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Kink solitons and optical shocks in dispersive nonlinear media

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The generalized nonlinear Schrödinger equation governs wave propagation in nonlinear dispersive media by including the effects of group-velocity dispersion, self-phase-modulation, stimulated Raman scattering, and self-steepening. This equation is shown to have solitary-wave solutions that correspond to an optical shock front moving at the group velocity. The properties of such kink-type solitary waves are discussed.

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I. INTRODUCTION

Wave propagation in dispersive nonlinear media has attracted considerable attention in recent years, in part motivated by its application to optical fibers [1–6]. Propagation of short optical pulses in optical fibers is governed by the well-known nonlinear Schrödinger equation (NSE) [1,2]. This equation can be solved exactly by the inverse scattering method [7]. Under certain conditions, an optical pulse can propagate as a fundamental soliton that remains unchanged during propagation as long as optical losses remain negligible. Pulselike solitons exist only when the group-velocity dispersion (GVD) is anomalous. In the case of normal GVD, the NSE has soliton solutions which appear in the form of a dip against a uniform background [7]. Such solitons are called dark solitons [1,2].

The NSE describes pulse propagation in optical fibers accurately only when the pulse width T_0 is relatively large ($T_0 \gg 1$ ps). Several higher-order effects become important for femtosecond optical pulses. Their evolution is governed by generalized NSE that includes these higher-order effects through additional terms which represent the effects of third-order dispersion, self-steepening, and intrapulse stimulated Raman scattering (SRS). Attempts have been made to find the solitary-wave solutions of the generalized NSE under varying degrees of approximations [8–12]. The inclusion of the SRS term in the generalized NSE appears to preclude the existence of solitary waves of both bright and dark types. Their absence can be attributed to the non-energy-conserving nature of the SRS term. This paper shows that the generalized NSE has solitary-wave solutions even when the SRS term is included, but these solutions correspond to a different class of solitons known as kink solitons [4]. Although kink solutions occur in many branches of physics (e.g., particle physics and solid-state physics), their occurrence in nonlinear optics is relatively rare [13,14]. In the context of nonlinear optics, a kink soliton represents a shock front that propagates undistorted inside the dispersive nonlinear medium.

This paper is organized as follows. The generalized NSE is given in Sec. II. This equation is normalized and rewritten in a form that is convenient for examining solitary-wave solutions. In Sec. III a solitary-wave solution to the generalized NSE is obtained by including intrapulse SRS, but neglecting third-order dispersion and self-steepening. The properties of this kink-type solitary wave are discussed and their implications for pulse propagation in optical fibers are explored. The results of Sec. III are expanded to include the effects of self-steepening in Sec. IV. Finally, the results are summarized in Sec. V.

II. THE GENERALIZED NSE

The properties of the NSE and its soliton solutions have been discussed extensively [1–7]. The generalized NSE includes three additional terms to the NSE which represent the effects of third-order dispersion, self-steeping, and intrapulse SRS. Its general form is given by [1,2]

$$\frac{\partial A}{\partial z}+\frac{1}{v_g}\frac{\partial A}{\partial t}+\frac{\alpha}{2}A+\frac{i}{2}\beta_2\frac{\partial^2 A}{\partial t^2}-\tfrac{1}{6}\beta_3\frac{\partial^3 A}{\partial t^3}
=i\gamma\left[|A|^2A+\frac{2i}{\omega_0}\frac{\partial}{\partial t}(|A|^2A)-T_R A\frac{\partial |A|^2}{\partial t}\right], \quad (1)$$

where $A(z,t)$ is the slowly varying envelope amplitude, v_g is the group velocity, α is the optical loss, β_2 is the GVD coefficient, β_3 is the third-order dispersion coefficient, γ is the nonlinear parameter, ω_0 is the carrier frequency, and the parameter T_R governs the effects of intrapulse SRS. T_R is related to the slope of the Raman-gain spectrum near ω_0 and is estimated to be in the range 6–10 fs for silica fibers [15]. We use the notation of Ref. [1] and refer to it for further details.

