnet 1		−2.242 89	
		−0.996 11	
		0.008 66	−0.339 26
		0.249 69	0.435 47
		0.427 38	−0.983 05
		0.534 66	0.396 87
		0.511 33	0.185 16
		0.456 37	0.261 94
Subnet 2		−0.715 40	
		1.837 17	
		0.421 79	
		−0.453 74	1.258 90
		−0.464 19	2.048 95
		−0.649 33	3.572 76
		−0.695 77	3.408 48
		−0.521 06	2.784 51
		−0.287 12	3.209 27

value, $\phi \sim 0.955$. The Kagomé results fall fairly smoothly between those for the alternating lattices. An overall estimate, taking all sequences into account, supports $\phi = 0.985 \pm 0.025$. Results obtained from applying the second-log-derivative scheme directly to estimate w (Table VI and Fig. 3) afford rather poor determination of the subdominant animal exponent. The irregularities of the sequences obtained are clearly seen from the table, with only the Kagomé lattice (site values included for illustration) showing some degree of stability and suggesting $w = 0.8 \pm 0.1$, which excludes the $1/n$ correction, suggesting that the leading one is indeed a scaling correction.

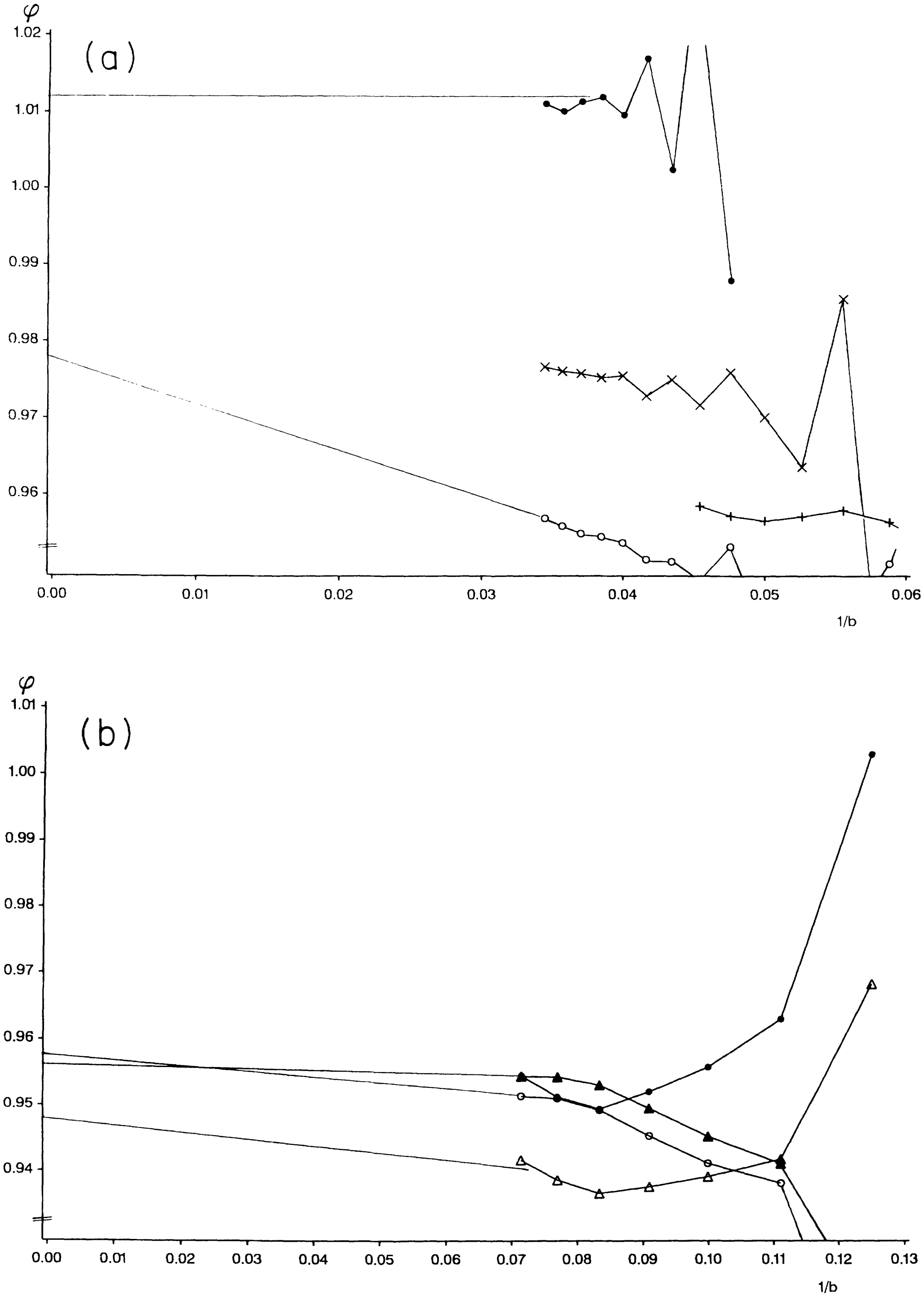

FIG. 2. (a) Three-point fit sequences for ϕ (modified exponent) honeycomb and Kagomé. ×, honeycomb animals $\langle t \rangle / b$; ○, honeycomb animals $\langle s \rangle / b$; ●, honeycomb trees $\langle t \rangle / b$; +, Kagomé animals $\langle s \rangle / b$. (b) Three-point