Rydberg-atom phase-sensitive detection and the quantum Zeno effect

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We present a scheme for experimentally observing the quantum Zeno effect using the quantum-nondemolition measurement recently proposed by Brune *et al.* [Phys. Rev. Lett. **65**, 976 (1990)]. The Zeno effect refers to the freezing of the (unitary) free dynamics of a system by rapid measurements. We generalize the Zeno effect to be any change in the survival probability of an initial state induced by very rapid measurements, when such measurements are the dominant source of fluctuations in the system. We derive a master equation for the evolution of a cavity mode when the photon number is monitored by this method. This equation describes a phase-diffusion process. We propose that this measurement scheme be used to monitor the exchange of a single photon between the cavity and a single Rydberg atom. We show that for very rapid monitoring the free oscillation of the atomic inversion is disrupted and the atom can be trapped close to the initial excited state. This is the quantum Zeno effect.

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I. INTRODUCTION

It has been known for some time that the unitary evolution of a quantum system with a discrete spectrum may be considerably disrupted when the system is subjected to measurement. In particular, a two-level system undergoing coherent oscillation between each of the two states, subjected to a sequence of instantaneous perfectly accurate (projective) measurements, can be frozen in an initially occupied state [1–3]. While this result is no doubt correct it assumes a highly idealized type of measurement. Real measurements are not instantaneous nor perfectly accurate. In Ref. [3] one of us showed that the effect of a sequence of inaccurate instantaneous measurements could be described by a Markov master equation for the system state. In an appropriate limit defining an efficient measurement the initial-state occupation probability decays exponentially linear in time with a very slow rate. In this paper we discuss a realistic scheme which is well modeled by this approach.

To date there has only been one attempt to demonstrate the Zeno effect experimentally [4]. Recently a very interesting photon-number quantum-nondemolition measurement was proposed by Brune *et al.* [5], based on the interaction between a Rydberg-atom transition detuned from a microwave cavity. This scheme opens up a more direct test of the Zeno effect in a two-level system with dynamics dominated by coherent oscillation (i.e., there is no need to consider spontaneous emission). The general scheme is as follows. A single two-level Rydberg-atom is placed in a high-Q microwave cavity resonant with the atomic transition. If the atom is prepared initially in the excited state the evolution of the atom-cavity system will consist in a coherent oscillation between two states corresponding to the atom initially excited and no photons in the field, and the atom in the ground state and one photon in the field. The initial state is monitored by a quantum-nondemolition (QND) measurement of the photon number in the cavity. The QND scheme is the Rydberg-atom phase-sensitive detection scheme proposed by Brune *et al.* As we shall show when the measurement is operating to extract the maximum amount of information at the greatest rate the coherent oscillation in the two-state system is suppressed and the evolution is dominated by a very slow decay of the initial state.

II. PHOTON-NUMBER QND MEASUREMENTS USING RYDBERG ATOMS

We now describe in some detail a photon-number QND scheme essentially equivalent to the scheme of Brune *et al.* As depicted in Fig. 1, a beam of Rydberg atoms passes into a microwave cavity of resonant frequency ω_c. The level structure of the atoms is also indicated in Fig. 1. The $|2\rangle \leftrightarrow |3\rangle$ transition is coupled to the cavity field but is widely detuned. This ensures that there is no absorption of photons from the cavity due to this

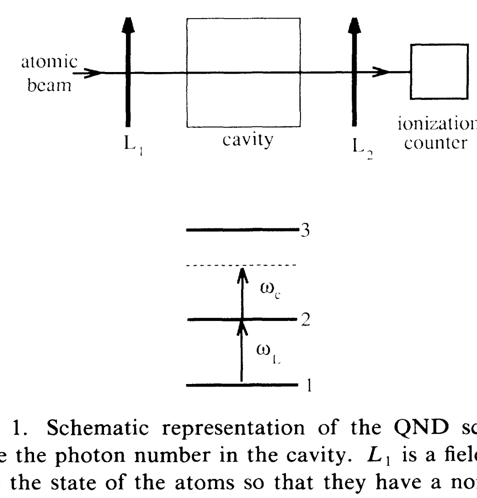

FIG. 1. Schematic representation of the QND scheme to measure the photon number in the cavity. L_1 is a field used to prepare the state of the atoms so that they have a nonzero dipole on entering the cavity, while L_2 ensures that the final ionization count will give information on the photon number in the cavity.

transition. Prior to entering the cavity the Rydberg atoms pass through an intense field L_1 resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition. This field is to prepare the atoms in a superposition of states $|1\rangle$ and $|2\rangle$. After leaving the cavity the atoms pass through a similar intense field $L_2 \pi/2$ out of phase with the first field, and then into an ionization counter which determines whether the atom is in state $|2\rangle$. In the scheme of Brune *et al.* the fields L_1, L_2 form a Ramsey fringe experiment; however, we will not view it in quite this way.

The interaction with the fields L_1, L_2 may be described in terms of a rotation of the Bloch vector representing the inversion and polarization of the $|1\rangle \leftrightarrow |2\rangle$ transition. The inversion and polarization operators are

$$J_z = \tfrac{1}{2}(|2\rangle\langle 2| - |1\rangle\langle 1|) , \tag{2.1}$$

$$J_y = -i\tfrac{1}{2}(|2\rangle\langle 1| - |1\rangle\langle 2|) , \tag{2.2}$$

$$J_x = \tfrac{1}{2}(|2\rangle\langle 1| + |1\rangle\langle 2|) . \tag{2.3}$$

The interaction with the fields L_1, L_2 are then described the unitary operators

$$R_1(\phi_1) = e^{-i\phi_1 J_x} , \tag{2.4}$$

$$R_2(\phi_2) = e^{-i\phi_2 J_y} . \tag{2.5}$$

The phase of precession ϕ_j is proportional to the product of the dipole moment for this transition, the interaction time, and the total field strength. We will assume that all atoms entering the field L_1 are in the ground state $|1\rangle$. Thus after passing through the first field the state of an atom is

$$|\psi_A\rangle = \cos(\phi_1/2)|1\rangle - i\sin(\phi_1/2)|2\rangle . \tag{2.6}$$

This first field may be regarded as a state preparation step for the probe atoms.

