Theory of a homogeneously broadened laser with arbitrary mirror outcoupling: Intrinsic linewidth and phase diffusion

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We present a theory of the homogeneously broadened laser that is based on the simple boundary conditions that govern the connection of intracavity traveling fields and the leaking vacuum field at the cavity mirrors. The approach is uniformly valid for arbitrary output-mirror transmissions. We compute the intrinsic linewidth of the laser both below and above threshold and for arbitrary mirror outcoupling. In the limit of complete output-mirror transmission, for which any semblance of the cavity to a lasing cavity is completely untenable and the laser is severely below threshold, we show that the laser linewidth reduces to the natural linewidth of spontaneous emission by atoms in free space. Above threshold, in contrast with most previously obtained results, the fundamental linewidth exhibits a power-independent contribution that arises from gain saturation that is spatially nonuniform.

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I. INTRODUCTION

Since the advent of the laser, many theoretical analyses have attempted to capture the subtlest of quantum-mechanical effects that characterize laser operation. One of the most important of these effects is the laser linewidths [1–7], since it determines the phase stability of the laser and therefore the sensitivity of the most precise interferometric measurements possible with a laser [8]. In stable operation well above threshold, the smallest possible linewidth of the output light is a consequence of the diffusion of phase arising from vacuum fluctuations entering the laser cavity through its output mirror and spontaneous emissions from active atoms. Thus the ultimate limitation on the phase stability of a laser is posed by effects that are purely quantum mechanical.

In recent times, the emergence of semiconductor lasers [9–12] has begun to have an impact on many fields of research, including optical communication and computing. As with traditional lasers, the ultimate limitation on phase-sensitive measurements done with semiconductor lasers will come from the quantum-mechanical diffusion of phase of the light field. In spite of many similarities between the traditional atomic or molecular lasers and semiconductor diode lasers, one of the important differences lies in the much larger output coupling of the latter. Such large output transmission makes any attempt to treat it as a distributed-damping mechanism, usually done in the context of traditional lasers with high Q cavities, invalid.

The principal objective of this paper [13] is to present a comprehensive theory of a homogeneously broadened laser and to treat quantum mechanically the intrinsic linewidth and the diffusion of the phase of the laser in the steady state. The key emphasis in the present work is on an exact treatment of mirror outcoupling, which, unlike previous treatments of the subject, is uniformly valid for all values of output-mirror transmission. Thus, as the transmission changes from being essentially zero (perfect cavity) to 100% (no cavity at all), we shall see that the intrinsic linewidth of the laser broadens from a very small value, described by the Schawlow-Townes linewidth formula [1], to the free-atom natural linewidth. Clearly, in the limit of 100% outcoupling, the feedback of light inside the cavity goes to zero and the active atoms essentially radiate in free space with the spectral linewidth nearly equal to the natural linewidth. The present approach makes essential use of mirror boundary conditions described via the reflection and transmission coefficients.

One of the important hurdles to any approach based on the concept of quasimodes [14], often simply called modes, is the lack of a rigorous definition for them inside leaky cavities. A distributed-damping theory, which assumes that the modes of a leaky cavity have the same spatial dependence as the corresponding modes of a perfect cavity, but unlike the latter have a damping constant or equivalently a linewidth associated with them, is obviously wrong when the outcoupling is large. For in a laser operating inside such a cavity, the field in steady state must have a nontrivial spatial dependence so that at the output mirror the loss of amplitude exactly compensates for the round-trip amplification.

An effort to remove this deficiency was made by Lang, Scully, and Lamb [15], who demonstrated how one may define rigorously the modes of an empty but leaky cavity and use those modes to quantize the electromagnetic field throughout space. They were able to establish the equivalence of the usual quasimode treatment of the cavity and the rigorous picture in the good-cavity limit. Further extension of this work was carried out in Ref. [16], in which the connection between the inside and outside quantum fluctuations of a leaky cavity was established in a rigorous manner.

Ujihara [17] extended the good-cavity analysis of Lang and co-workers to lasers with arbitrary output-coupling constants. In a series of several papers, he treated both the steady-state intensity and the quantum noise leading to the linewidth. The essential difference from all the other previous treatments arose from the nontrivial spatial dependence of the field in a low Q cavity. His analysis, based on the rigorously defined modes, the so-called modes of the universe, seems to the present author to obscure the essential physics by an excessive use of mathematical formalism. Moreover, as we shall see later, his analysis contains an implicit assumption that in effect restricts the applicability to strictly two-level active atoms for which the sum of the occupation probabilities of the two levels is always 1, regardless of the field strength. Other approaches to the problem, notably of Henry [11] and of Hamel and Woerdman [18], make use of semiclassical analyses and reproduce the answers of Ujihara when saturation is ignored.

With Abbott, the present author adopted a traveling-wave analysis [19] of the linewidth problem, although they stayed within the single-quasimode formalism. In this analysis, the cavity damping due to mirror outcoupling was correctly treated in a nondistributed fashion that takes place once every round trip from the output mirror. However, this approach proved suitable only for high to moderately high Q cavities.

The analysis presented here obviates the need for the intermediate step of introducing any modes whatever of the leaky lasing cavity. Yet, the treatment is in effect multimode, since the spatial and temporal variations of the field are correctly analyzed. The cavity damping of the field at the output mirror is carefully treated by the boundary conditions at the mirror, which can be specified in terms of its transmission and reflection coefficients $\tilde{t}$ and $\tilde{r}$, respectively. Thus there is no need to have a reservoir model for the cavity losses. The fluctuation-dissipation theorem is automatically enforced for such losses by the boundary conditions.

It is important to point out here that the approach described in this paper defines a general formalism for any matter-light resonant interaction problem inside a cavity in which quantum-mechanical phenomena—as well as classical ones—are treated accurately. The method of boundary conditions espoused here addresses the connection between the fields inside and outside without requiring any special problem-dependent modifications for its implementation. Indeed, this approach has been gainfully used by Abbott and the present author [20] in a quantitatively correct description of quantum noise and squeezing in a degenerate parametric oscillator with arbitrary mirror outcoupling.