It is useful to write Eq. (1) in a normalized form. For this purpose, we introduce the normalized quantities

$$\tau=\frac{t-z/v_g}{T_0}, \quad \xi=\frac{z|\beta_2|}{T_0^2}, \quad U=\frac{A}{(P_0)^{1/2}}, \quad (2)$$

where T_0 and P_0 set the time scale and the power scale, respectively. In the case of optical pulses T_0 and P_0 are chosen, respectively, as the width and the peak power of the pulse. We neglect loss by setting $\alpha=0$. The GVD parameter β_2 is assumed to be negative. Equation (1) then takes the following form [1]:

$$i\frac{\partial U}{\partial \xi}+\frac{1}{2}\frac{\partial^2 U}{\partial \tau^2}+N^2|U|^2U = i\delta\frac{\partial^3 U}{\partial \tau^3}-isN^2\frac{\partial}{\partial \tau}(|U|^2U)+N^2\tau_R U\frac{\partial |U|^2}{\partial \tau}\ , \quad (3)$$

where

$$\delta=\frac{\beta_3}{6|\beta_2|T_0}\ ,\quad s=\frac{2}{\omega_0 T_0}\ ,\quad \tau_R=\frac{T_R}{T_0}\ , \quad (4)$$

and

$$N^2=\gamma P_0 T_0^2/|\beta_2|\ . \quad (5)$$

The parameters δ, s, and τ_R govern, respectively, the effects of third-order dispersion, self-steepening, and intrapulse SRS. For relatively long optical pulses ($T_0>10$ ps), all three parameters are so small that the right side of Eq. (3) can be set to zero. Equation (3) then reduces to the standard NSE that can be solved by the inverse scattering method [7]. For integer values of N, the solutions are in the form of solitons. The fundamental soliton corresponds to $N=1$, and its amplitude is given by $U(\xi,\tau)=\mathrm{sech}(\tau)\exp(i\xi/2)$.

The generalized NSE governs the evolution of optical fields whose envelope varies on a time scale shorter than a picosecond ($T_0<1$ ps). Considerable attention has been paid [8–12] to obtaining the solitonlike solutions of Eq. (3). In the absence of the SRS term ($\tau_R=0$), pulselike soliton solutions of Eq. (3) have been found in a certain range of parameters. In practice, however, the SRS term dominates and can affect temporal evolution even for pulses as wide as a few picoseconds [15,16]. We therefore consider the opposite limit in which we keep the intrapulse SRS term but neglect the effects of third-order dispersion and self-steeping by setting $\delta=0$ and $s=0$ in Eq. (3). The effect of self-steepening is included in a later section.

III. SOLITARY-WAVE SOLUTIONS

We follow the standard approach [8–12] and assume that the solitary-wave solutions of Eq. (3) can be written as

$$U(\xi,\tau)=F(\tau)\exp[i(K\xi-\Omega\tau)]\ , \quad (6)$$

where Ω represents a frequency shift from the carrier frequency ω_0 ,K is the corresponding change in the propagation constant, and the envelope function $F(\tau)$ is assumed to remain invariant on propagation. By substituting Eq. (6) in Eq. (3), $F(\tau)$ is found to satisfy

$$\frac{d^2F}{d\tau^2}-4N^2\tau_R F^2\frac{dF}{d\tau}+2N^2F^3-(2K-\Omega^2)F=0 \quad (7)$$

together with $\Omega(dF/d\tau)=0$, if we assume F to be real. For a nontrivial solution $dF/d\tau$ cannot be zero for all τ; hence Ω should be set to zero in Eq. (7). This equation does not appear to have a pulselike solution satisfying the boundary condition $F(\tau)=0$ as $|\tau|$ tends toward infinity. However, if the boundary condition is relaxed so that $F(\tau)$ becomes constant as $|\tau|$ increases, Eq. (7) is found to have the following solution:

$$F(\tau)=\exp\left[-\frac{3\tau}{4\tau_R}\right]\left[\mathrm{sech}\left[\frac{3\tau}{2\tau_R}\right]\right]^{1/2}\ . \quad (8)$$