Inside the cavity sufficiently far from resonance, the interaction is described by the Hamiltonian [6]

$$H_I = \hbar\chi\sigma_z^{23} a^\dagger a , \tag{2.7}$$

where

$$\sigma_z^{23} = \tfrac{1}{2}(|3\rangle\langle 3| - |2\rangle\langle 2|) \tag{2.8}$$

and $a^\dagger a$ is the photon number operator for the intracavity field. Clearly $a^\dagger a$ is a QND variable for the cavity field, and H_I represents a back-action evasion coupling.

To see how the QND measurement works consider the Heisenberg equations of motion for the dipole moment operators on the $|1\rangle \leftrightarrow |2\rangle$ transition, Eqs. (2.2) and (2.3),

$$\frac{dJ_x}{dt} = \frac{\chi}{2} J_y a^\dagger a , \tag{2.9}$$

$$\frac{dJ_y}{dt} = -\frac{\chi}{2} J_x a^\dagger a . \tag{2.10}$$

The solutions for an interaction time τ are

$$J_x(\tau) = \cos(\theta a^\dagger a) J_x(0) + \sin(\theta a^\dagger a) J_y(0) , \tag{2.11}$$

$$J_y(\tau) = \cos(\theta a^\dagger a) J_y(0) - \sin(\theta a^\dagger a) J_x(0) , \tag{2.12}$$

where $\theta = \chi\tau/2$. Thus a measurement of J_x or J_y will yield information on $a^\dagger a$, provided the state of the atom is such that there is a nonzero dipole on the transition $|1\rangle \leftrightarrow |2\rangle$ (i.e., $\langle J_x\rangle, \langle J_y\rangle \neq 0$). The purpose of the initial field is to ensure that this is the case. Unfortunately the ionization counter at the output of the experiment effectively measures J_z and not the dipole moment operators. The purpose of the second field is to rotate the information in the dipole into a component in the inversion.

If we view the experiment as a whole (Fig. 1), it effectively transforms the input Bloch vector components of the $|1\rangle \leftrightarrow |2\rangle$ transition into output components. The final measurement is made on the z component of the output. The total transformation of the z component is

$$\begin{aligned} J_z^0 = {} & [\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2 \sin(\theta a^\dagger a)] J_z^i \\ & - \sin\phi_2 \cos(\theta a^\dagger a) J_x^i \\ & + [\sin\phi_1 \cos\phi_2 - \sin\phi_2 \cos\phi_1 \sin(\theta a^\dagger a)] J_y^i . \end{aligned} \tag{2.13}$$

Note that if the atom is not first prepared in a superposition state by the first laser field, $\phi_1 = 0$, and $\langle J_y^i\rangle = 0$ so no information on $a^\dagger a$ is obtained, i.e., no measurement has taken place. If we now take $\phi_1 = \phi_2 = \pi/2$,

$$J_z^0 = \sin(\theta a^\dagger a) J_z^i - \cos(\theta a^\dagger a) J_x^i . \tag{2.14}$$

Effectively with this choice of phase the transformation is a precession about the y axis of the Bloch sphere, a result easily confirmed by a geometric representation of each rotation.

The mean signal at the detector is

$$\langle J_z^0\rangle = -\tfrac{1}{2}\langle \sin(\theta a^\dagger a)\rangle . \tag{2.15}$$

If θ is small and only low photon numbers are excited in the cavity,

$$\langle J_z^0\rangle \approx -\frac{\theta}{2}\langle a^\dagger a\rangle . \tag{2.16}$$

This is clearly a measurement of the photon number in the cavity.

III. INTRACAVITY DYNAMICS

We now determine the evolution equation for the cavity field state and show that the measurement leads to a rapid diagonalization of the state of the field in the number basis; that is, the state is reduced as a result of the measurement. The state of each atom entering the cavity is the state after the field L_1,

$$|\psi_A\rangle = c_1|1\rangle + c_2|2\rangle , \tag{3.1}$$

where $c_1 = \cos(\phi_1/2)$, $c_2 = -i\sin(\phi_1/2)$. The change in the state of the field due to the interaction of a single atom for a time τ is

$$\begin{aligned} \rho' &= \Phi_\tau \rho \\ &\equiv \mathrm{tr}_A[U(\tau)\rho \otimes |\psi_A\rangle\langle\psi_A| U^\dagger(\tau)] \qquad (3.2) \\ &= |c_1|^2\rho + |c_2|^2 e^{i\theta a^\dagger a}\rho e^{-i\theta a^\dagger a} . \qquad (3.3) \end{aligned}$$

In this equation $U(\tau)$ is the time-evolution operator following from the interaction Hamiltonian in Eq. (2.7).

The transformation defined by Eq. (3.3) is nonunitary. The resulting state is a statistical mixture of a field which has undergone a phase jump of θ and a state which is left unchanged. The probability that a given probe atom will induce a phase change is $|c_2|^2$. Note that the phase jump, if it occurs, is always of the same size and direction. The nonunitarity arises because we do not know whether a given probe atom has induced a phase jump or not.

