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Analytic solution for inversion and intensity of the Jaynes-Cummings model with cavity damping

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An analytic expression for the inversion and intensity of the Jaynes-Cummings model with cavity damping is derived in the rotating-wave approximation for vanishing thermal photon numbers. Using the s-parametrized quasiprobability distributions of Cahill and Glauber [Phys. Rev. **177**, 1882 (1969)], the equation of motion for the density operator is transformed into c-number equations for the quasiprobability functions. By a suitable expansion into a Fourier series and into Laguerre functions, we obtain ordinary tridiagonal coupled differential equations for the expansion coefficients. By an appropriate choice of a scaling parameter and by a proper elimination procedure, it is shown that the coefficients that determine the inversion and the mean intensity are only coupled to coefficients with the same index and to coefficients with the next upper index. Because of this coupling the Laplace transform can be given analytically. Furthermore, it is shown that the eigenvalues and eigenvectors can also be calculated, thus leading to an analytic solution for the inversion of the Jaynes-Cummings model.

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I. INTRODUCTION

One of the simplest models describing the interaction of light and matter is the Jaynes-Cummings (JC) model [1], where a single two-level atom interacts with one light mode of the cavity. In the rotating-wave approximation this model can be solved exactly in the absence of cavity damping. Starting with a cavity field in a coherent state $|\alpha_0\rangle$ and with the atom in its upper state, it was found that repeated decays and revivals of Rabi oscillations occur, see for instance Refs. [1–5]. The predicted collapses and revivals of the inversion oscillations are in agreement with the experiments done with Rydberg atoms in a microwave cavity, see Ref. [6–8].

In the experiments, the damping of the cavity mode is not negligibly small. Thus for a detailed comparison with experiments, cavity damping must be included in a treatment of the JC model. With cavity damping one has to solve an appropriate equation for the density operator which describes the system. This equation of motion for the density operator is more difficult to solve. As far as we know, no analytic solution for this model has been published for general initial conditions. For the initial state of the light field being a vacuum state, an analytic solution was given by Agarwal and Puri [9]. An even simpler model, the so-called Raman coupled model [10], was solved analytically with the inclusion of cavity damping by two of us [11]. In the JC model damping was already included by some approximate technique valid for small damping and vanishing thermal quanta $n_{\text{th}} = 0$ (dressed-atom approximation), see Refs. [12–14], or numerically without any such approximation, see Refs. [15–18]. In Ref. [15] an initial intensity of the coherent field of $I_0 = |\alpha_0|^2 = 2$ has been used. This number is too small to give pronounced repeated decays and revivals of the Rabi oscillations without detuning. For an appreciable detuning, however, the revivals can be seen even for this low initial intensity [15]. With the method presented in [16–18] initial intensities up to $I_0 = 30$ could be handled. The cavity-damped JC model with an additional Kerr medium has also been calculated [19] using this technique.

In the present paper we show that for vanishing thermal quanta $n_{\text{th}} = 0$, an analytic expression of the inversion and the mean intensity can be obtained within the rotating-wave approximation. Here the damping constant can have any arbitrary value within the approximations used in deriving the Markovian reduced density operator equation of motion. In the limit of vanishing damping the usual expressions are recovered. The main steps for obtaining this analytical expression are the following. First we transform the equation of motion for the density operator into c-number equations by using the s-parametrized quasiprobability distributions of Cahill and Glauber [20, 21] as done in Refs. [16–19]. Next these distribution functions are expanded into a Fourier series and into Laguerre functions. In this way we obtain ordinary tridiagonal coupled differential equations for the expansion coefficients. In Refs. [16–19] a similar expansion was used. However, in the present work we do not employ the six combinations of the matrix elements as done in the previous work. As it was shown by one of us [22] the six combinations used previously lead to some superfluous additional variables, which give rise to unphysical positive eigenvalues. For physical initial conditions these unphysical eigenvalues have zero weights and thus can only show up because of numerical inaccuracies. By employing a more suitable expansion we remove the superfluous variables; nevertheless, we have been able to decouple the equations for the expansion coefficients in the Fourier index as done previously. By a suitable choice of a scaling parameter and by a proper elimination procedure, it is shown next that the coefficients which determine the inversion and the mean intensity are only coupled to coefficients with the same Laguerre index and to coefficients with the next upper Laguerre index. Because of this coupling the Laplace transform can be given analytically. Furthermore, it turns out that also the eigenvalues and eigenvectors follow analytically, thus leading to an analytic solution for the inversion and the mean intensity of the Jaynes-Cummings model with cavity damping. It should be mentioned that the same expansion procedure can also be used if thermal quanta are taken into account [22]. In this case, however, we have not been able to derive analytical results for the eigenvalues and their weights.

The present paper is organized as follows. In Sec. II we introduce the equation of motion for the zero-temperature reduced density operator for the Jaynes-Cummings model. Four field operators are introduced to take into account the atomic degree of freedom. In Sec. III we introduce the s-parametrized quasiprobability distributions which satisfy four coupled partial differential equations. These distributions are expanded over a complete set of functions in Sec. IV and lead to a tridiagonal recurrence relation for the expansion coefficients. In Sec. V we calculate the Laplace transform of this recurrence relation. The poles of this Laplace transform are the eigenvalues, the weights also follow from this expression. In Sec. VI we calculate the Laplace transform of the intensity $I(t)$ and the inversion $D(t)$ for the damped Jaynes-Cummings model for arbitrary initial conditions. By an inverse Laplace transform this result yields the time dependence of the intensity and inversion. In Sec. VII we derive the series representation for a special initial state, which can be used to calculate the time dependence of the intensity $I(t)$ and inversion $D(t)$ for that particular initial condition. In Sec. VIII the limit of a vanishing damping is demonstrated.