By substituting this solution in Eq. (7) it is easy to verify that the solution exists only when $K=9/(8\tau_R^2)$ and N is related to τ_R through

$$N=\frac{3}{4\tau_R}\ . \quad (9)$$

The solitary-wave solution (8) belongs to a class of solitons appearing in the form of kinks. Kink solitons have been extensively studied in many physical systems whose dynamics are governed by the sine-Gordon equation [4]. In the present context, the kink soliton represents an optical front or optical shock which preserves its shape when propagating through an optical fiber. Figure 1 shows the shock profiles by plotting $F^2(\tau)$ for several values of τ_R; $F(\tau)$ is normalized such that $F(0)=1$. The steepness of the shock depends on the parameter τ_R defined as $\tau_R=T_R/T_0$. The shock front becomes increasingly steep as τ_R is reduced. The parameter N given by Eq. (9) increases when τ_R decreases. However, the power level P_0 (defined as the power at $\tau=0$) remains the same. This can be seen by combining Eqs. (5) and (9) and noting that

$$P_0=\frac{9}{16}\frac{|\beta_2|}{\gamma T_R^2}\ . \quad (10)$$

By using $T_R=10$ fs and typical parameter values for $|\beta_2|$ and γ, P_0 is estimated to be in the range 1–10 kW.

The shock front shown in Fig. 1 corresponds to the switch-on behavior of an optical field since shock occurs at the leading edge. There is another kind of shock front that corresponds to the mirror image of the one shown in Fig. 1. It can be obtained by changing τ to $-\tau$ in Eq. (8) and is sometimes referred to as an antikink soliton [4]. It is easy to verify that $F(-\tau)$ is not a solution of Eq. (7) because of the presence of the first-derivative term related

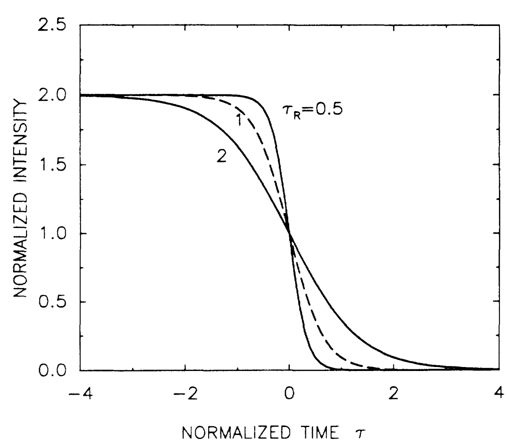

FIG. 1. Temporal profiles of an optical shock representing kink solitons for several values of τ_R.

to intrapulse SRS. Numerical results obtained by solving Eq. (3) confirm this behavior. When the initial amplitude $U(0,\tau)$ is set equal to $F(\tau)$ given by Eq. (8), $|U(\xi,\tau)|$ does not vary with the propagation distance ξ. However, when $U(0,\tau)$ is chosen to be $F(-\tau)$, the shock profile changes with propagation. The combination of a kink and antikink can be used to provide a pulselike input. Such a pulse may propagate without much change in its shape when the kink and antikink are well separated. To check this possibility, the generalized NSE was solved with the following input:

$$U(0,\tau)=F(\tau-\tau_0)H(\tau)+F(-\tau-\tau_0)H(-\tau) , \qquad (11)$$

where $H(\tau)$ is the Heaviside function defined to be 1 for $\tau \geq 0$ and 0 otherwise. The parameter $2\tau_0$ governs the spacing between the kink and the antikink. Figure 2 shows the evolution of such a pulse over the range $\xi=0$ to 1 by choosing $\tau_R=1$, $\tau_0=5$, and $N=\frac{3}{4}$. The trailing edge of the pulse does not change significantly on propagation since it travels approximately as a kink soliton of Eq. (8). The leading edge, however, begins to deform for $\xi>0.2$ simply because it does not propagate as a solitary wave. Deformation is in the form of a buildup of oscillations which move towards the trailing edge. The qualitative feature remains the same for other values of τ_0. The pulse spectra shown in Fig. 3 indicate that a part of the pulse energy has shifted towards the red side. Such a shift is due to intrapulse SRS and has been well studied in the context of optical fibers [16].