The layout of the paper is as follows. In the next section, the basic equations of motion for the dynamics of two-active-level atoms and the coherent field with which the atoms interact are derived. The nature of incoherent spontaneous emission and its effects on these equations are clarified. An improved adiabatic approximation is introduced to eliminate the atomic variables in order to obtain an equation for the field alone. In this approximation, the quantum noise operators representing incoherent emission are properly treated. In Sec. III, the theory of the laser operating below threshold is presented, which treats the cavity damping as well as the vacuum fluctuations causing spontaneous emission into the longitudinal on-axis modes via the output-mirror boundary conditions. The most interesting result of this section is an expression for the power spectrum of light that goes from the usual Schawlow-Townes description to spontaneous emission by atoms in free space as the output-mirror transmission is increased. In Sec. IV, the steady-state operating point (amplitude and frequency) of the laser oscillating above threshold is derived, on which the calculations of Sec. V on phase diffusion and linewidth are predicated. The intrinsic linewidth of the laser operating above threshold exhibits several interesting features, most notably the presence of a power-independent term arising from gain saturation and its spatial nonuniformity. Finally, in Sec. VI, we summarize the conclusions of this paper.

II. FORMALISM

We consider a single-ended cavity of cross section A bounded by a perfectly reflecting mirror at one end, $z=l$, and a partially transmitting output mirror at the other, $z=0$, with real outside-to-inside reflection coefficient $\tilde{r}$ and transmission coefficient $\tilde{t}$. This cavity contains N_A randomly distributed active atoms, as in a gas, which are assumed to be pumped incoherently into the upper and lower laser levels, a and b, at the rates Λ_a and Λ_b per unit volume per unit time. We assume in the following that only axial modes of the electromagnetic field are coherently excited and that the coherent field is linearly polarized in the x direction.

Let σ^a, σ^b, and $\sigma^{\pm}$ denote the atomic level population and dipole operators, which at time $t=0$ have the following representation:

$$\sigma^a=|a\rangle\langle a|, \quad \sigma^b=|b\rangle\langle b|$$

and

$$\sigma^-=|b\rangle\langle a|, \quad \sigma^+=|a\rangle\langle b| ,$$

Their commutation rules are easy to derive by exploiting the orthonormality of the states $|a\rangle$ and $|b\rangle$. Since they are well known, we shall not write them down here.

We may express the pumping rates Λ_a and Λ_b in terms of the rates λ_a and λ_b at which an individual atom is pumped: $\lambda_a=\Lambda_a/n_0$ and $\lambda_b=\Lambda_b/n_0$, where $n_0=N_A/Al$ is the gas density. We assume that the randomly distributed atoms are at a sufficiently low temperature and density so that their motion or collisions are of no significance over the lifetimes of their excited states, during which they interact with the field. This is the limit of homogeneous broadening to which we restrict our calculations.

The interacting atom-field system may be described in terms of the following Hamiltonian:

$$H=H_F+H_A+H_{AF} , \tag{2.1}$$

where H_F is the free-field Hamiltonian, H_A is the free-atom Hamiltonian given by

$$H_A=\frac{\hbar\omega_0}{2}\sum_{i=1}^{N_A}(\sigma_i^a-\sigma_i^b) , \tag{2.2}$$

and H_{AF} is the atom-field dipole interaction potential in the energy-conserving rotating-wave approximation (RWA)

$$H_{AF}=-\boldsymbol{\mu}\cdot\sum_{i=1}^{N_A}[\sigma_i^+\mathbf{E}^{(+)}(\mathbf{r}_i,t)+\sigma_i^-\mathbf{E}^{(-)}(\mathbf{r}_i,t)] . \tag{2.3}$$

In the preceding equations, ω_0 is the resonance frequency of the atomic transition, $\boldsymbol{\mu}$ is the dipole matrix element between the two laser levels, and $\mathbf{E}^{(+)}$ and $\mathbf{E}^{(-)}$ are the positive and negative frequency parts of the full transverse electric-field operator.

The interaction part of the Hamiltonian H_{AF} describes all coherent and most of the incoherent radiative processes (within the dipole approximation) in our atom-field system. In the present context, "coherent" refers to the essentially forward-scattered field, which evolves via stimulated emission and absorption processes and therefore maintains its phase. On the other hand, "incoherent" refers to the off-axis scattered field generated mostly by spontaneous emission by atoms, a field that is essentially isotropic. It is worth noting here that even in the forward direction there is spontaneous emission, which is amplified by the inverted active atoms. This on-axis spontaneous emission can be construed as arising from the fluctuations of the vacuum field outside that leaks into the laser cavity through the partially transmitting output mirror.

In other words, there are three kinds of fields that make up the total radiation field (i) the forward-propagating coherent field, (ii) the forward-propagating incoherent field (transmitted and amplified, fluctuating vacuum field), and (iii) the off-axis incoherent field ("spontaneous emission"). The off-axis field (iii) may be conveniently eliminated from the full field at any location by a spatial averaging [21] of the full field over a thin transverse slice of atoms centered at that location. The thickness Δz of such a slice is taken to be small compared to the wavelength of emission so that the full wave structure of the propagating fields is not compromised. The spatial averaging leaves the coherent and incoherent parts of the forward-propagating field [contributions (i) and (ii)] intact. At this point one may identify the coherent part of the forward-propagating field as that arising from the in-phase excited atomic dipoles, while its incoherent part may be separately treated as arising from a random-phase excitation of the atomic dipoles by the outside vacuum field that is transmitted into the cavity and is amplified by the gain medium, as stated before. This separation of the forward-propagating field into coherent and incoherent parts may seem a bit *ad hoc* at the moment, but we shall justify it later by means of the boundary conditions on the fields at the output mirror.

In what follows, we shall treat the forward-propagating coherent field dynamically via the spatially smoother version of Hamiltonian (2.1), while contribution (ii) to the field will be incorporated rigorously via mirror boundary conditions. Lastly, the contribution (iii) to the field will be included indirectly into our dynamical equations via the Wigner-Weisskopf formalism.

Since Maxwell's equations of electromagnetism are linear in the fields, it follows that in the Heisenberg picture the field operators formally obey the same wave equation that the classical field obeys. The positive-frequency part of the coherent electric-field operator satisfies the following wave equation:

$$\left[\frac{\partial^2}{\partial z^2}-\frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right]E^{(+)}(z,t)=\frac{1}{\epsilon_0 c^2}\frac{\partial^2}{\partial t^2}P^-(z,t) , \tag{2.4}$$

in which the macroscopically coherent polarization operator $P^-(z,t)$ is defined as the spatially smoothed version [21] of the microscopic operator $P_{\rm mic}^-(\mathbf{r},t)$

$$P_{\rm mic}^-(\mathbf{r},t)=\mu\sum_i\sigma_i^-(t)\delta(\mathbf{r}-\mathbf{r}_i) \tag{2.5}$$

and n is the refractive index of the medium in which the active atoms reside. As we have indicated before, the spatial smoothing is done by averaging the preceding expression over the entire cross section A and over a longitudinal interval $(z-\Delta z/2,z+\Delta z/2)$, which, although much smaller than the wavelength $\lambda\gg\Delta z$, still contains many atoms. Thus,

$$P^-(z,t)=\frac{\mu}{A\,\Delta z}\int_Z d\rho\int_{z-\Delta z/s}^{z+\Delta z/2}dz'P_{\rm mic}^-(\mathbf{r}',t)$$

$$=\frac{\mu}{A\,\Delta z}\sum_{i_z}^{\Delta z}\sigma_{i_z}^-(t) , \tag{2.6}$$

in which the symbol Δz over the summation sign indicates that i_z labels atoms located in the slice $(z-\Delta z/2,z+\Delta z/2)$.