II. THE EQUATIONS OF MOTION

The Hamilton operator for the Jaynes-Cummings model in the rotating-wave approximation reads

$$H/\hbar = \omega_c a^\dagger a + \omega_a \sigma_z/2 + g(a^\dagger \sigma^- + a\,\sigma^+), \tag{2.1}$$

where ω_a and ω_c are the frequencies of the atom and cavity mode, respectively, $a^\dagger$ and a are the creation and annihilation operators of the field mode, σ_z, σ^+, and σ^- are the Pauli spin matrices, and g is the atom-cavity coupling constant. The equation of motion for the reduced density operator $\rho = \rho(t)$ of the Jaynes-Cummings model with cavity damping takes the form

$$\dot\rho = -(i/\hbar)\,[H,\rho] + \kappa L_{\rm ir}(\rho). \tag{2.2}$$

The damping is characterized by κ, the decay rate of the photon numbers in the cavity, and $L_{\rm ir}$, which describes the irreversible motion of ρ caused by the environment. Neglecting the influence of the thermal photons in the cavity, $L_{\rm ir}$ is given by

$$L_{\rm ir}(\rho) = 2a\rho a^\dagger - \rho a^\dagger a - a^\dagger a \rho. \tag{2.3}$$

Transforming to the interaction picture in the usual way

$$\tilde\rho(t) = e^{i\omega_c(a^\dagger a + \sigma_z/2)t}\rho(t)e^{-i\omega_c(a^\dagger a+\sigma_z/2)t}\,, \tag{2.4}$$

the equation of motion for $\tilde\rho$ is of a similar form as equation (2.2), but with

$$\tilde H/\hbar = \Delta\sigma_z/2 + g(a^\dagger\sigma^- + a\,\sigma^+), \tag{2.5}$$

where $\Delta = \omega_a - \omega_c$ is the detuning of the atomic frequency ω_a from the cavity frequency ω_c. The matrix elements with respect to the atomic states are combined to the four new operators

$$\begin{aligned}\rho_1 &= \langle\uparrow|\tilde\rho|\uparrow\rangle + \langle\downarrow|\tilde\rho|\downarrow\rangle, \quad \rho_2 = \langle\uparrow|\tilde\rho|\uparrow\rangle - \langle\downarrow|\tilde\rho|\downarrow\rangle \\ \rho_3 &= \langle\uparrow|\tilde\rho|\downarrow\rangle, \quad \rho_4 = \langle\downarrow|\tilde\rho|\uparrow\rangle.\end{aligned} \tag{2.6}$$

They are still operators with respect to the light mode. Their equations of motion follow from (2.2) as

$$\begin{aligned}\dot\rho_1 &= ig(\rho_3 a^\dagger - a^\dagger\rho_3 + \rho_4 a - a\rho_4) + \kappa L_{\rm ir}(\rho_1), \\ \dot\rho_2 &= ig(\rho_3 a^\dagger + a^\dagger\rho_3 - \rho_4 a - a\rho_4) + \kappa L_{\rm ir}(\rho_2), \\ \dot\rho_3 &= -i\Delta\rho_3 + ig(\rho_1 a - a\rho_1 + \rho_2 a + a\rho_2)/2 \\ &\quad +\kappa L_{\rm ir}(\rho_3), \\ \dot\rho_4 &= i\Delta\rho_4 + ig(\rho_1 a^\dagger - a^\dagger\rho_1 - \rho_2 a^\dagger - a^\dagger\rho_2)/2 \\ &\quad +\kappa L_{\rm ir}(\rho_4)\,.\end{aligned} \tag{2.7}$$

III. QUASIPROBABILITY DISTRIBUTIONS

We use the s-parametrized quasiprobability distributions $W(\alpha,\alpha^*,s,t)$ introduced by Cahill and Glauber [21] to get a c-number representation of the operator equation (2.7). We define the characteristic functions $\chi_i(\xi,\xi^*,s,t)$ for each operator ρ_i, $i\in\{1,\dots,4\}$, by

$$\chi_i(\xi,\xi^*,s,t) = {\rm Tr}\left[e^{\xi a^\dagger - \xi^* a + s|\xi^2|/2}\rho_i(t)\right]\,, \tag{3.1}$$

which yields the quasiprobability distribution W_i by a Fourier transformation

$$W_i(\alpha,\alpha^*,s,t) = \int \chi_i(\xi,\xi^*,s,t)e^{\alpha\xi^* - \alpha^*\xi}\frac{d^2\xi}{\pi^2}. \tag{3.2}$$

Inserting this relation into the operator equation (2.7) and applying the relations in Table I of Ref. [23], we arrive at the following system of partial differential equations for the distributions $W_1 - W_4$:

$$\begin{aligned}
\dot{W}_1 &= ig\left[\frac{\partial}{\partial\alpha}W_3 - \frac{\partial}{\partial\alpha^*}W_4\right] + \kappa\mathcal{L}_{\rm ir}W_1,\\
\dot{W}_2 &= ig\left[\left(2\alpha^* - s\frac{\partial}{\partial\alpha}\right)W_3 - \left(2\alpha - s\frac{\partial}{\partial\alpha^*}\right)W_4\right]\\
&\quad +\kappa\mathcal{L}_{\rm ir}W_2,\\
\dot{W}_3 &= -i\Delta W_3 - \frac{ig}{2}\left[\frac{\partial}{\partial\alpha^*}W_1 - \left(2\alpha - s\frac{\partial}{\partial\alpha^*}\right)W_2\right]\\
&\quad +\kappa\mathcal{L}_{\rm ir}W_3,\\
\dot{W}_4 &= i\Delta W_4 + \frac{ig}{2}\left[\frac{\partial}{\partial\alpha}W_1 - \left(2\alpha^* - s\frac{\partial}{\partial\alpha}\right)W_2\right]\\
&\quad +\kappa\mathcal{L}_{\rm ir}W_4,
\end{aligned} \tag{3.3}$$

with the shorthand notation $\mathcal{L}_{\rm ir}$ for the damping operator:

$$\mathcal{L}_{\rm ir} = \frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^* + (1-s)\frac{\partial^2}{\partial\alpha\partial\alpha^*}\,. \tag{3.4}$$

The Hermiticity of the density operator ρ is reflected by the following constraints:

$$W_1 = W_1^*\,,\quad W_2 = W_2^*\,, \tag{3.5}$$

$$W_3 = W_4^*\,,\quad W_4 = W_3^*\,, \tag{3.6}$$

for the distribution functions W_i.