We first consider the rather idealized case where θ does not vary from atom to atom and all the atoms arrive in the cavity at equally spaced time intervals. In this case Eq. (3.3) can be iterated N times to give

$$\rho^{(N)}=\sum_{r=0}^{N}\begin{bmatrix}N\\ r\end{bmatrix}|c_2|^{2r}|c_1|^{2(N-r)}e^{ir\theta a^\dagger a}\rho^{(0)}e^{-ir\theta a^\dagger a} \tag{3.4}$$

(we ignore for the moment the free evolution between measurements). To understand the effect of the measurements on the field consider the matrix elements of $\rho^{(N)}$ in the number basis

$$\begin{aligned}\rho_{p,q}^{(N)}&=\langle p|\rho^{(N)}|q\rangle\\ &=\rho_{p,q}^{(0)}e^{i(\theta/2)(p-q)N}(|c_1|^2e^{-i(\theta/2)(p-q)}\\ &\qquad+|c_2|^2e^{i(\theta/2)(p-q)})^N .\end{aligned} \tag{3.5}$$

In the case $|c_1|=|c_2|=1/\sqrt{2}$, we find

$$\rho_{p,q}^{(N)}=\rho_{p,q}^{(0)}e^{i(\theta/2)(p-q)N}\left[\cos\left[\frac{\theta}{2}(p-q)\right]\right]^N . \tag{3.6}$$

We now consider a continuous regular limit in which θ is small but N is large. (That is, each measurement is not very effective but we make very many to compensate.) To second order in θ we find

$$\rho_{p,q}^{(N)}\approx\rho_{p,q}^{(0)}e^{i(\theta/2)(p-q)N}\exp\left[\frac{-N\theta^2}{8}(p-q)^2\right] . \tag{3.7}$$

Clearly there is a decay of coherence in the number basis, a result to be expected for a scheme designed to measure the number. If we assume that the time between kicks is T then $N=t/T$ and

$$\rho_{p,q}^{(N)}\approx\rho_{p,q}^{(0)}e^{i(\theta/2)(p-q)\gamma t}\exp\left[\frac{-\gamma t\theta^2}{8}(p-q)^2\right] , \tag{3.8}$$

where $\gamma=T^{-1}$ is the frequency of the measurements, effectively the bandwidth of the measurement. In this form the decay of coherence is typical of the decay of coherence in continuous measurement models [7]. Clearly a good measurement corresponds to $\gamma\theta^2$ large.

Away from the continuous limit coherence decay is more complicated. In fact for finite θ there are certain coherences which do not decay. For example, if $\theta=\pi/2$ coherence between states $|p\rangle$ and $|q\rangle$ will not decay whenever $p-q=4n$ for n an integer. Even so if N is large $\rho_{p,q}^{(N)}$ decreases very quickly away from these special values.

The source of this coherence decay is the need to prepare the initial state of the atom to be a superposition of the ground and excited states, as noted in Sec. II. This is easily seen in Eq. (3.5) by setting either c_1 or c_2 to zero. The field then sees a random sequence of excited and unexcited atoms which leads to a stochastic variation of the refractive index inside the cavity. The need to prepare the atom in a state suitable for a measurement to take place leads directly to state reduction in the energy basis. This is a totally unavoidable nonunitary effect of the measurement.

However, other sources of nonunitarity may also play a role. For example, the atoms may arrive in the cavity at random times rather than at regular time intervals as assumed above. In the case of Poisson-distributed arrival times an evolution equation may be derived for the cavity field [3],

$$\begin{aligned}\left[\frac{d\rho}{dt}\right]_{\text{meas}}&=\gamma(\Phi_\tau\rho-\rho)\\ &=\gamma|c_2|^2(e^{i\theta a^\dagger a}\rho e^{-i\theta a^\dagger a}-\rho) .\end{aligned} \tag{3.9}$$

In addition the interaction time τ may vary from atom to atom due to the velocity profile. In this case we must average over a distribution for θ. Assuming a Gaussian distribution of mean $\bar{\theta}$ and a variance of Δ, such that $\bar{\theta}\gg\sqrt{\Delta}$, then to first order in the variance

$$\begin{aligned}\left[\frac{d\rho}{dt}\right]_{\text{meas}}&=\gamma|c_2|^2(e^{i\bar{\theta}a^\dagger a}\rho e^{-i\bar{\theta}a^\dagger a}-\rho)\\ &\quad-\Gamma_\Delta[a^\dagger a,[a^\dagger a,e^{i\bar{\theta}a^\dagger a}\rho e^{-i\bar{\theta}a^\dagger a}]] ,\end{aligned} \tag{3.10}$$

where

$$\Gamma_\Delta=\frac{\Delta\gamma|c_2|^2}{2} . \tag{3.11}$$

If the average phase shift is small and there are few photons in the cavity we may write

$$\left[\frac{d\rho}{dt}\right]_{\text{meas}}=i\delta[a^\dagger a,\rho]-\Gamma_\Delta[a^\dagger a,[a^\dagger a,\rho]] , \tag{3.12}$$

where

$$\delta=\gamma|c_2|^2\bar{\theta} \tag{3.13}$$

is the linear deterministic phase shift. The effect of the first term in Eq. (3.12) is to induce a fixed detuning of the cavity field from the empty cavity frequency. This deterministic phase shift is the average phase shift induced by a beam of probe atoms which enter the cavity at random, Poisson-distributed, times. The second term leads to a diffusion in the phase of the cavity field. More importantly this last term also causes the density operator for the field to become diagonal in the photon number basis, a result to be expected for a measurement of the photon number. A good measurement thus corresponds to large Γ, as in the regularly measured case.