Just like the macroscopic polarization operator above, we also define macroscopic population density operators $R_a(z,t)$ and $R_b(z,t)$ as follows:

$$R^a(z,t)=\frac{1}{A\,\Delta z}\sum_{i_z}^{\Delta z}\sigma_{i_z}^a \tag{2.7a}$$

and

$$R^b(z,t)=\frac{1}{A\,\Delta z}\sum_{i_z}^{\Delta z}\sigma_{i_z}^b . \tag{2.7b}$$

By employing the Hamiltonian (2.1) and making explicit the incoherent pumping, damping, and associated noise terms, we may write down the following atomic evolution equations:

$$\frac{\partial}{\partial t}P^-(z,t)=-(i\omega_0+\gamma)P^-(z,t)+\frac{\mu^2}{i\hbar}[R_a(z,t)-R_b(z,t)]E^{(+)}(z,t)+\mu F^-(z,t)\ , \tag{2.8a}$$

$$\frac{\partial}{\partial t}R^a(z,t)=\Lambda_a-\gamma_a R^a(z,t)+\frac{i}{\hbar}[P^+(z,t)E^{(+)}(z,t)-P^-(z,t)E^{(-)}(z,t)]+F^a(z,t)\ , \tag{2.8b}$$

and

$$\frac{\partial}{\partial t}R^b(z,t)=\Lambda_b-\gamma_b R^b(z,t)-\frac{i}{\hbar}[P^+(z,t)E^{(+)}(z,t)-P^{(-)}(z,t)E^{(-)}(z,t)]+F^b(z,t)\ , \tag{2.8c}$$

where the noise operators F^-, F^a, and F^b are related to the individual atom noise operators f_i^-, f_i^a, and f_i^b by relations of form (2.6)

$$F^\alpha(z,t)=\frac{1}{A\,\Delta z}\sum_{i_z}^{\Delta z}f_{i_z}^\alpha,\quad \alpha=-,a,b\ . \tag{2.9}$$

The pumping terms Λ_a and Λ_b represent the excitation of atoms from other levels to the levels a and b, under the assumption that the other levels are not significantly depleted. The damping and associated noise terms are a result of a Wigner-Weisskopf type of analysis of spontaneous emission in the Heisenberg picture and represent the effects of the off-axis incoherent field, as mentioned earlier.

Equations (2.4) and (2.8) constitute the system of equations that, when supplemented with the boundary conditions appropriate to the perfect reflector at $z=l$ and the partially transmitting mirror at $z=0$ [to incorporate the on-axis incoherent field contribution (ii) above], describe fully the temporal and spatial forms of the radiation field. However, since the equations are nonlinear, solving them exactly is not possible. We shall invoke the resonant nature of the coupling of the atoms to the field to approximate Eq. (2.4) by a first-order differential equation. Below threshold, the field is sufficiently weak that nonlinearities in the equations can be ignored. Above threshold, the nonlinearity mostly affects the field amplitude in steady state, not its phase evolution, and can be treated classically since the field intensity is large. In the latter case, we shall linearize the quantum fluctuations of the field around such a classically describable amplitude in order to calculate the phase diffusion and the linewidth of the laser. These assumptions are the usual ones [6] that describe a stable laser oscillating well above threshold.

Since the radiation field emitted by the active atoms is quasimonochromatic, we may decompose its electric field as a product of a fast varying time exponential $\exp(-i\Omega t)$ and a much more slowly varying function of time. The nominal operating frequency Ω of the field is expected to be close to a bare-cavity resonance frequency and the atomic transition frequency ω_0. Its actual value is governed by a competition between the cavity resonance and the atomic resonance. This phenomenon of frequency pulling is contained in our equations, as we shall see later.

A similar decomposition should also obtain for the spatial variation of the electric field, where the fast spatial variation is governed by the wave vector $k\equiv n\Omega/c$. Since the preceding statements regarding fast and slow variations can also be made about the polarization operator P^-, we may express both the field and polarization operators as follows:

$$E^{(+)}(z,t)=[e_+(z,t)e^{ikz}+e_-(z,t)e^{-ikz}]e^{-i\Omega t} \tag{2.10a}$$

and

$$P^-(z,t)=[p_+(z,t)e^{ikz}+p_-(z,t)e^{-ikz}]e^{-i\Omega t}\ . \tag{2.10b}$$

The subscripts $+$ and $-$ refer to the right- and left-going waves, respectively, whose amplitudes, denoted by lower-case letters, are slowly varying both in time (over a scale of the inverse frequency) and in space (over a scale of the inverse wave vector).

When the forms (2.10) are substituted into Eqs. (2.4) and (2.8), and the second derivatives of $e_\pm$ and $p_\pm$ as well as terms modulated by factors such as $\exp(\pm 2ikz)$ are ignored, one obtains the following equations:

$$\left[\frac{\partial}{\partial z}\pm\frac{n}{c}\frac{\partial}{\partial t}\right]e_\pm(z,t)=\pm\frac{i\Omega^2}{2\epsilon_0 kc^2}p_\pm(z,t)\ , \tag{2.11a}$$

$$\left[\frac{\partial}{\partial t}+i\Delta+\gamma\right]p_\pm(z,t)=-\frac{i\mu^2}{\hbar}[R_a(z,t)-R_b(z,t)]\times e_\pm(z,t)+\mu f_\pm(z,t)\ , \tag{2.11b}$$

$$\left[\frac{\partial}{\partial t}+\gamma_a\right]R_a(z,t)=\Lambda_a+\frac{i}{\hbar}(p_+^\dagger e_+ +p_-^\dagger e_- -p_+e_+^\dagger -p_-e_-^\dagger)+F_a(z,t)\ , \tag{2.11c}$$

and

$$\left[\frac{\partial}{\partial t}+\gamma_b\right]R_b(z,t)=\Lambda_b-\frac{i}{\hbar}(p_+^\dagger e_+ +p_-^\dagger e_- -p_+e_+^\dagger -p_-e_-^\dagger)+F_b(z,t)\ . \tag{2.11d}$$

The noise forces $f_\pm$ in Eq. (2.11b) are obtained by averaging the noise term $F^-(z,t)$ in Eq. (2.8a) multiplied by $\exp(\mp ikz)$ over a wavelength. Noting that $F^-(z,t)$ is already smoothed, this additional spatial averaging yields the result (see Appendix A)

$$f_\pm(z,t)=\frac{1}{A\lambda}\left[\sum_{i_z}^{\lambda}f_{i_z}^-(t)e^{\mp ikz_{i_z}}\right]e^{i\Omega t}\ , \tag{2.12}$$

where the index λ over the summation sign indicates that i_z labels those atoms that lie in the range

$(z-\lambda/2, z+\lambda/2)$.