To obtain expectation values of s-ordered products $\{(a^\dagger)^n a^m\}_s$ one simply has to integrate the quasiprobabilities $W(\alpha,\alpha^*,s)$ over the complex α-plane according to

$$\langle\{(a^\dagger)^n a^m\}_s\rangle = \int (\alpha^*)^n\alpha^m W(\alpha,\alpha^*,s)\,d^2\alpha\,. \tag{3.7}$$

Therefore the intensity $I(t)$ can be expressed by

$$I(t) = \langle a^\dagger a\rangle = \int \alpha^*\alpha W_1(\alpha,\alpha^*,s)\,d^2\alpha - \tfrac{1}{2}(1-s)\,, \tag{3.8}$$

while the inversion $D(t)$ of the atom, after taking the trace over the atomic states, is given by

$$\begin{aligned}
D(t) &= {\rm Tr}_{AF}(\sigma_z\rho) = {\rm Tr}_F(\rho_2)\\
&= \int W_2(\alpha,\alpha^*,s,t)\,d^2\alpha\,.
\end{aligned} \tag{3.9}$$

IV. EXPANSION OF THE QUASIPROBABILITY DISTRIBUTIONS

To handle the set of partial differential equations (3.3) we expand the various distributions $W_1 - W_4$ in a complete set of functions $F_{n,m}(I,\phi)$ over the complex α plane with appropriate decay behavior. For convenience we will use the radial variables I (intensity) and angle ϕ (phase) instead of α and α^*, i.e.,

$$\alpha = \sqrt{I}e^{i\phi},\quad \alpha^* = \sqrt{I}e^{-i\phi}. \tag{4.1}$$

Let us start with the expansions

$$\begin{aligned}
W_1(I,\phi,s,t) &= \sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty} c_{n,m}^{(1)}(t)F_{n,m}(I,\phi),\\
W_2(I,\phi,s,t) &= \sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty} c_{n,m}^{(2)}(t)F_{n,m}(I,\phi),\\
W_3(I,\phi,s,t) &= \sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty} c_{n-1,m}^{(3)}(t)F_{n,m}(I,\phi),\\
W_4(I,\phi,s,t) &= \sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty} c_{n+1,m}^{(4)}(t)F_{n,m}(I,\phi)\,,
\end{aligned} \tag{4.2}$$

over the complete set of functions

$$F_{n,m}(I,\phi) = e^{in\phi}(I/\tilde{I})^{|n|/2}L_m^{|n|}(I/\tilde{I})\,e^{-I/\tilde{I}}\,. \tag{4.3}$$

Here the $L_m^{|n|}(x)$ are the following generalized Laguerre polynomials:

$$L_m^{|n|}(x) = \sum_{k=0}^{m}(-1)^k\binom{m+|n|}{m-k}\frac{x^k}{k!}\,, \tag{4.4}$$

and $\tilde{I}$ is an arbitrary scaling parameter. We have chosen the indices of the expansion coefficients $c_{n,m}^{(i)}$ so that the resulting recurrence relation will decouple in the index n.

After inserting the above expansion (4.2) into the partial differential equation (3.3) and by using the orthogonality properties of the generalized Laguerre polynomials $\mathrm{L}_m^{|n|}$ [24] we can derive a system of tridiagonal recurrence relations for the coefficients $c_{n,m}^{(i)}$ of the expansion (4.2). This recurrence relation can be simplified by choosing the positive scaling factor $\tilde{I}$ according to

$$\tilde{I} = (1-s)/2\,. \tag{4.5}$$

Because in this paper we are only interested in the dynamics of the inversion $D(t)$ and intensity $I(t)$ we can focus on the resulting equations for $n = 0$. Introducing the real quantities

$$\begin{aligned}
x_m(t) &= c_{0,m}^{(1)}(t)\,, & u_m(t) &= 2g\tilde{I}^{-1/2}\,{\rm Im}[c_{0,m}^{(3)}(t)]\,,\\
y_m(t) &= c_{0,m}^{(2)}(t)\,, & v_m(t) &= 2g\tilde{I}^{-1/2}\,{\rm Re}[c_{0,m}^{(3)}(t)]\,,
\end{aligned} \tag{4.6}$$

one finally obtains the system of equations

$$\begin{aligned}
\dot{x}_m &= -2m\kappa x_m - mu_{m-1},\\
\dot{y}_m &= -2m\kappa y_m - 2\tilde{I}(m+1)u_m + mu_{m-1},\\
\dot{u}_m &= -\Delta v_m - (2m+1)\kappa u_m\\
&\qquad +(g^2/\tilde{I})(x_m+y_m) - 2g^2y_{m+1},\\
\dot{v}_m &= \Delta u_m - (2m+1)\kappa v_m\,.
\end{aligned} \tag{4.7}$$