It is likely that optical pulses with a nearly rectangular shape may propagate without much distortion as long as the propagation distance is of the order of a dispersion length. We have used a super-Gaussian profile of the form

$$U(0,\tau)= \exp\left[-\left[\frac{\tau}{5\tau_R}\right]^4\right] \qquad (12)$$

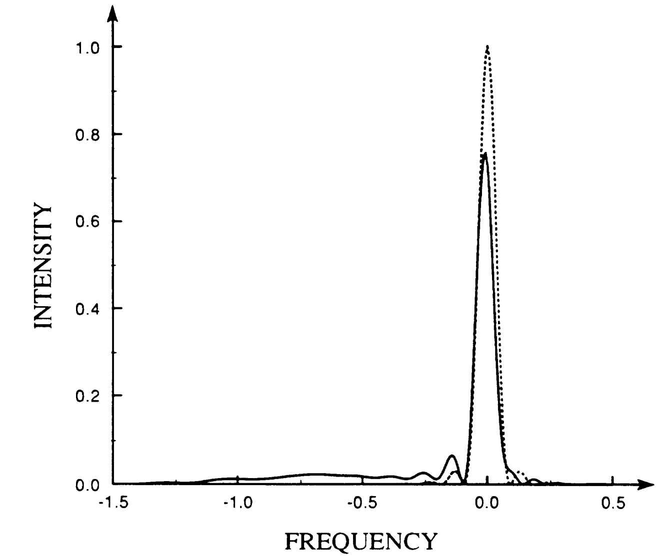

FIG. 3. Intensity spectrum at $\xi=1$ showing that a part of the pulse energy has shifted toward the red side. Input spectrum is shown by a dotted curve for comparison.

since they have a nearly rectangular shape at the input. Figure 4 shows evolution of such pulses over the range $\xi=0$ to 5 by choosing $\tau_R=1$ and $N=\frac{3}{4}$. The corresponding pulse spectra are shown in Fig. 5. The pulse shape and spectrum remain nearly the same over the range $\xi=0$ to 1 but begin to change for $\xi>1$. The main difference from a previous study [17] in which τ_R was much smaller ($\tau_R=0.01$) is that the spectral shift towards the red side is almost suppressed. These numerical results suggest that femtosecond pulses of relatively short duration (50 fs or less) may experience less red shift than pulses whose duration is much longer than 100 fs when such pulses are transmitted through an optical fiber.

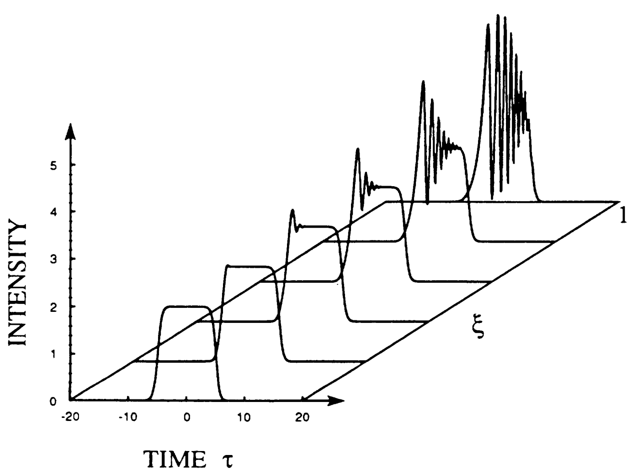

FIG. 2. Evolution of a pulse composed of two optical shocks (representing a kink-antikink pair) over one dispersion length ($\xi=0$ to 1) for $\tau_R=1$, $N=\frac{3}{4}$, and $\tau_0=5$.

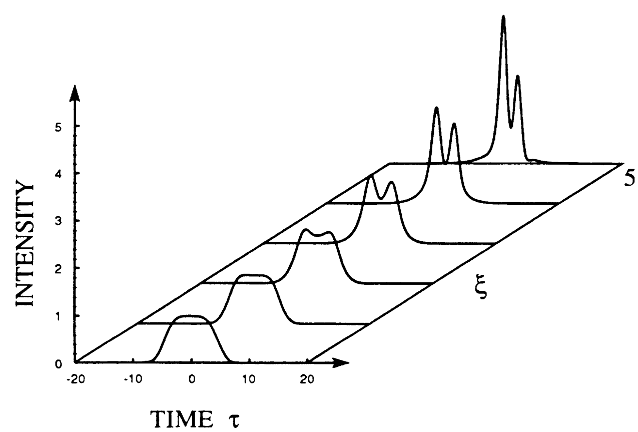

FIG. 4. Evolution of a super-Gaussian pulse over five dispersion lengths ($\xi=0$ to 5). Other parameters are identical to those of Fig. 2.