The calculation of linewidth, which is the central theme of the present work, involves several steps and approximations, which begin with the preceding nonlinear system of operator equations. We first note that since the noise operators F_a and F_b governing the evolution of the population operators R_a and R_b implicitly enter the polarization equation (2.11b) quadratically in the coupling constant μ, they will be dominated by the noise term $\mu f_{\pm}(z,t)$ in that equation. We shall assume, as traditionally done [6], that F_a and F_b may in effect be ignored. In other words, the population operators may be treated in the expectation-value sense without much effect on our final results.

Thus the noise operators $f_{\pm}(z,t)$ associated with the radiative spontaneous relaxation of the atomic dipoles represent nearly the entire spontaneous-emission contribution to the interaction dynamics. Furthermore, since the linewidth is determined from the second-order autocorrelation function of the field, we shall only need two-time correlations of the noise operators $f_{\pm}(z,t)$. Since the characteristic correlation time of spontaneous emission from atoms is of the order of the fundamental period of the emitted light, it is quite appropriate, on the much longer time scales characteristic of phase diffusion and other relaxation phenomena, to assume that $f_{\pm}(z,t)$ are δ correlated. We shall see in Appendix B that the diffusion constants for $f_{\pm}$, which govern the strength of the δ correlation, linearly involve the saturated level populations [see also Eqs. (20.21) and (20.22) of Ref. [6].]

To proceed further, we assume atomic decay rates much greater than the cavity damping rate of the field, so that the field does not change much over an atomic decay period. This lets us integrate Eq. (2.11b) formally as

$$p_{\pm}(z,t) \approx \frac{-i\mu^2}{\hbar(\gamma+i\Delta)}\mathcal{N}e_{\pm}(z,t) + \mu\int_0^{\infty} f_{\pm}(z,t-t')e^{-(\gamma+i\Delta)t'}dt' , \tag{2.13}$$

in which $\mathcal{N}$ is the population difference: $\mathcal{N}=\langle R_a\rangle-\langle R_b\rangle$. Atomic memory "colors" the δ-correlated (white) noise $f_{\pm}$ via the integral in Eq. (2.13). This represents a departure from the traditional treatments of laser noise, which wrongly assume in effect that the fluctuations $f_{\pm}$ do not vary much over an atomic lifetime, γ^{-1}. This refinement, which we call an improved adiabatic approximation, allows, as we shall see, the atomic linewidth to serve as the limiting value for the intrinsic laser linewidth for large outcoupling.

III. LASER OPERATING BELOW THRESHOLD

Below threshold, the laser is very noisy with strong amplitude and phase fluctuations. However, in steady state the field is generated by spontaneous emission alone and is quite weak. For such weak fields, the populations $\langle R_a\rangle$ and $\langle R_b\rangle$ may be assumed to be unchanged from their zero-field values, the so-called unsaturated values denoted by appending the superscript (0). In steady state, these are given by setting the noise terms, the time derivatives, and $e_{\pm}$ to zero in Eqs. (2.11c) and (2.11d),

$$R_a^{(0)}=\Lambda_a/\gamma_a, \quad R_b^{(0)}=\Lambda_b/\gamma_b . \tag{3.1}$$

By substituting these values in Eq. (2.13) and combining the resulting equation with Eq. (2.11a), we obtain the following quantum Langevin equation for the time evolution of the full fluctuating field:

$$\left[\frac{\partial}{\partial z}\pm\frac{n}{c}\frac{\partial}{\partial t}\right]e_{\pm}(z,t)=\pm\alpha_0 e_{\pm}(z,t)+\phi_{\pm}(z,t) , \tag{3.2}$$

where α_0 given by

$$\alpha_0\equiv\frac{\mu^2\Omega^2\mathcal{N}_0}{2\epsilon_0 kc^2\hbar(\gamma+i\Delta)} \tag{3.3a}$$

is the usual linear gain coefficient proportional to the *unsaturated* population inversion density

$$\mathcal{N}_0=\langle R_a^{(0)}\rangle-\langle R_b^{(0)}\rangle=\Lambda_a/\gamma_a-\Lambda_b/\gamma_b \tag{3.3b}$$

and the fluctuating forces $\phi_{\pm}$ are given by

$$\phi_{\pm}(z,t)\equiv\pm\frac{i\mu\Omega^2}{2\epsilon_0 kc^2}\int_{-\infty}^{t} f_{\pm}(z,t')e^{-(\gamma+i\Delta)(t-t')}dt' . \tag{3.4}$$

Equations (3.2) may be reduced to ordinary differential equations in single independent variables by transforming to the retarded times $\tau_+=t-nz/c$ (for e_+) and $\tau_-=t-n(l-z)/c$ (for e_-), since these transformations imply the following derivative transformations:

$$\frac{\partial}{\partial z}\pm\frac{n}{c}\frac{\partial}{\partial t}\to\frac{\partial}{\partial z} .$$

The resulting differential equations are easy to integrate and yield the following integral solutions for Eqs. (3.2):

$$e_+(z,t)=e_+(0,t-nz/c)e^{\alpha_0 z} + \int_0^z dz' e^{\alpha_0(z-z')}\phi_+(z',t-n(z-z')/c) \tag{3.5a}$$

and

$$e_-(z,t)=e_-(l,t-n(l-z/c))e^{\alpha_0(l-z)} - \int_z^l dz' e^{\alpha_0(z'-z)}\phi_-(z',t-n(z'-z)/c) , \tag{3.5b}$$

as one may also verify by direct substitution into Eqs. (3.2). Equations (3.5) describe how the radiation field amplifies in the rightward traveling direction [Eq. (3.5a)] and on the return trip in the leftward traveling direction [Eq. (3.5b)].