It is worth noticing that because the density matrix ρ obeys the boundness condition

$$0 \le {\rm Tr}(\rho^2) = |\rho_{\uparrow\uparrow}|^2 + 2|\rho_{\uparrow\downarrow}|^2 + |\rho_{\downarrow\downarrow}|^2 \le 1\,, \tag{4.8}$$

where $|\rho| = \sqrt{{\rm Tr}_F(\rho^\dagger\rho)}$ is the Hilbert-Schmidt norm [20], one may check that each of the expansion coefficients $c_{n,m}^{(i)}$ must vanish for $m \to \infty$. Since the intensity $I(t)$ and the inversion $D(t)$ can be expressed by (3.8) and (3.9), we can derive the following relation to the coefficients $x_1(t), y_0(t)$:

$$I(t) = \tfrac{1}{2} \int_0^\infty \int_0^{2\pi} I\, W_1(I,\phi,s,t)\, d\phi\, dI = -\pi \tilde{I}^2 x_1(t), \tag{4.9}$$

$$D(t) = \tfrac{1}{2} \int_0^\infty \int_0^{2\pi} W_2(I,\phi,s,t)\, d\phi\, dI = \pi \tilde{I} y_0(t)\,. \tag{4.10}$$

V. THE LAPLACE TRANSFORM OF THE SOLUTION

In this section we calculate the eigenvalues of the equation of motion (4.7). We will use them to reexpress the Laplace transform of the intensity $I(t)$ and the inversion $D(t)$ as a partial fraction decomposition over its poles. In order to keep track of the initial conditions let us apply a Laplace transformation to the recurrence relation (4.7), i.e.,

$$X(t) \to \hat{X}(z) = \int_0^\infty e^{-zt} X(t)\, dt\,, \tag{5.1}$$

where we use a hat to denote the Laplace-transformed quantities. Furthermore, let us denote the initial values by

$$\begin{aligned} x_m(0) &= x_m^0\,, & u_m(0) &= u_m^0\,, \\ y_m(0) &= y_m^0\,, & v_m(0) &= v_m^0\,. \end{aligned} \tag{5.2}$$

Therefore the Laplace transform of (4.7) yields the set of recurrence equations

$$\begin{aligned} z\hat{x}_m - x_m^0 &= -2m\kappa\hat{x}_m - m\hat{u}_{m-1}, \\ z\hat{y}_m - y_m^0 &= -2m\kappa\hat{y}_m - 2\tilde{I}(m+1)\hat{u}_m + m\hat{u}_{m-1}, \\ z\hat{u}_m - u_m^0 &= -\Delta\hat{v}_m - (2m+1)\kappa\hat{u}_m \\ &\quad + (g^2/\tilde{I})(\hat{x}_m + \hat{y}_m) - 2g^2\hat{y}_{m+1}, \\ z\hat{v}_m - v_m^0 &= \Delta\hat{u}_m - (2m+1)\kappa\hat{v}_m. \end{aligned} \tag{5.3}$$

It should be mentioned here that the special case $\Delta = 0$ could be treated separately to simplify the further calculations. In that case the last equation of (5.3) decouples from the first three equations and can be solved immediately. Therefore only the first three equations need to be handled further on in the case of vanishing detuning.

As one may check, it is possible to eliminate the variables $\hat{u}_m, \hat{v}_m$ in (5.3), leading to the set of equations

$$\begin{aligned} z\hat{x}_0 &= x_0^0, \\ \hat{x}_{m+1} &= \left[Q_m^x(z)\right]^{-1} (\hat{x}_m + \hat{y}_m) + P_m^x, \\ \hat{y}_{m+1} &= \left[Q_m^y(z)\right]^{-1} (\hat{x}_m + \hat{y}_m) + P_m^y\,. \end{aligned} \tag{5.4}$$

Adding the last two equations one gets a recurrence relation where only the quantity $\hat{H}_m = \hat{x}_m + \hat{y}_m$ occurs,

$$\hat{H}_{m+1}(z) = \left[Q_m(z)\right]^{-1} \hat{H}_m(z) + P_m(z)\,. \tag{5.5}$$

The solution of these equations determines the Laplace-transformed expansion coefficients $\hat{x}_m$, $\hat{y}_m$ via equation (5.4).

For our purpose it is very useful to get the explicit values of the poles of $Q_m(z)$. To achieve this we had to find the roots of a fourth-order polynomial in z. Using this result we can express $Q_m(z)$ and $P_m(z)$ in the following way:

$$Q_m(z) = \frac{4g^2\tilde{I}(m+1)(z+2m\kappa+2\kappa)(z+2m\kappa+\kappa)}{(z-\lambda_m^{(1)})(z-\lambda_m^{(2)})(z-\lambda_m^{(3)})(z-\lambda_m^{(4)})}, \tag{5.6}$$

$$\begin{aligned} P_m(z) = &\frac{x_{m+1}^0 - H_m^0/(2\tilde{I})}{z+2m\kappa+2\kappa} \\ &- \frac{1}{2\tilde{I}} \frac{z+2m\kappa+\kappa+\Delta^2/(z+2m\kappa+\kappa)}{2g^2(m+1)} H_m^0 \\ &- \frac{\Delta v_m^0}{2g^2(z+2m\kappa+\kappa)} + \frac{u_m^0}{2g^2} \end{aligned}$$

with the various abbreviations

$$\begin{aligned} H_m^0 &= x_m^0 + y_m^0, \\ \lambda_m^{(1,2)} &= -(2m+1)\kappa \pm \kappa\Delta/\omega_m, \\ \lambda_m^{(3,4)} &= -(2m+1)\kappa \pm i\omega_m, \\ \omega_m &= \left[\left(\sqrt{\Omega_m^4 + 4\kappa^2\Delta^2} + \Omega_m^2\right)/2\right]^{1/2}, \\ \Omega_m^2 &= \Delta^2 + 4g^2(m+1) - \kappa^2\,. \end{aligned} \tag{5.7}$$

Because $\hat{H}_m(z)$ should tend to zero for $m \to \infty$ and fixed z as mentioned in Sec. IV, the eigenvalues must be the poles of $Q_m(z)$, i.e., the eigenvalues of Eq. (5.4) are $\lambda_m^{(i)}$, $i \in \{1,2,3,4\}$, $m \geq 0$. From the first equation of (5.4) one obtains the additional eigenvalue

$$\lambda^0 = 0\,, \tag{5.8}$$

which describes the stationary solution. It should be mentioned that the eigenvalues for $m = 0$ follow from (2.12) of the work of Agarwal and Puri [9].