IV. CONDITIONAL STATE OF THE CAVITY

Given that an atom is detected in state $|2\rangle$, what is the state of the cavity field conditioned on this result? In this section we give the answer to this question, which is needed to simulate the evolution of a particular measurement sequence.

The probability to detect an atom in the state $|2\rangle$ after passing through the final field L_2 is

$$P_2 = \mathrm{tr}_F(\Phi_\tau^{(2)}\rho) , \tag{4.1}$$

where

$$\Phi_\tau^{(2)}\rho = \mathrm{tr}_A[|2\rangle\langle 2|R_2(\phi_2)U(\tau)\rho\otimes\rho_A U^\dagger(\tau)R_2^\dagger(\phi_2)] \tag{4.2}$$

$$= \langle 2|R_2(\phi_2)U(\tau)\rho\otimes\rho_A U^\dagger(\tau)R_2^\dagger(\phi_2)|2\rangle \tag{4.3}$$

and where ρ_A is the state of the atom after passing through L_1 [given in Eq. (3.1)]. The state transformations in the final rotation of the Bloch vector are

$$R_2(\phi_2)|1\rangle = d_1|1\rangle - d_2|2\rangle , \tag{4.4}$$

$$R_2(\phi_2)|2\rangle = d_2|1\rangle + d_1|2\rangle , \tag{4.5}$$

where $d_1 = \cos(\phi_2/2)$ and $d_2 = \sin(\phi_2/2)$. Using Eqs. (3.1), (4.3), (4.4), and (4.5) we find

$$\Phi_\tau^{(2)}\rho = |c_1|^2|d_1|^2\rho + |c_2|^2|d_2|^2 e^{i\theta a^\dagger a}\rho e^{-i\theta a^\dagger a} - c_1c_2^* d_2 d_1^* \rho e^{-i\theta a^\dagger a} - c_1^* c_2 d_1^* d_2 e^{i\theta a^\dagger a}\rho . \tag{4.6}$$

Thus

$$P_2 = |c_1|^2|d_1|^2 + |c_2|^2|d_2|^2 - (c_1c_2^* d_2 d_1^* \langle e^{-i\theta a^\dagger a}\rangle + \text{c.c.}) , \tag{4.7}$$

where c.c. denotes the complex conjugate. Note that if either c_1 or c_2 is zero, no information on the cavity phonon number is obtained. In the case of $\phi_1 = \phi_2 = \pi/2$ the coefficients are $c_1 = 1/\sqrt{2}$, $c_2 = -i/\sqrt{2}$, $d_1 = d_2 = 1/\sqrt{2}$, and

$$\Phi_\tau^{(2)}\rho = \tfrac{1}{4}(\rho + e^{i\theta a^\dagger a}\rho e^{-i\theta a^\dagger a} - i\rho e^{-i\theta a^\dagger a} + i e^{i\theta a^\dagger a}\rho) , \tag{4.8}$$

and thus

$$P_2 = \tfrac{1}{2} - \tfrac{1}{2}\langle \sin\theta a^\dagger a\rangle \tag{4.9}$$

as $\langle J_z^0\rangle = P_x - \frac{1}{2}$. This result agrees with the result in Eq. (2.15). If the field is in a coherent state $|\alpha\rangle$ we find

$$\langle \sin\theta a^\dagger a\rangle = \exp[-|\alpha|^2(1-\cos\theta)]\sin(|\alpha|^2\sin\theta) , \tag{4.10}$$

which for $\theta \ll 1$ becomes approximately $\sin(|\alpha|^2\theta)$, the semiclassical result.

The conditional sate of the field is then given by

$$\rho^{(2)} = (P_2)^{-1}\Phi_\tau^{(2)}\rho = \frac{\rho - i\rho e^{-i\theta a^\dagger a} + i e^{i\theta a^\dagger a}\rho + e^{i\theta a^\dagger a}\rho e^{-i\theta a^\dagger a}}{2(1-\langle \sin\theta a^\dagger a\rangle)} . \tag{4.11}$$

The reduced photon-number distribution in particular is

$$P^{(2)}(n) = \frac{P(n)(1-\sin\theta n)}{1-\langle \sin\theta a^\dagger a\rangle} . \tag{4.12}$$

This will have holes at values of n such that $n\theta = (\pi/2)(4m+1)$ for m an integer. In a long sequence of measurements the net effect of superposing different interference patterns corresponding to the stochastic sequence of results is that the reduced probability distribution converges to a single peak at a random value of n, as demonstrated in Brune *et al.* [5]. In fact for certain values of θ, conditional measurements can result in a photon-number distribution that is multiply peaked. This would require a beam for which the standard deviation of θ is much less than the mean. Atomic beams can be prepared with a velocity distribution width of 4% [8]. For completeness, the operation for the conditional state of the field given that the atom was not detected in state $|2\rangle$ is

$$\Phi_\tau^{(1)}\rho = \tfrac{1}{4}(\rho + e^{i\theta a^\dagger a}\rho e^{-i\theta a^\dagger a} + i\rho e^{-i\theta a^\dagger a} - i e^{i\theta a^\dagger a}\rho) . \tag{4.13}$$

Thus

$$P_1 = \tfrac{1}{2} + \tfrac{1}{2}\langle \sin\theta a^\dagger a\rangle . \tag{4.14}$$

If $\theta = \pi/2$ and there is one photon in the field we find that $P_2 = 0$ and $P_1 = 1$. That is, the atom will never be ionized from state $|2\rangle$, only from state $|1\rangle$ in the final ionization state readout. It then seems that the best way to detect a single photon would be to arrange to have $\theta = \pi/2$. This is of relevance to the quantum-Zeno-effect determination discussed in Sec. V.