The boundary conditions on the fields $e_{\pm}$ at the mirrors, namely,

$$e_+(l,t)e^{ikl}=-e_-(l,t)e^{-ikl} \tag{3.6a}$$

and

$$e_+(0,t)=-\tilde{r}e_-(0,t)+\tilde{t}e_+^{(\mathrm{vac})}(0,t)/n , \tag{3.6b}$$

in which $e_+^{(\mathrm{vac})}(0,t)$ is the incoming on-axis vacuum field amplitude, provide the needed connections between the two traveling fields, which when employed with Eqs. (3.5) permit one to determine how the field evolves over a complete round trip. We also note from Eq. (3.6b) that the on-axis vacuum field, whose fluctuations drive the on-axis spontaneous emission from excited atoms, enters our analysis via the leaky-mirror boundary condition.

The derivation of the complete round-trip evolution equation for either traveling field, say $e_-(z,t)$, now proceeds in four steps. Set $z=0$ in Eq. (3.5b). Replace $e_-(l,t-nl/c)$ on the right-hand side of that equation in terms of $e_+(l,t-nl/c)$ via Eq. (3.6a). Follow this step with the replacement of $e_+(l,t-nl/c)$ in terms of $e_+(0,t-2nl/c)$ via Eq. (3.5a). Finally, employ the boundary condition (3.6b). The net result of these four steps is the following round-trip equation (a trivial overall time shift by the round-trip time is also involved):

$$e_-(0,t+t_R)=\tilde{r}e^{2(\alpha_0+ik)l}e_-(0,t)+G(t)\ , \qquad (3.7)$$

where $t_R\equiv 2ln/c$ is the light round-trip time in the cavity. The noise operator $G(t)$ is a sum of the amplified vacuum field entering the cavity and the amplified dipole fluctuations (related to $\phi_\pm$) resulting from off-axis spontaneous emission

$$\begin{aligned}G(t)=&-\frac{\tilde{t}}{n}e^{2(\alpha_0+ik)l}e_+^{(\mathrm{vac})}(0,t)\\&-\frac{i\mu\Omega^2}{2\epsilon_0kc^2}e^{2(\alpha_0+ik)l}\int_0^l dz'\int_0^\infty dt' f_+(z',t+nz'/c-t')e^{-\alpha_0z'}e^{-(\gamma+i\Delta)t'}\\&+\frac{i\mu\Omega^2}{2\epsilon_0kc^2}\int_0^l dz'\int_0^\infty dt' f_-(z',t+t_R-nz'/c-t')e^{\alpha_0z'}e^{-(\gamma+i\Delta)t'}\ .\end{aligned} \qquad (3.8)$$

The structure of the noise expression (3.8) is quite clear. The vacuum field enters at time t through the mirror at $z=0$ at time t and then amplifies through the entire round trip before contributing to $e_-(0,t+t_R)$. As for the dipole fluctuations, their left-going piece $f_-(z',t')$ amplifies through a distance z' before arriving at the mirror at $z=0$, while their right-going piece $f_+(z',t')$ amplifies through a distance $2l-z'$ before arriving at the mirror at $z=0$.

In the good-cavity limit, since fields do not change much over a single round trip, Eq. (3.7) may be approximated by the usual differential Langevin equation of single-mode theories by writing $e_-(0,t+t_R)-e_-(0,t)$ as $t_R de_-(0,t)/dt$. However, for an arbitrary Q cavity, the difference equation (3.7) has to be treated accurately. In what follows, we look at the expectation value of intensity of the leftward traveling light wave as well as its power spectrum just to the right of the output mirror at $z=0$. We note that the power spectrum so calculated is the same, up to an overall power transmission factor, as the power spectrum of the observed output field. In order to compute these expectation values, which involve second-order, normally ordered field products, we need, as indicated by Eq. (3.7), to know what the second-order, normally ordered moment of $G(t)$ is. The calculation of $\langle G^\dagger(t)G(t')\rangle$ is presented in Appendix B, where it is shown that the correlation time for $G(t)$ is about γ^{-1}.

A. Average intensity in the leftward-traveling wave

In steady state, the average intensity is independent of time. If we define the average intensity of the leftward-traveling wave at $z=0^+$ in steady state by the relation $i_-(0)=\langle e_-^\dagger(0,t)e_-(0,t)\rangle$, then from Eq. (3.7) it follows that

$$i_-(0)=\tilde{r}^2e^{4\,\mathrm{Re}(\alpha_0)l}i_-(0)+\langle G^\dagger(t)G(t)\rangle\ , \qquad (3.9)$$

in which use was also made of the fact that $\langle G^\dagger(t)G(t\pm t_R)\rangle$ is negligible, since $\gamma t_R\gg 1$. We may now write with the aid of Eqs. (3.9) and (B16)

$$i_-(0)=\frac{(e^{4\,\mathrm{Re}(\alpha_0)l}-1)}{(1-\tilde{r}^2e^{4\,\mathrm{Re}(\alpha_0)l})}\frac{\hbar\Omega}{2\epsilon_0cnA}\frac{\langle R_a^{(0)}\rangle}{\langle R_a^{(0)}\rangle-\langle R_b^{(0)}\rangle}\times\left[\frac{\gamma^2+\Delta^2}{2\gamma}\right]. \qquad (3.10)$$

B. Power spectrum of outgoing light

To compute the power spectrum of the outgoing light, which is proportional to the power spectrum of the e_- field at $z=0$, we take a finite Fourier transform of Eq. (3.7) defined as

$$A_T(\delta\omega)=\int_{-T/2}^{T/2}e_-(0,t)e^{i\delta\omega t}dt\ , \qquad (3.11)$$

in which T is the measurement time (inverse of the spectral resolution of the spectrum analyzer) to be set to ∞ in the end. In the limit $T\to\infty$, such a Fourier transform of Eq. (3.7) yields

$$A_T(\delta\omega)=\frac{\tilde{G}_T(\delta\omega)}{e^{-i\delta\omega t_R}-\tilde{r}e^{2(\alpha_0+ik)l}}\ , \qquad (3.12)$$

where $\tilde{G}_T(\delta\omega)$ is the Fourier transform of $G(t)$.