Finally, using Eqs. (4.9), (4.10), and the first two equations of (5.4), we obtain the Laplace transformation of the intensity $\hat{I}(z)$ and the inversion $\hat{D}(z)$ in the following form:

$$\hat{I}(z) = -\frac{\pi\tilde{I}}{2} \frac{z\hat{H}_0(z) + 2\tilde{I}x_1^0 - H_0^0}{z+2\kappa}, \tag{5.9}$$

$$\hat{D}(z) = \pi\tilde{I}\hat{H}_0(z) - 1/z\,. \tag{5.10}$$

Although the Laplace-transformed intensity $\hat{I}(z)$ seems to have a pole at $z = -2\kappa$, one can show that the limit

$$\lim_{z \to -2\kappa} |\hat{I}(z)| < \infty \tag{5.11}$$

is still bounded. Now it remains to work out the explicit solution for $\hat{H}_0(z)$, therefore let us rewrite (5.5) as

$$\hat{H}_m(z) = Q_m(z)\left[\hat{H}_{m+1}(z) - P_m(z)\right] . \tag{5.12}$$

Iterating this equation leads to the following expression, since $\hat{H}_m(z)$ must vanish for $m \to \infty$ as mentioned in Sec. IV:

$$\hat{H}_0(z) = -\sum_{\mu=0}^{\infty}\left[P_\mu(z)\prod_{l=0}^{\mu} Q_l(z)\right] . \tag{5.13}$$

Let us write $\hat{H}_0(z)$ as a partial fraction decomposition over its poles as

$$\hat{H}_0(z) = \sum_{\nu,\mu}\frac{H_{0,\nu}^{(\mu)}}{z-\lambda_\nu^{(\mu)}} . \tag{5.14}$$

The weights $H_{0,\nu}^{(\mu)}$ can now be extracted by taking the limit

$$H_{0,\nu}^{(\mu)} = \lim_{z\to\lambda_\nu^{(\mu)}}(z-\lambda_\nu^{(\mu)})\hat{H}_0(z) . \tag{5.15}$$

After some calculations one obtains the explicit expression for the weights of $\hat{H}_0(z)$

$$H_{0,\nu}^{(\mu)} = a_\nu^{(\mu)}\left[\prod_{l=0}^{\nu-1} q_{l,\nu}^{(\mu)}\right]\sum_{k=\nu}^{\infty}\left[p_{k,\nu}^{(\mu)}\prod_{l=\nu+1}^{k} q_{l,\nu}^{(\mu)}\right] , \tag{5.16}$$

with the various coefficients defined by

$$a_\nu^{(1,2)} = -\frac{2\tilde{I}g^2(\nu+1)(\kappa\pm\kappa\Delta/\omega_\nu)}{\sqrt{\Omega_\nu^4+4\kappa^2\Delta^2}}, \tag{5.17}$$

$$a_\nu^{(3,4)} = \frac{2\tilde{I}g^2(\nu+1)(\kappa\pm i\omega_\nu)}{\sqrt{\Omega_\nu^4+4\kappa^2\Delta^2}}, \tag{5.18}$$

$$q_{l,\nu}^{(\mu)} = Q_l\left(\lambda_\nu^{(\mu)}\right) , \tag{5.19}$$

$$p_{k,\nu}^{(\mu)} = P_k\left(\lambda_\nu^{(\mu)}\right) . \tag{5.20}$$

In the special case $\Delta = 0$ the eigenvalues $\lambda_m^{(1)}$ and $\lambda_m^{(2)}$ are identical and one can remove the common factor $z + 2l\kappa + \kappa$ from the numerator and denominator of the Q_l, which expresses the fact that the last equation of (5.3) is decoupled from the first three equations of (5.3). To get the correct formula for the $p_{k,\nu}^{(\mu)}$ one has to set $\Delta = 0$ in the $P_k(z)$ first and then insert the $\lambda_\nu^{(\mu)}$. In the following section we derive the explicit dependence of the weights of the poles of the intensity $\hat{I}(z)$ and the inversion $\hat{D}(z)$ on the initial conditions.

VI. DYNAMICS OF THE INVERSION AND INTENSITY

In order to get the time dependence of the inversion and the intensity, we have to use the solution (5.13) for $\hat{H}_0(z)$ with the given initial condition x_m^0, y_m^0, u_m^0, v_m^0. Similar as in the preceding section, we perform a partial fraction decomposition for the intensity $\hat{I}(z)$ and the inversion $\hat{D}(z)$. Transforming these expressions back to the time domain according to

$$\hat{I}(z) \to I(t) = \sum_{\nu,\mu} I_\nu^{(\mu)}\, e^{\lambda_\nu^{(\mu)}t} , \tag{6.1}$$

$$\hat{D}(z) \to D(t) = -1 + \sum_{\nu,\mu} D_\nu^{(\mu)}\, e^{\lambda_\nu^{(\mu)}t} , \tag{6.2}$$

this will give us the time dependence of the quantities of interest. One arrives at the following results for the weights $I_\nu^{(\mu)}$ of the intensity

$$I_\nu^{(\mu)} = -\frac{\pi\tilde{I}}{2}\frac{\lambda_\nu^{(\mu)}}{\lambda_\nu^{(\mu)}+2\kappa}H_{0,\nu}^{(\mu)} , \tag{6.3}$$

and in a similar way the expression for the weights $D_\nu^{(\mu)}$ of the inversion