Although in a particular sequence of measurements a random time distribution of ionization counts will be observed, the average ionization rates are easily calculated. The average ionization rates from state $|1\rangle, |2\rangle$ are given by

$$i_{1,2}(t) = \gamma P_{1,2} = \frac{\gamma}{2}(1 \pm \langle \sin\theta a^\dagger a\rangle) , \tag{4.15}$$

where γ is the atom injection rate.

V. A TEST OF THE ZENO EFFECT

We now assume that the microwave cavity initially contains a single two-level Rydberg atom, referred to as the object atom, with a transition frequency resonant with the cavity frequency. The excited state of this atom will be denoted $|e\rangle$ while the ground state is $|g\rangle$. We assume also that the atom is prepared initially in the excited state and that the cavity contains no photons. Under free evolution the quantum of energy is periodically exchanged between the object atom and the cavity field. By monitoring the photon number in the cavity we can monitor the evolution of the atom away from its initial excited state. We will discuss separately the two cases of regular probe injection and Poisson probe injection. We show in both cases that the measurement necessarily disrupts the free oscillation of the atom-field system.

A. Regular probe injection

In Sec. III we showed that the measurement necessarily introduces a minimum degree of nonunitary change in the field state, described by Eq. (3.3). This change is interpreted as a statistical mixture of a field which has undergone a phase change and a field which has not. In the case $|c_1|^2=|c_2|^2=0.5$ there is an equal probability for each of these events. Each probe atom has a 50% chance of causing a fixed phase change in the state of the field. In this section we show that this minimal level of nonunitarity induced by the measurement causes a change in the free evolution of the cavity–object-atom system which becomes more disruptive the greater the rate of measurement.

Under free evolution the object atom will periodically emit and absorb one photon, an entirely coherent process in the absence of spontaneous emission (we assume that the rate of spontaneous emission for this configuration is much smaller than the coherent oscillation frequency). The Hamiltonian for this interaction is

$$H_I=\frac{\hbar\kappa}{2}(a^\dagger\sigma_-+a\sigma_+)\ , \tag{5.1}$$

where $\sigma_\pm$ are the dipole raising and lowering operators for the $|e\rangle\leftrightarrow|g\rangle$ transition. The initial state is $|e\rangle\otimes|0\rangle_F$, i.e., the atom is in the excited state and the field is in the vacuum state. The state at any time lies within the subspace spanned by

$$|a\rangle=|e\rangle\otimes|0\rangle_F\ , \tag{5.2}$$

$$|b\rangle=|g\rangle\otimes|1\rangle_F\ . \tag{5.3}$$

In this basis the matrix elements of the density operator obey

$$\frac{d\mathbf{S}}{dt}=\mathbf{B}\times\mathbf{S}\ , \tag{5.4}$$

where

$$\mathbf{S}=(X,Y,Z)\ , \tag{5.5}$$

$$\mathbf{B}=(\kappa,0,0)\ , \tag{5.6}$$

and

$$X=\tfrac{1}{2}(\langle a|\rho|b\rangle+\langle b|\rho|a\rangle)\ , \tag{5.7}$$

$$Y=\frac{i}{2}(\langle b|\rho|a\rangle-\langle a|\rho|b\rangle)\ , \tag{5.8}$$

$$Z=\tfrac{1}{2}(\langle b|\rho|b\rangle-\langle a|\rho|a\rangle)\ . \tag{5.9}$$

This equation represents the precession of $\mathbf{S}$ around the direction defined by $\mathbf{B}$, i.e., the x axis.

We will assume that the interaction of the probe atom with the cavity field occurs on a much faster time scale than the coherent interaction between the cavity field and the object atom. This requires that the probe atoms pass through the cavity at a very high rate. This point is further discussed in the concluding paragraph. With this assumption the effect of the probe atom on the system may be modeled as a "kick," which periodically interrupts the free dynamics. The dynamics is then represented by a map for the vector $\mathbf{S}$, which comprises a free precession for a fixed time followed by a nonunitary jump as the probe atom passes through the cavity. To determine this map we first note that the interaction Hamiltonian may be written as

$$H_I=\hbar\kappa\sigma_x\ , \tag{5.10}$$

where the angular-momentum operators are defined with respect to the effective two-level system by

$$\sigma_x=\tfrac{1}{2}(|b\rangle\langle a|+|a\rangle\langle b|)\ , \tag{5.11}$$

$$\sigma_y=\frac{-i}{2}(|b\rangle\langle a|-|a\rangle\langle b|)\ , \tag{5.12}$$

$$\sigma_z=\tfrac{1}{2}(|b\rangle\langle b|-|a\rangle\langle a|)\ . \tag{5.13}$$

Thus the free dynamics is simply represented as a precession around the x axis by an angle $\phi=\kappa\tau$, where τ is the time between each probe atom. To determine the effect of a probe atom we note that the photon-number operator in the effective two-level system is formally identical to σ_z,

$$a^\dagger a\otimes I_A=\sigma_z\ , \tag{5.14}$$

where I_A is the identity operator for the object atom. (This result may be verified by checking the commutation relations.) A phase shift of θ in the field is thus represented by a precession of θ about the z axis.

The resulting map for the state of the system is

$$\rho_{n+1}=\tfrac{1}{2}(e^{-i\phi\sigma_x}\rho_n e^{i\phi\sigma_x}+e^{-i\theta\sigma_z}e^{-i\phi\sigma_x}\rho_n e^{i\phi\sigma_x}e^{i\theta\sigma_z})\ , \tag{5.15}$$

where θ is the phase change induced by a probe atom and $\phi=\kappa\tau$, where τ is the period of time between each probe atom passing through the cavity.