The power spectrum $S(\delta\omega)$ of the e_- field at $z=0$ now follows on taking the following limit:

$$S(\delta\omega)=\lim_{T\to\infty}\frac{1}{T}\langle A_T^\dagger(\delta\omega)A_T(\delta\omega)\ ,$$

which with the help of Eq. (3.12) produces the result

$$S(\delta\omega)=\frac{i_-(0)[1-\tilde{r}^2e^{4\,\mathrm{Re}(\alpha_0)l}]}{1+\tilde{r}^2e^{4\,\mathrm{Re}(\alpha_0)l}-2\tilde{r}\eta l\cos[\delta\omega t_R+2kl+2\,\mathrm{Im}(\alpha_0)l]}\,\frac{2\gamma}{\gamma^2+(\delta\omega-\Delta)^2}\ . \tag{3.13}$$

In order to derive Eq. (3.13), we also used the fact that the power spectrum of $G(t)$ given by

$$\lim_{T\to\infty}\frac{1}{T}\langle\tilde{G}^\dagger(\delta\omega)\tilde{G}(\delta\omega)\rangle$$

may, via the Fourier-transform definition, be written as

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt_1\int_{-T/2}^{T/2}dt_2\langle G^\dagger(t_1)G(t_2)\rangle e^{-i\delta\omega(t_1-t_2)}\ ,$$

which in turn via relations (B16) and (3.10) may be expressed in the form

$$i_-(0)\left[1-\tilde{r}^2e^{4\,\mathrm{Re}(\alpha_0)l}\right]\frac{2\gamma}{\gamma^2+(\delta\omega-\Delta)^2}\ .$$

By now employing the trigonometric identity

$$\cos\theta=1-2\sin^2\frac{\theta}{2}\ ,$$

we may rewrite the power spectrum given by Eq. (3.13) as a product of the power spectrum of $G(t)$, which is a Lorentzian with width equal to the atomic linewidth γ as seen above, and a Fabry-Pérot-like resonant factor

$$S(\delta\omega)\sim\frac{\gamma^2}{\gamma^2+(\omega-\omega_0)^2}\,\frac{1}{1+F\sin^2(\delta\omega t_R/2)}\ , \tag{3.14}$$

in which we have suppressed the constant $\mathrm{Im}(\alpha_0)l+kl$ from within the argument of the sine, since it is expected to be close to a multiple of π anyway [see Eq. (4.10b)]. Even when that is not the case, the suppressed term merely represents an overall frequency shift of the spectrum. We have also suppressed an overall proportionality factor in writing Eq. (3.14), but that may be easily written down by comparing Eqs. (3.13) and (3.14). The coefficient F given by

$$F\equiv\frac{4\tilde{r}e^{2\,\mathrm{Re}(\alpha_0)l}}{(1-\tilde{r}e^{2\,\mathrm{Re}(\alpha_0)l})^2}$$

determines the linewidth.

In the limit of large γ and near threshold, $\tilde{r}\exp[2\,\mathrm{Re}(\alpha_0)l]\lesssim 1$, F can be quite large, so that the power spectrum consists of many individual lines (or quasimodes) separated successively by $\Delta\omega_m\equiv 2\pi/t_R$ and sitting under the atomic line shape envelope with half-width γ, as shown in Figs. 1(a) and 1(b). With increasing mirror transmission the individual lines broaden until their width becomes comparable to $\Delta\omega_m$. Then they lose their individuality and the spectrum begins to resemble the natural atomic line shape with half-width γ. In this limit the laser cavity no longer confines light.

If $\tilde{r}$ is not too large, the full width at half maximum (FWHM) $\Delta\Omega=4/t_R\sqrt{F}$ of each individual line may be expressed in a form involving the total output power. To do so, note from Eq. (3.10) that

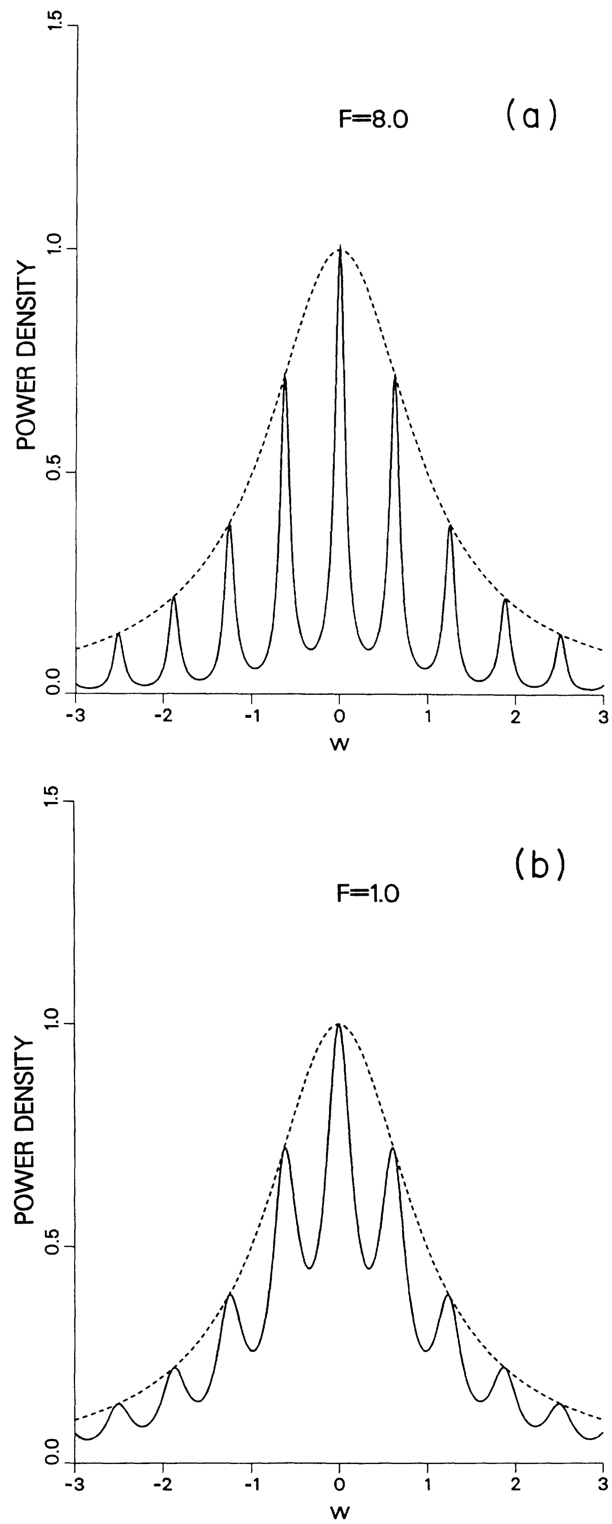

FIG. 1. Power spectrum below threshold. The detuning w is in units of γ, and $\Omega=\omega_0$, $\gamma t_R=10$, and (a) $F=8$ and (b) $F=1$.