$$D_\nu^{(\mu)} = \pi\tilde{I}H_{0,\nu}^{(\mu)} . \tag{6.4}$$

To get explicit numeric values for inversion and intensity a computer must be used to evaluate the above formulas. The calculation of the eigenvalues and weights with the presented formula is about 20 times faster than the calculation using a numerical method. Once the eigenvalues and weights are calculated, the computation time is proportional to the number of points of time the inversion and intensity has to be calculated. Using a Runge-Kutta method [18] for the integration of the system of differential equations (4.7), the computation time is proportional to the maximum time up to which one wants to calculate the inversion and intensity. Thus for larger times the analytic method is faster whereas for short enough times the Runge-Kutta method is faster. For example, to calculate data of the inversion from $gt = 0$ up to $gt = 500$ with at least 1000 points and a Runge-Kutta step width of $gdt = 0.025$ the analytic way is more than 40 times faster whereas for obtaining these data from $gt = 0$ up to only $gt = 5$ the Runge-Kutta method is about two times faster.

VII. INITIAL VALUES

In this section we derive the series representation of a special initial state, which can be used to calculate the time dependence of the intensity $I(t)$ and inversion $D(t)$ for that particular initial condition. Let us assume that the atom is in the upper state and the field mode is in a coherent state $|\alpha_0\rangle$ at $t = 0$, therefore the operators ρ_1–ρ_4 take the form

$$\begin{aligned}\rho_1(0) &= \rho_2(0) = |\alpha_0\rangle\langle\alpha_0|,\\ \rho_3(0) &= \rho_4(0) = 0.\end{aligned} \tag{7.1}$$

The initial conditions for the distributions $W_i(\alpha,\alpha^*,s,t)$ may be obtained by inserting (7.1) in (3.1) and performing the Fourier transformation (3.2)

$$\begin{aligned}W_1(\alpha,\alpha^*,s) &= W_2(\alpha,\alpha^*,s)\\ &= \frac{2}{\pi(s-1)}\exp\left\{-\frac{2}{1-s}|\alpha-\alpha_0|^2\right\},\\ W_3(\alpha,\alpha^*,s) &= W_4(\alpha,\alpha^*,s) = 0 .\end{aligned} \tag{7.2}$$

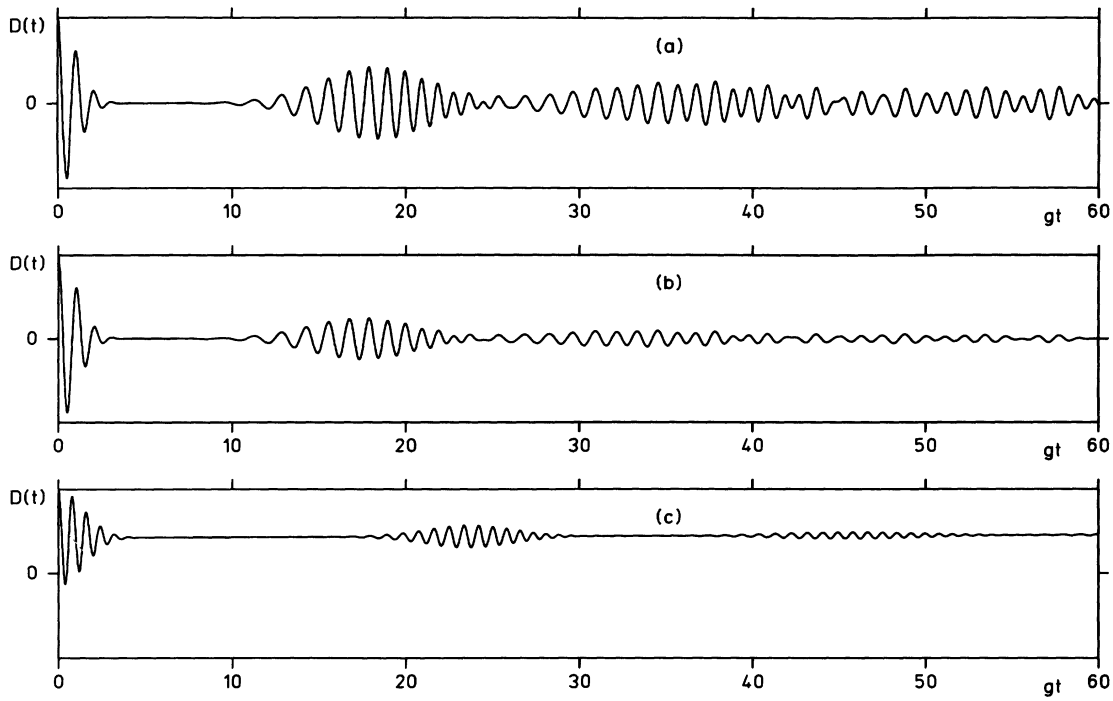

FIG. 1. The inversion $D(t)$ as a function of gt for (a) $\kappa/g = 0.001$, $\Delta/g = 0$, (b) $\kappa/g = 0.003$, $\Delta/g = 0$, (c) $\kappa/g = 0.003$, $\Delta/g = 5$ and for $I_0 = 8$.

The expansion coefficients $x_m^0, \ldots, v_m^0$ may be calculated by using the orthogonality relations of the generalized Laguerre polynomials $L_m^{|n|}(x)$, which then yields the explicit form, see Ref. [18],

$$x_m^0 = y_m^0 = \frac{1}{\pi \tilde{I}} \frac{1}{m!} \left(-\frac{I_0}{\tilde{I}} \right)^m ,$$
$$u_m^0 = v_m^0 = 0 . \tag{7.3}$$

Inserting these initial conditions into (5.6) we obtain the weights of the eigenvalue expansions (6.1) and (6.2) . As one can easily check these weights do not depend on the scaling intensity $\tilde{I}$, or, because of (4.5), they do not depend on the parameter s of the quasiprobability distribution. The eigenvalues (5.7) do not depend on s either and therefore the inversion and the mean intensity are independent of s. This is physically obvious, since these quantities cannot depend on the representation used for the density operator.