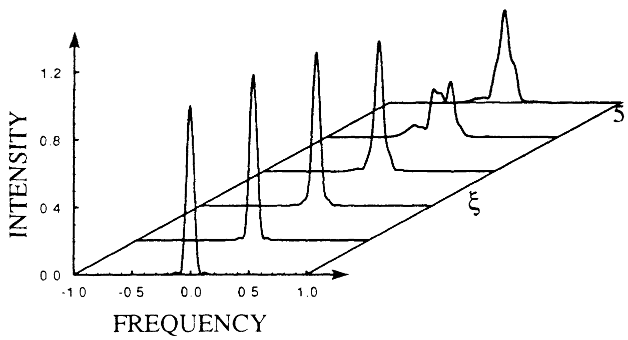

FIG. 5. Pulse spectra corresponding to pulse shapes of Fig. 4.

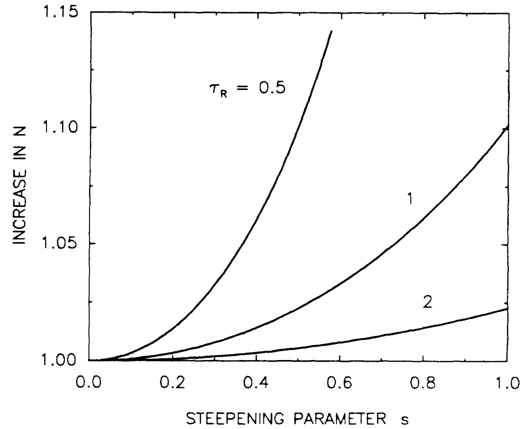

FIG. 6. Increase in the value of N required to maintain the kink soliton in the presence of self-steepening.

IV. EFFECT OF SELF-STEEPENING

In this section we study the effect of self-steepening on the kink soliton found in Sec. II by retaining the term containing the parameter s in Eq. (3) together with the intrapulse SRS term containing τ_R. It is no longer possible to assume a solution of the form (6) since the phase of $U(\xi,\tau)$ becomes intensity dependent. We therefore use

$$U(\xi,\tau)=F(\tau)\exp[i\phi(\tau)+iK\xi] , \tag{13}$$

where the phase $\phi(\tau)$ is yet to be determined. By substituting Eq. (13) in Eq. (3) we find after some simplifications [8] that $F(\tau)$ satisfies

$$\frac{d^2F}{d\tau^2}-4N^2\tau_R F^2\frac{dF}{d\tau}+\tfrac{3}{4}s^2N^4F^5+2N^2F^3-2KF=0 . \tag{14}$$

The phase $\phi(\tau)$ is obtained by solving

$$\frac{d\phi}{d\tau}=-3sN^2F^2 . \tag{15}$$

In the absence of self-steepening ($s=0$), the solitary-wave solution of Eq. (14) is given by Eq. (8). The functional form of the solution remains unchanged when s is nonzero. The solution of Eq. (14) can be rewritten as

$$F(\tau)=\exp\left[-\frac{p\tau}{2\tau_R}\right]\left[\operatorname{sech}\left[\frac{p\tau}{\tau_R}\right]\right]^{1/2} , \tag{16}$$

where

$$p=\frac{2+(16-9s^2)^{1/2}}{4-3s^2} . \tag{17}$$

Both K and N also become s dependent and are given by

$$K=\frac{p^2}{2\tau_R^2} , \tag{18}$$

$$N=\frac{1}{\tau_R s}[\tfrac{1}{3}(4+3s^2p^2)^{1/2}-\tfrac{2}{3}]^{1/2} . \tag{19}$$

In the limit $s=0$, $p=\frac{3}{2}$, and the solution (16) reduces to that obtained in Sec. III, as it should. The qualitative features of the intensity profiles shown in Fig. 1 remain unchanged in the presence of self-steepening. The numerical value of N is larger than $3/(4\tau_R)$ for $s\neq 0$. The increase is, however, by only a few percent in most cases. Figure 6 shows the increase in N as a function of s for several values of τ_R.