$$1-\tilde{r}^2 e^{4\,\mathrm{Re}(\alpha_0)l}=\frac{K}{i_-(0)}\ , \tag{3.15}$$

where

$$K=\frac{\hbar\Omega}{2\epsilon_0 cnA}\frac{\langle R_a^{(0)}\rangle}{\langle R_a^{(0)}\rangle-\langle R_b^{(0)}\rangle}\left[\frac{\gamma^2+\Delta^2}{2\gamma}\right] \times(e^{4\,\mathrm{Re}(\alpha_0)l}-1)\ . \tag{3.16}$$

If the laser is not too far off threshold, $\tilde{r}e^{2\,\mathrm{Re}(\alpha_0)l}\lesssim 1$, we may rewrite Eq. (3.15) as

$$1-\tilde{r}e^{2\,\mathrm{Re}(\alpha_0)l}=\frac{K}{2i_-(0)}\ ,$$

so that

$$\Delta\Omega=\frac{4}{t_R}\frac{1-\tilde{r}e^{2\,\mathrm{Re}(\alpha_0)l}}{2(\tilde{r}e^{2\,\mathrm{Re}(\alpha_0)l})^{1/2}}\approx\frac{K}{i_-(0)t_R}\ . \tag{3.17}$$

We may reexpress this result in terms of the total output power P_{out}, which is equal to $P_{\text{out}}=[2A\epsilon_0 ni_-(0)c](1-\tilde{r}^2)$. Thus, the linewidth $\Delta\Omega$ becomes

$$\Delta\Omega\approx\frac{2KnA\epsilon_0 c(1-\tilde{r}^2)}{t_R P_{\text{out}}}\ ,$$

which by definition (3.16) of K is equal to

$$\Delta\Omega\approx\frac{\hbar\Omega}{t_R^2}\frac{(1-\tilde{r}^2)^2}{\tilde{r}^2}\frac{\langle R_a^{(0)}\rangle}{\langle R_a^{(0)}\rangle-\langle R_b^{(0)}\rangle}\frac{\gamma t_R/2}{P_{\text{out}}}\left[1+\frac{\Delta^2}{\gamma^2}\right]\ . \tag{3.18}$$

At this point, it is worth reminding the reader that P_{out} is the *total* output power that is partitioned among *all* quasimodes oscillating under the atomic linewidth profile contained in Eq. (3.14). It is usual to express $\Delta\Omega$ not in terms of P_{out} but in terms of P_1, the output power contained in the *single* central quasimode. The relation between P_{out} and P_1, namely,

$$P_{\text{out}}\approx P_1(\gamma t_R/2)\ , \tag{3.19}$$

has been derived in Appendix C. So finally, in terms of P_1, we obtain the intrinsic laser linewidth as

$$\Delta\Omega\approx\frac{\hbar\Omega}{P_1 t_R^2}\frac{(1-\tilde{r}^2)^2}{\tilde{r}^2}\frac{\langle R_a^{(0)}\rangle}{\langle R_a^{(0)}\rangle-\langle R_b^{(0)}\rangle}\left[1+\frac{\Delta^2}{\gamma^2}\right]\ , \tag{3.20}$$

which has the same form as the result derived by Ujihara [17] and Henry [11] for an above-threshold laser. Note also the presence of the detuning dependent factor $(1+\Delta^2/\gamma^2)$, which is the analog of Henry's linewidth enhancement factor [11] for semiconductor lasers. This factor goes back to the work of Lax [2] in the late 1960s.

In terms of the total photon number $\bar{n}_{\text{ph}}$ in the central quasimode, which one may relate to $i_-(0)$ via the integral

$$\hbar\Omega\bar{n}_{\text{ph}}=An^2\epsilon_0\int_0^l dz(\langle e_+^\dagger e_+\rangle+\langle e_-^\dagger e_-\rangle)\ ,$$

Eq. (3.20) may be shown to be equivalent to

$$\Delta\Omega=\frac{\langle R_a^{(0)}\rangle}{\langle R_a^{(0)}\rangle-\langle R_b^{(0)}\rangle}\frac{(1-\tilde{r}^2)^2}{2\ln(1/\tilde{r})\tilde{r}^2 t_R}\frac{1}{\bar{n}_{\text{ph}}}\left[1+\frac{\Delta^2}{\gamma^2}\right]\ . \tag{3.21}$$

This expression tends to the correct value $\gamma_Q/\bar{n}_{\text{ph}}$ for high inversion, zero detuning, and the good-cavity limit ($\tilde{r}\rightarrow 1$), $\gamma_Q\approx(1-\tilde{r}^2)/t_R$ being the cavity decay rate. We shall, however, not present the arduous but straightforward demonstration of this equivalence here.

IV. STEADY-STATE OSCILLATION ABOVE THRESHOLD

If the laser is oscillating well above threshold, then its field intensity has an average that is large compared to its fluctuations. This is the characteristic regime of stable laser operation where questions concerning quantum-limited phase diffusion and linewidth are of experimental relevance. We shall therefore assume such operation. In this regime, the calculation of the average intensity in steady state can be performed semiclassically. We note, however, that the calculations of this section permit no time dependence of the fields and so cannot describe phase diffusion, which we shall cover in the following section.

On taking the expectation values of Eqs. (2.11) and setting the time derivatives as well as correlations equal to zero, we obtain the following simple algebraic equations:

$$\frac{\partial}{\partial z}e_\pm(z)=\pm\frac{i\Omega^2}{2\epsilon_0 kc^2}p_\pm(z)\ , \tag{4.1a}$$

$$(i\Delta+\gamma)p_\pm(z)=-\frac{i\mu^2}{\hbar}[R_a(z)-R_b(z)]e_\pm(z)\ , \tag{4.1b}$$

$$\gamma_a R_a(z)=\Lambda_a+\frac{i}{\hbar}(p_+^\dagger e_+ + p_-^\dagger e_- - p_+ e_+^\dagger - p_- e_-^\dagger)\ , \tag{4.1c}$$

and

$$\gamma_b R_b(z)=\Lambda_b-\frac{i}{\hbar}(p_+^\dagger e_+ + p_-^\dagger e_- - p_+ e_+^\dagger - p_- e_-^\dagger)\ . \tag{4.1d}$$

In the preceding equations, we have dropped the expectation value signs for brevity of notation.