The state ρ_{n+1} is a statistical mixture of states, one of which undergoes a precession of ϕ around the x axis and one which undergoes a precession of ϕ around the x axis followed by a precession of θ around the z axis. Any given probe atom will induce one or the other of these processes with equal probability. The corresponding map for the vector $\mathbf{S}$ is

$$\mathbf{S}_{n+1}=\tfrac{1}{2}(R_1+R_2)\mathbf{S}_n\ , \tag{5.16}$$

where

$$R_1=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi & \cos\phi\end{bmatrix}\ , \tag{5.17}$$

$$R_2=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \cos\phi\sin\theta & \cos\phi\cos\theta & -\sin\phi\\ \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\phi\end{bmatrix}\ . \tag{5.18}$$

In the case of θ small the total map matrix may be written approximately as

$$R=\begin{bmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & 0\\ \cos\phi\sin\frac{\theta}{2} & \cos\phi\cos\frac{\theta}{2} & -\sin\phi\\ \sin\phi\sin\frac{\theta}{2} & \sin\phi\cos\frac{\theta}{2} & \cos\phi\end{bmatrix}\times\begin{bmatrix}e^{-\theta^2/8} & 0 & 0\\ 0 & e^{-\theta^2/8} & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{5.19}$$

which is correct to second order in θ only. In this form we see that the probe atoms on average induce an extra precession of $\theta/2$ about the z axis in addition to the unitary precession around the x axis, and they also induce a nonunitary decay of the "coherences" S_x and S_y at the rate $\theta^2/8$. This decay is of course the same as that given in Eq. (3.7). The measurement has both a systematic effect, as determined by $\theta/2$, and a random or diffusive effect determined by θ^2.

The probability to find the cavity–object-atom system in the initial state after n probe kicks is given by

$$p_{a,n}=\tfrac{1}{2}-S_{z,n} \; . \tag{5.20}$$

In Figs. 2 and 3 we plot this probability versus $t=n\tau$ for a number of cases. In Figs. 2(a)–2(c) we consider the case of fixed θ (i.e., fixed measurement strength), but vary the time between each probe atom. This shows the effect of increasing the rate of measurement. It is apparent that the oscillations in the probability p_a are suppressed as the rate of measurement increases, and the system remains longer near the initial state. The measurement thus disrupts the free evolution as the measurement rate increases. This is what is meant by the Zeno effect. In the explicit model of this paper we are able to trace the origin of this effect to the physical effect of the measuring apparatus (the probe atoms) on the measured system (the object-atom–field system). We can distinguish two physical explanations for the effect in the model. The first is the average extra rotation induced by the probe atoms, which has the effect of tilting the precession direction ways from the x axis as described by the first factor in Eq. (5.19). This causes a frequency shift in the evolution of p_a that is apparent in comparing Figs. 2(a)–2(c). This is more evident in Fig. 3 where we have considered a "weak-coupling" limit in which θ is small but the product of θ and the measurement rate is greater than one. In this case the effective detuning dominates the evolution of the survival probability, although a very slow decay is just discernible. This limit will be discussed in more detail in Sec. V D.

The second explanation is the destruction of coherence, or the decay of polarization, reflected in the second factor in Eq. (5.19). This decay is due to the fact that a given probe atom may or may not cause a change in the phase of the field in the cavity, with each event equiprobable. It may of course be possible, in some experiments, to eliminate the systematic effect of the measurement, perhaps by arranging for θ to fluctuate between a positive and negative value rather than between a positive value and zero as in this paper, but nothing can be done about the fluctuations induced by the measurement.

It is always possible to view the effect of measurement in terms of an ensemble dynamics, as above, or as a stochastic trajectory conditioned on the history of what each particular probe atom actually does. In the model of this paper the actual phase change induced by a probe atom determines the measured state of the probe atom after it has exited the cavity and is thus known (in principle at least) to the experimenter. This gives another way to view the dynamics of the vector $\mathbf{S}$ as the measurement proceeds by directly simulating the random process represented by the passage of the probe atoms. In this case we toss a coin to determine whether to apply R_1 or R_2 at each step of the map. The results of such a simulation are shown in Fig. 4. It is now quite apparent that the measurement causes random-phase disturbances to