A major qualitative change induced by self-steepening is related to the phase $\phi(\tau)$. If we substitute $F(\tau)$ from Eq. (16) in Eq. (15) and carry out the integration, the phase is given by

$$\phi(\tau)=\frac{3s\tau_R N^2}{p}\ln\left[1+\exp\left[-\frac{2p\tau}{\tau_R}\right]\right] . \tag{20}$$

Thus the phase varies with time across the entire soliton, a phenomenon referred to as frequency chirping. Such solitons are often called chirped solitons. The frequency chirp is related to the phase derivative as $\Delta\omega_c=-d\phi/d\tau$, and is given by

$$\Delta\omega_c=3sN^2F^2(\tau)=6sN^2\left[1+\exp\left[\frac{2p\tau}{\tau_R}\right]\right]^{-1} . \tag{21}$$

The maximum value of $\Delta\omega_c$ is $6sN^2$ and occurs for large negative values of τ. It reduces to $3sN^2$ at the front center located at $\tau=0$ and vanishes for $\tau\gg 0$.

V. CONCLUDING REMARKS

In this paper we have obtained solitary-wave solutions of the generalized NSE. Although we did not find bright or dark solitons, we were able to obtain a kink-type soliton that represents an optical shock. Recently, shock-type solitary-wave solutions were obtained [14] for the case of nonlinear interaction of two waves in a Raman medium. In this geometry, two optical fields at different wavelengths are incident at the nonlinear medium. They copropagate in such a way that the longer-wavelength field can be amplified through SRS by gaining energy from the other optical field. Furthermore, dispersive effects were assumed to be absent in Ref. [14]. The situation is quite different from the one considered in this paper where only a single optical field propagates through the dispersive nonlinear medium. However, SRS appears to be necessary for the existence of kink solitons in both cases.

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[1] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic, San Diego, 1989).

[2] A. Hasegawa, Optical Solitons (Springer-Verlag, New York, 1989).

[3] G. P. Agrawal, in *Contemporary Nonlinear Optics,* edited by G. P. Agrawal and R. W. Boyd (Academic, Boston, 1992).

[4] R. K. Dodd, J. C. Eilbeck, J. D. Gibbons, and H. C. Morris, *Solitons and Nonlinear Wave Equations* (Academic, Orlando, 1982).

[5] E. M. Dianov, P. V. Memyshev, A. M. Prokhorov, and V. N. Serkin, *Nonlinear Effects in Optical Fibers* (Harwood, Academic, Chur, Switzerland, 1989).

[6] *Soltions in Physics, Mathematics, and Nonlinear Optics,* edited by P. J. Olver and D. H. Sattinger (Springer-Verlag, New York, 1990).

[7] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)]; **64**, 1627 (1973) [**37**, 823 (1973)].

[8] D. Anderson and M. Lisak, Phys. Rev. A **27**, 1393 (1983).

[9] D. N. Christodoulides and R. I. Joseph, Electron. Lett. **20**, 659 (1984).

[10] V. N. Serkin, Pis'ma Zh. Tekh. Fiz. **13**, 772 (1987) [Sov. Tech. Phys. Lett. **13**, 320 (1987)].

[11] M. J. Potasek, J. Appl. Phys. **65**, 941 (1989).

[12] A. B. Grudinin, V. N. Men'shov, and T. N. Fursa, Zh. Eksp. Teor. Fiz. **97**, 449 (1990) [Sov. Phys. JETP **70**, 249 (1990)].

[13] C. Headley III, G. P. Agrawal, and A. C. Reardon, in *SPIE Proceedings on Nonlinear Optics and Materials,* edited by C. D. Cantrell and C. M. Bowden (SPIE, Bellingham, WA 1991), Vol. 1497.

[14] D. N. Christodoulides, Opt. Commun. **86**, 431 (1991).

[15] J. P. Gordon, Opt. Lett. **11**, 662 (1986).

[16] F. M. Mitschke and F. L. Mollenauer, Opt. Lett. **11**, 659 (1986).

[17] G. P. Agrawal, Opt. Lett. **15**, 226 (1990).