Figure 1 shows the influence of damping and detuning on the inversion using the above initial condition. It is seen that the height of the revival oscillations decreases with increasing damping or detuning and with increasing time. Further on the detuning leads to clearly separated revivals and the value of the inversion in between the first collapse and the first revival increases for increasing detuning. Figure 2 shows the inversion for various truncations of the infinite set of eigenfunctions. The long-time behavior can be described using only a few real eigenvalues whereas many more eigenvalues must be taken into account in order to obtain the collapse and revival of the Rabi oscillations. Figure 3 shows the collapses and revivals as well as the decay of the intensity.

A. Short- and long-time behavior of the intensity

For $I_0 \gg 1$ the short-time behavior is determined by the asymptotic decay rate of $I(z) \approx 1/(z + 2\kappa)$ (5.9) for large values of z leading approximately to the $\exp(-2\kappa t)$ decay of the empty cavity.

For $I_0 \gg 1$ the long-time behavior is determined by the lowest nonvanishing real eigenvalue. [The contribution

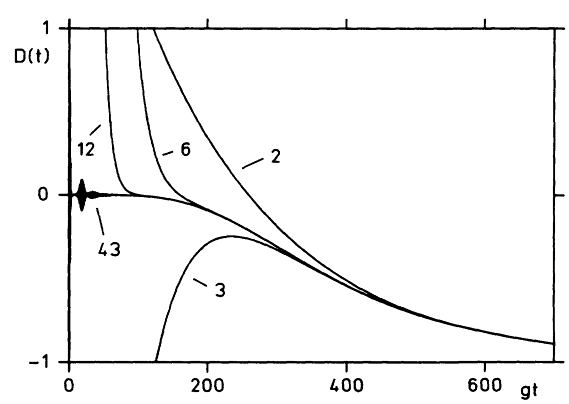

FIG. 2. The inversion $D(t)$ for $\kappa/g = 0.005$, $\Delta/g = 0$, $I_0 = 10$ using 2, 3, 6, 12, 43 different eigenvalues including the stationary one. For the contributions 2, 3, 6, 12 only the real eigenvalues need to be taken into account.

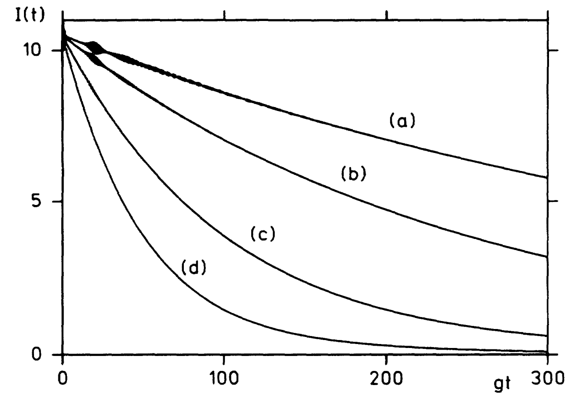

FIG. 3. The mean intensity $I(t)$ as a function of gt for various damping constants (a) $\kappa/g = 0.001$, (b) $\kappa/g = 0.002$, (c) $\kappa/g = 0.005$, (d) $\kappa/g = 0.01$ and for $\Delta/g = 0$ and $I_0 = 10$.

from the complex eigenvalue with the lowest real part is of the order $\exp(-I_0)$.] Thus the long-time decay has the form

$$I(t) \propto \exp(-r\kappa t)\,, \tag{7.4}$$

with

$$r = 1 - |\Delta|/\omega_0\,, \tag{7.5}$$

where ω_0 is defined by the fourth equation of (5.7) for $m = 0$. For small κ this relation simplifies to

$$r = 1 - |\Delta|/\sqrt{\Delta^2 + 4g^2}\,. \tag{7.6}$$

In Fig. 4 we have plotted the intensity in a logarithmic scale without detuning. The behavior of the short-time decay $\exp(-2\kappa t)$ and the long-time decay $\exp(-\kappa t)$ is clearly seen. In order to understand this effect it is very useful to look at the ratio $R(t)$ of the energy in photon numbers of the atomic state, i.e., $[1 + D(t)]/2$, to the field, i.e., $I(t)$. For large t this ratio takes the form

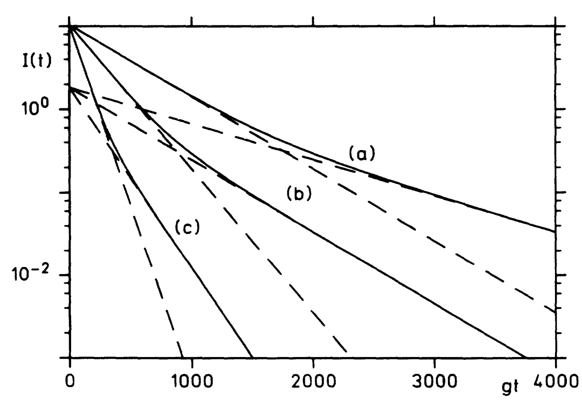

FIG. 4. The intensity in logarithmic scale as a function of time. The short- and long-time contributions are shown by dashed lines for various damping constants (a) $\kappa/g = 0.001$, (b) $\kappa/g = 0.002$, (c) $\kappa/g = 0.005$ and for $\Delta/g = 0$ and $I_0 = 10$.