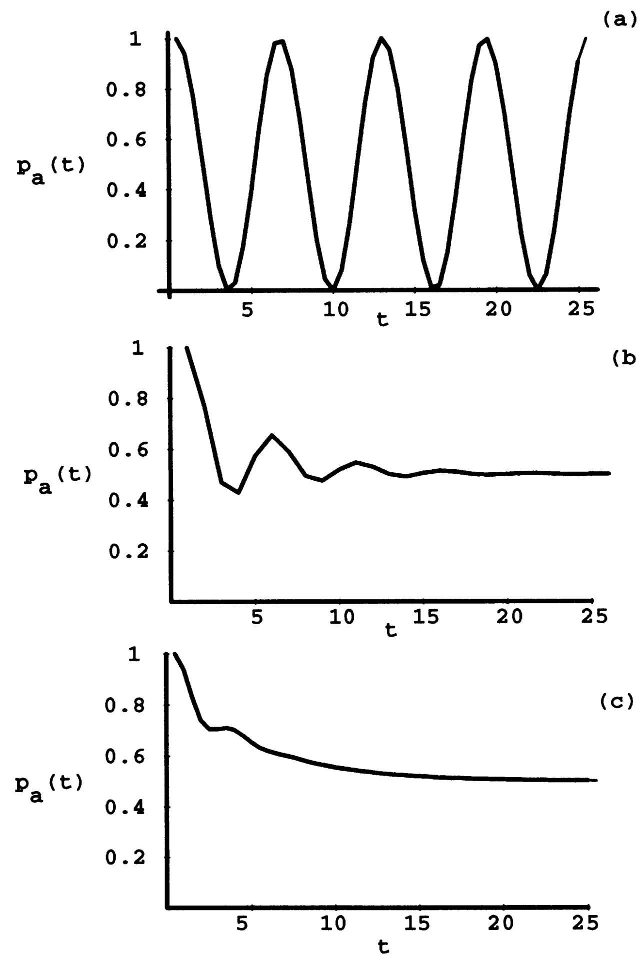

FIG. 2. A plot of the survival probability $p_a(t)$ vs time for a regular injection probe. The effect of increasing the measurement rate is shown in (a) through to (c). The time t is defined in terms of the number of injections by $t=n\phi$, where n is the number of injected probe atoms to that point and ϕ is the time of interaction of the probe atom in units of the one-photon Rabi period (κ^{-1}). (a) $\theta=0$ (no measurement), (b) $\theta=\pi/2$, $\phi=1.0$, (c) $\theta=\pi/2$, $\phi=0.5$.

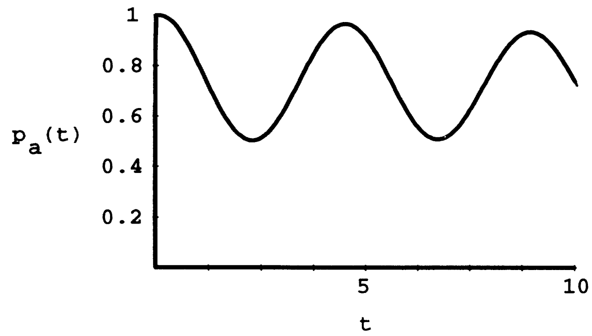

FIG. 3. The survival probability $p_a(t)$ is plotted vs time for the regular probe injection case. $\theta=0.05$, $\phi=0.05$. The effective measurement is 20.

the evolution, much in the way collisions disrupt the polarization dynamics for atomic transitions. Of course we also know that there is a systematic shift in the precession axis but this is hard to see in a single trajectory.

B. Poisson distributed probe injection

We now assume that the probe atoms arrive in the cavity at Poisson-distributed time intervals. This is probably a more realistic assumption than the case of regular arrival times discussed above. However, the general picture of the dynamics is not changed much. One considerable advantage of the Poisson model is that it enables one to write down an explicit evolution equation for the state (or vector **S**), rather than a map. The interaction time of the probe atom in the cavity remains fixed.

The general form of the evolution equation for a system subjected to Poisson distributed kicks is [9]

$$\frac{d\rho(t)}{dt}=\frac{-i}{\hbar}[H_0,\rho(t)]+\gamma[\mathcal{K}\rho(t)-\rho(t)] , \tag{5.21}$$

where $\mathcal{K}$ is the operation describing the effect of the kick on the density operator and γ is the average injection rate. In the case of this paper $\mathcal{K}$ is defined by Eq. (3.3). With $|c_1|^2=|c_2|^2=0.5$ the evolution equation becomes

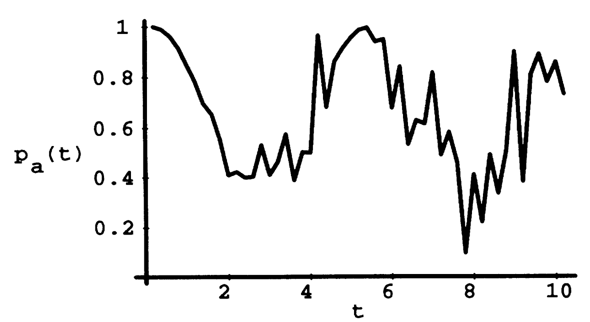

FIG. 4. A stochastic simulation of the survival probability $p_a(t)$ when account is taken of each probe atom. At each step there is a probability of 0.5 that a probe atom will cause a phase shift in the field, or do nothing. $\theta=0.3$, $\phi=0.2$. The effective measurement rate is 5.

$$\frac{d\rho}{dt}=-i\kappa[\sigma_x,\rho]+\frac{\gamma}{2}(e^{-i\theta\sigma_z}\rho e^{i\theta\sigma_z}-\rho) . \tag{5.22}$$

This corresponds to a sequence of Poisson distributed rotations around the z axis at half the injection rate γ; not a surprising result as, on average, only half the atoms are effective in producing a phase change in the field.

The resulting dynamics for the vector **S** with time measured in units of κ, is given by

$$\frac{d}{dt}\begin{bmatrix}X\\Y\\Z\end{bmatrix}=\begin{bmatrix}-A & B & 0\\ -B & -A & -1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}X\\Y\\Z\end{bmatrix} , \tag{5.23}$$

where

$$A=\beta\mu , \tag{5.24}$$

$$B=(\beta A-A^2)^{1/2} , \tag{5.25}$$

and we have defined the dimensionless parameters

$$\mu=\sin^2\frac{\theta}{2} , \tag{5.26}$$

$$\beta=\frac{\gamma}{\kappa} . \tag{5.27}$$

It is clear from Eq. (5.23) that the measurement has detuned the object atom from resonance with the cavity. The precession direction is now $\mathbf{B}=(\kappa,0,-B)$. This detuning is due to the systematic phase shift averaged over a large number of probe atoms. Indeed for θ small the detuning is $\Delta=\gamma\theta/2$, which is consistent with Eq. (5.19). Furthermore, there is a decay of the x and y components which, for small θ, goes as $\gamma\theta^2/4$, again consistent with Eq. (5.19).