fit sequences for ϕ (modified exponent) Archimedean (3,3,3,4,4). Sublattice 1: ●, $\langle t \rangle / b$; ○, $\langle s \rangle / b$. Sublattice 2: ▲, $\langle t \rangle / b$; △, $\langle s \rangle / b$.

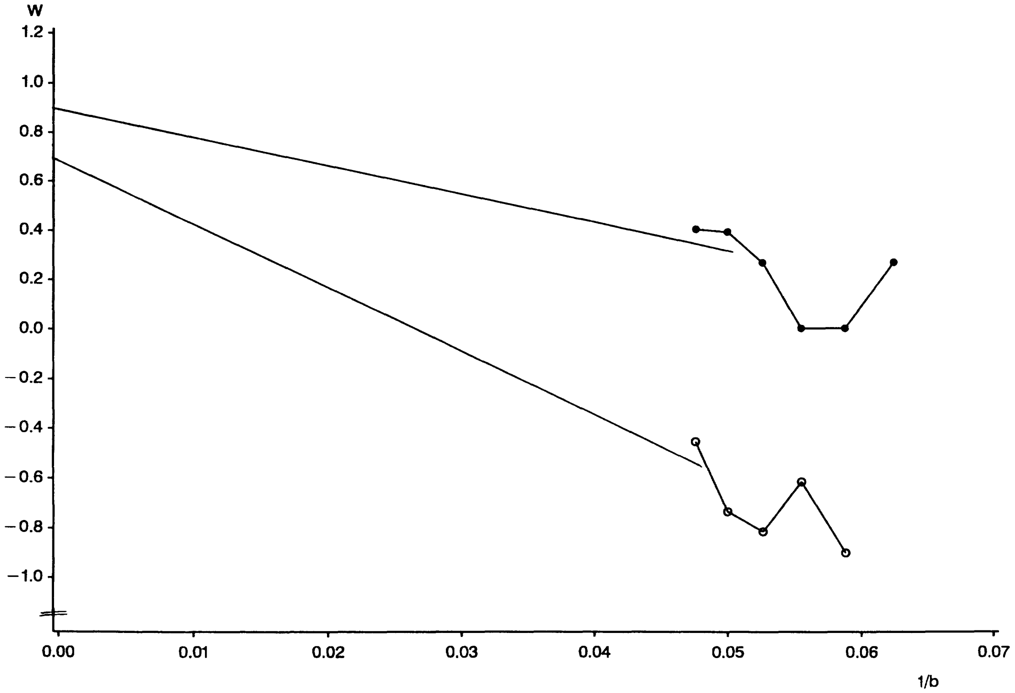

FIG. 3. Kagomé sequences for w (second log-derivative scheme). ●, bond; ○, site.

While the precision is not particularly good, it is comparable with that attained for studies on the Lee-Yang edge singularities and other allied problems [16], with w in broad agreement with subsequent estimates [3,11].

From the central limiting values of the ratios given in Table III, we find the effective coordination number for the directed honeycomb lattice to be $\tilde{z}=1.450+-0.050$, which is low compared with the percolation series results [13]. These were shown to support a central value as high as $\tilde{z}=1.75$, but with the smoother earlier sequences favoring the lower bound, $\tilde{z}=1.5$, the large uncertainties being ascribed to the characteristic honeycomb series oscillation of sign for larger n. Unfortunately, the confluence effects present in the animal data in this instance also inhibit a more precise interpretation, although the evidence again suggests that a value close to 1.5 (roughly equivalent to that found for percolation data) is more likely to be correct.