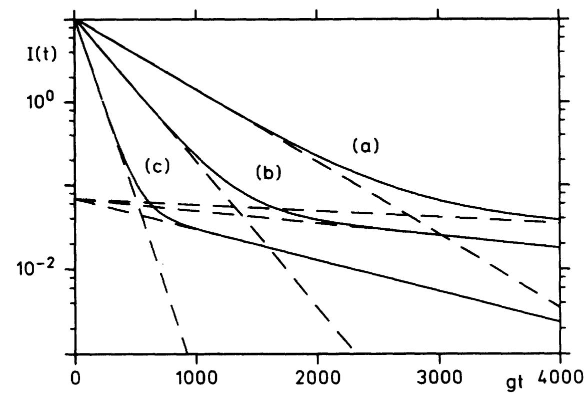

FIG. 5. Same as Fig. 4 but with $\Delta/g = 3$.

$$R(\infty) = \lim_{t\to\infty}[1 + D(t)]/[2I(t)] = (2 - r)/r\,. \tag{7.7}$$

For vanishing detuning this ratio $R(\infty)$ is equal to 1. The small decay of the intensity for large times may thus be interpreted in the following way. The photon probability in the two-level atom and in the field mode is the same. Because the two-level atom is not damped, no decay is possible of the photon energy stored in the atom. Therefore the decay rate is half of the decay rate of the empty cavity. A similar effect of this line narrowing has been discussed by Carmichael *et al.* [25]. In that reference, however, only the atomic system was damped. For appreciable detuning $R(\infty)$ is much larger leading even to a much smaller decay of the intensity which is demonstrated in Fig. 5.

VIII. LIMIT OF VANISHING DAMPING

Using the initial condition from Sec. VIII it is shown in this section that the usual expressions for the intensity and for the inversion are obtained for vanishing damping. For $\kappa = 0$ the eigenvalues in (5.7) and (5.8) take the form

$$\begin{aligned} \lambda^0 &= 0, \\ \lambda_m^{(1,2)} &= 0, \\ \lambda_m^{(3,4)} &= \pm i\omega_m\,, \end{aligned} \tag{8.1}$$

with

$$\omega_m = \sqrt{\Delta^2 + 4g^2(m+1)}\,. \tag{8.2}$$

Because $\hat{H}_0(z)$ is an analytical function of κ it is sufficient to calculate $H_{0,\nu}^{(3,4)}$. The contribution of the zero eigenvalues is just a constant which is fixed by the initial condition. Evaluating Eq. (5.16) leads to

$$H_{0,\nu}^{(3,4)} = \frac{1}{\pi\tilde{I}}\frac{2g^2(\nu+1)}{\Delta^2 + 4g^2(\nu+1)}\frac{(I_0)^\nu}{\nu!}e^{-I_0}\,, \tag{8.3}$$

thus the inversion is given by the well-known expression

$$D(t) = \sum_{\nu=0}^{\infty}\frac{(I_0)^\nu}{\nu!}e^{-I_0}\frac{\Delta^2 + 4g^2(\nu+1)\cos\omega_\nu t}{\Delta^2 + 4g^2(\nu+1)}\,. \tag{8.4}$$

From (6.3) and (6.4) it follows that

$$I_{0,\nu}^{(3,4)} = -\tfrac{1}{2} D_{0,\nu}^{(3,4)}, \tag{8.5}$$

and from (6.1) and (6.2) we obtain

$$I(t) + \tfrac{1}{2} D(t) = I_0 + \tfrac{1}{2}, \tag{8.6}$$

where we have used the initial condition $I(0) = I_0$ and $D(0) = 1$.

Comparing our result with the approximation of Barnett and Knight, i.e., Eqs. (11) and (12) of [12], one can see that their expression for the inversion is similar to our expression for the special case of vanishing detuning ($\Delta = 0$). The real eigenvalues and the real parts of the complex eigenvalues are the same. Their imaginary parts of the complex eigenvalues do not depend on κ whereas our exact ones depend on κ according to

$$\omega_m(\Delta = 0) = \sqrt{4g^2(m+1) - \kappa^2}, \tag{8.7}$$

see (5.7) for $\Delta = 0$. (To be precise this is only true for $\kappa < 2g$; for $\kappa \geq 2g$ some of the former complex eigenvalues even become real.) Thus only for vanishing damping ($\kappa = 0$) we obtain their frequencies of the Rabi oscillations, which are identical to (8.2) for $\Delta = 0$. Furthermore, their weights of both the real and the complex eigenvalues are only identical to our weights in the limit $\kappa \to 0$. Thus their result is an extension of (8.4) for small damping constants for the special case of vanishing detuning. In their expressions some terms appear as in our exact expressions whereas others, as the frequency shift of the Rabi oscillations induced by κ, are not taken into account.

IX. SUMMARY

We have found an analytic expression for the inversion and the intensity of the Jaynes-Cummings model, including cavity damping for vanishing thermal quanta. In solving the Markovian reduced density operator equation of motion for the system no further approximations besides the usual rotating-wave approximation have been made.

The essential steps have been the following ones: Instead of using the density operator of the full system, we introduced the four atomic matrix elements of the density operator. These matrix elements are still operators with respect to the field. The equations of motions of these four operators are transformed into a c-number representation using the quasiprobability distributions of Cahill and Glauber. By expanding the four quasiprobability distributions into Laguerre polynomials and into a Fourier series we obtained a system of ordinary differential equations for the expansion coefficients $c_{n,m}^{(i)}$. This system has the advantage that it is decoupled in the Fourier index n and it is only tridiagonal coupled in the other index m. In order to express the inversion and the intensity only the system with the index $n = 0$ needs to be taken into account. By a suitable choice of the scaling parameter $\tilde{I}$ and by a proper elimination procedure we could express the Laplace transform of the solution in terms of a quantity $\hat{H}_m$, which is only coupled to $\hat{H}_{m+1}$. This coupling makes it possible to get an analytic expression for the Laplace transform of the inversion and intensity. Performing a partial fraction decomposition of $\hat{H}_0$ all the eigenvalues and their weights could be determined analytically leading finally to an analytic solution for inversion and intensity.

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