C. Weak-measurement limit, $\beta\ll 1$

In this case the rate of probe injection is much less than the time scale of coherent evolution of the system away from the initial state. The eigenvalues of the dynamics are

$$\lambda_1=-\mu\beta , \tag{5.28}$$

$$\lambda_{2,3}=-\frac{\mu\beta}{2}\pm i . \tag{5.29}$$

The survival probability exhibits a slowly damped oscillation, see Fig. 5(a). In the case of this limit occurring with θ small, the eigenvalues, in dimensioned units, are $\lambda_1=-\gamma\theta^2/4$, $\lambda_{2,3}=-\gamma\theta^2/8\pm i\kappa$, which are consistent with the regular injection result in the same limit. This behavior is evident in the small-β region of Fig. 5(b).

D. Strong-measurement–weak-coupling limit, $\beta\rightarrow\infty$, $\mu\rightarrow 0$, $\mu\beta=\Delta$

In this limit we consider the situation in which the probe atom shifts the frequency by a very small amount at each injection, but there are many probe injections. Furthermore, the average detuning induced is fixed at $\Delta<1$. The eigenvalues are given by

$$\lambda_1 = 0.0 \ , \tag{5.30}$$

$$\lambda_{2,3} = \pm i\sqrt{\beta\mu} \ . \tag{5.31}$$

In this case the survival probability exhibits small oscillations near unity and decays on a very slow time scale. Thus the system remains close to the initial state for long times. This behavior is shown in Fig. 5(b) in the large-β region.

The special case of $\theta = \pi$ ($\mu = 1.0$) is worthy of comment. Inspection of Eq. (4.15) shows that this case does not correspond to a measurement of $a^\dagger a$ in the cavity, and thus cannot be used to monitor the evolution away from the initial state. However, the effect on the evolution on the system is still quite considerable. In fact the eigenvalues of the dynamics are

$$\lambda_1 = -\beta \ , \tag{5.32}$$

$$\lambda_{2,3} = -\frac{\beta}{2} \pm i \left[1 - \frac{\beta^2}{4} \right]^{1/2} \ . \tag{5.33}$$

The dynamics are characterized by a transition from underdamped to overdamped motion in the case of $\beta > 2$. While no measurement can be said to have taken place the effect of the probe atoms on the effective two-level dynamics is equivalent to pure phase decay. Indeed the sequence of probe atoms is equivalent to a random sequence of phase reversals in the two-level density matrix.

Is the strong-measurement limit $\beta \gg 1$ achievable? The answer is yes but it necessarily requires that θ be small. The size of γ is immediately given by the requirement that we have only one atom in the cavity at a time. If the length of the cavity is L_c and the velocity of the atoms is v then $\gamma = v/L_c$. We thus require an object atom with single-atom Rabi frequency κ such that $\kappa \ll v/L_c$. However, θ is proportional to the time of flight of the atom through the cavity, which cannot be greater than γ^{-1} without violating the assumption that there is only one atom in the cavity at a time. Increasing γ will necessarily decrease θ. In other words, the strong-measurement limit necessarily implies the weak-coupling regime. Typically single-photon Rabi frequencies for Rydberg atoms are in the range [10] $10^4 < \kappa < 10^5$ s^{-1}. Thus we require $\gamma > 10^5$ s^{-1}. For a 1-cm-length cavity this means a velocity of the order of 1000 m/s. The velocity of probe atoms is limited by the source. It could possibly be increased by reversing a laser-cooling scheme, i.e., propagating a resonant laser field along the beam of probe atoms. For example, using a scheme of two slightly detuned counterpropagating waves [11], velocities of the order of 1000 m/s could be achieved for sodium.

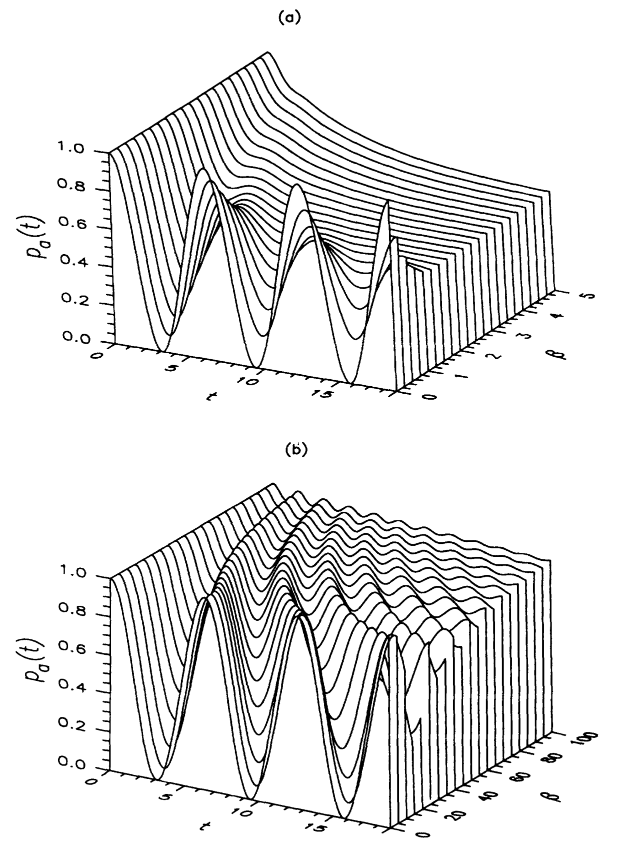

FIG. 5. A plot of the survival probability vs time for the case of Poisson injected probe atoms, with different injection rates. (a) $\theta = \pi/2$, (b) $\theta = 0.01$.

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