From the central limiting value for the average perimeter-to-bond ratio following Duarte [11], we predict the *asymptotic* value of the growth parameter for the honeycomb lattice to be $\lambda=2.362+-0.007$. For the Archimedean lattice, the central limiting value for $\langle t \rangle /b$ quoted by Ruskin [13] predicts an asymptotic value of $\lambda=5.002+-0.008$. The higher asymptotic values for both honeycomb and Archimedean lattices are predictable, or course, since the region of interest is now $a > a_c$ [Eq. (5)].

CONCLUSION

Investigation of the dominant exponent behavior for trees and animals on selected "exotic" lattices with both regular and alternating periodicity provides evidence in support of a hierarchical exponent structure, which is unaffected by the periodicity, with θ for unrestricted animals equivalent to θ_0 for trees. Estimates of the growth parameters for the trees on selected lattices and for the honeycomb animal data are also given, and an analysis of the site-to-bond ratios for the honeycomb series suggests that the "effective" coordination number governing the lattice behavior is lower than that originally conjectured by percolation analyses. We quote $\tilde{z}=1.450+-0.050$. The effect of confluent singularities on the exponent estimates for the exotic lattices is investigated for the animal and tree series, but while numerical evidence supports an exponent ϕ for the averages in the region of 1.0 (particularly for longer sequences), results of estimating the animal correction term w directly are not conclusive. The best estimates for w are obtained from the Kagomé lattice and broadly support those of previous works, with $w=0.8\pm0.1$.

ACKNOWLEDGMENTS

Thanks are due to Professor D. Weaire of Trinity College, University of Dublin for first drawing "exotic" lattices to our attention and to Dr. J. Duarte for useful discussions and generous provision of copies of his work prior to publication.

APPENDIX: RECURSION RELATIONSHIP

In terms of the enumeration of clusters, the linkages of the different site types can be vitally important, and the establishment of recursion relationships for cluster growth on the different sublattices can provide a substantial reduction of effort required. Taking the Kagomé lattice as an example, it is well known that the isotropic lattice has three sublattices defined by three different root-site types. For the directed (anisotropic) Kagomé lattice, however, there is redundancy in site types, with two of the three being equivalent, owing to preferred directions of linkage. Hence, the directed Kagomé lattice has two sublattices, dependent on the two distinct site types, which we can denote I and II. As the Kagomé lattice is built up from hexagons and triangles, we can distinguish between these, with type I on the triangle having two incoming bonds, and type II having one incoming and one outgoing bond. Consequently, type-I sites can be exactly counted for a tree configuration of given size s. Further, since the Kagomé trees are bond-to-site transformations from the honeycomb lattice (i.e., self-avoiding walks), we can establish a recursion relationship as follows. From any type-I site, we can progress to either a type-I or a type-II site but not both, since we are interested in self-avoiding walks (SAW's). Thus, the number of SAW's for a type-I site is given by

$$N_s^{\mathrm{I}} = N_{s-1}^{\mathrm{I}} + N_{s-1}^{\mathrm{II}} \; . \tag{A1}$$

Further, since the type-II progression represents a branch from a type-I site en route to another type-I site, we have

$$N_{s-1}^{\mathrm{II}} = N_{s-2}^{\mathrm{I}} \; ,$$

i.e.,

$$N_s^{\mathrm{I}} = N_{s-1}^{\mathrm{I}} + N_{s-2}^{\mathrm{I}} \; . \tag{A2}$$

In other words, only one site type is relevant for enumeration purposes, with initial conditions given by $N_1^{\mathrm{I}} = 1$, $N_2^{\mathrm{I}} = 2$. For type-II sites as defined above, trees are not equivalent to SAW's, since the root site has two outgoing bonds, i.e., two branches. Clearly, since we can regard type-II sites as intermediary points on the type-I sublattice, SAW's are one branch trees in the type-II sublattice, and we have directly for SAW's

$$N_s^{\mathrm{II}} = 2N_{s-1}^{\mathrm{I}} \; . \tag{A3}$$

For trees with more than one branch, an exact relationship is not straightforward, because as branches grow there is the possibility of interaction leading to a configuration that is not a tree. Growth parameters on the two sublattices will be equivalent, but in this case also, we have from Eqs. (A1) and (A2) the result that only type-I sites need to be enumerated and that $\lambda_s^{\mathrm{II}} = \lambda_s^{\mathrm{I}} \, \forall s > 3$.

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