In the absence of the radiation field, the population inversion (R_a-R_b) would have assumed the unsaturated value $\mathcal{N}_0$ given by Eq. (3.3b), which arises from the balance of pumping and damping. However, the coherent radiation field saturates the inversion and therefore the gain via stimulated emission and absorption processes. It is easy to solve Eqs. (4.1b)–(4.1d) for (R_a-R_b) in terms of $\mathcal{N}_0$ and the total saturating intensity proportional to $(|e_+|^2+|e_-|^2)$

$$R_a(z)-R_b(z)=\frac{\mathcal{N}_0}{1+[|e_+(z)|^2+|e_-(z)|^2]/I_s}\ ,\qquad (4.2)$$

where the saturation parameter I_s is defined by the relation

$$1/I_s\equiv\frac{2\mu^2}{\hbar^2}\left[\frac{1}{\gamma_a}+\frac{1}{\gamma_b}\right]\frac{\gamma}{\gamma^2+\Delta^2}\ .\qquad (4.3)$$

Eliminating $p_\pm$ between Eqs. (4.1a) and (4.1b) and then using Eq. (4.2), we obtain the following equation describing the steady-state spatial dependence of the radiation field inside the lasing cavity:

$$\frac{d}{dz}e_\pm(z)=\pm\frac{\alpha_0}{\{1+[|e_+(z)|^2+|e_-(z)|^2]/I_s\}}e_\pm(z)\ ,\qquad (4.4)$$

where α_0, the complex linear gain coefficient, is given by Eq. (3.3a). The sign of the right-hand side of (4.4) is such that for an inverted active medium $\mathcal{N}_0>0$, the rightward wave e_+ increases in amplitude as z increases from $z=0$ to $z=l$, while the leftward wave e_- increases in amplitude as z decreases from $z=l$ to $z=0$. In a traveling-wave picture [19], this corresponds to the wave field amplifying in time during a round trip.

The spatial dependence of the field is determined unequivocally by Eq. (4.4) and by the boundary conditions appropriate to the partially transmitting mirror at $z=0$ and the perfectly reflecting mirror at $z=l$

$$e_+(0)=-\tilde{r}e_-(0)\qquad (4.5)$$

and

$$e_+(l)e^{ikl}=-e_-(l)e^{-ikl}\ .\qquad (4.6)$$

We have assumed here that no coherent field is injected from outside through the mirror at $z=0$. Otherwise, the first of the boundary conditions above is modified by an additive term proportional to $\tilde{t}$ times the injected field.

A. Good-cavity limit

If $\tilde{r}$ is not far from 1, then $e_\pm(z)$ will not have a strong spatial dependence inside the cavity. The field does not have to amplify much to compensate for the small mirror loss. In that case we may assume that the saturation term in the denominator of (4.4) has no z dependence. In this good-cavity limit, one may solve Eq. (4.4) approximately as

$$e_\pm(z)\approx e_\pm(0)e^{\pm A_s z}\ ,\qquad (4.7)$$

in which A_s is the saturated gain coefficient defined in terms of the spatially averaged field intensity $\langle |e_+|^2+|e_-|^2\rangle_s$

$$A_s=\frac{\alpha_0}{1+[\langle |e_+|^2+|e_-|^2\rangle_s]/I_s}\ .\qquad (4.8)$$

By substituting $z=l$ into (4.7) we may demonstrate easily that the boundary conditions (4.5) and (4.6) will be satisfied if and only if the following condition is met:

$$\tilde{r}e^{2A_s l}e^{2ikl}=1\ .\qquad (4.9)$$

Taking the modulus and phase of this relation, we arrive at the steady-state amplitude and operating frequency conditions

$$\mathrm{Re}(A_s)=\frac{1}{2l}\ln\frac{1}{\tilde{r}}\qquad (4.10a)$$

and

$$kl+\mathrm{Im}(A_s)l=p\pi,\quad p=\text{integer}\ .\qquad (4.10b)$$

By noting that in the good-cavity limit $\tilde{r}\approx 1$, $\ln(1/\tilde{r})\approx(1-\tilde{r})\approx\tilde{t}^2/2$ to which the cavity damping rate γ_c is proportional, we recover from (4.10a) the usual steady-state condition of the saturated gain being equal to the cavity loss. The actual frequency of oscillation Ω is determined from (4.10b).

B. Cavity with arbitrary output coupling

Even for arbitrary values of the mirror reflectivity $\tilde{r}$, it is still possible to solve Eq. (4.4) exactly. As we shall see presently, this will yield the spatial dependence of the field intensity in an implicit form and the operating frequency in an explicit form. The results presented in this subsection are essentially identical to those of Rigrod [22], who considered saturation effects in an arbitrary-transmission lasing cavity; but we include them since we shall use them to calculate the linewidth in Sec. V.

The solution is facilitated by first writing the fields $e_\pm(z)$ in terms of their amplitude and phase as

$$e_\pm(z)=r_\pm(z)e^{i\theta_\pm(z)}\ .\qquad (4.11)$$

Equation (4.4) transforms to the following equations for $r_\pm$ and $\theta_\pm$:

$$\frac{d}{dz}r_\pm(z)=\pm\frac{\mathrm{Re}(\alpha_0)r_\pm(z)}{1+[r_+^2(z)+r_-^2(z)]/I_s}\qquad (4.12a)$$

and

$$\frac{d}{dz}\theta_\pm(z)=\pm\frac{\mathrm{Im}(\alpha_0)}{1+[r_+^2(z)+r_-^2(z)]/I_s}\ .\qquad (4.12b)$$

The boundary conditions (4.5) and (4.6) transform to the following relations:

$$r_+(0)=\tilde{r}r_-(0),\quad \theta_+(0)-\theta_-(0)=\pi\ \mathrm{mod}2\pi\qquad (4.13a)$$

and

$$r_+(l)=r_-(l),\quad \theta_+(l)-\theta_-(l)+2kl=\pi\ \mathrm{mod}2\pi\ .\qquad (4.13b)$$

Now, by dividing Eq. (4.12a) by Eq. (4.12b) we obtain a simple differential equation connecting $\theta_\pm(z)$ to $r_\pm(z)$. On solving this equation in conjunction with the spatial invariance of $\theta_+(z)+\theta_-(z)$, a result that follows from Eq. (4.12b), we get the following result:

$$\theta_+(l)-\theta_+(0)=-[\theta_-(l)-\theta_-(0)]=\frac{\mathrm{Im}(\alpha_0)}{\mathrm{Re}(\alpha_0)}\ln\frac{r_+(l)}{r_+(0)} . \tag{4.14}$$

By first subtracting the phase part of the boundary condition Eq. (4.13a) from that of Eq. (4.13b) and then by applying result (4.14) we achieve the operating-frequency condition

$$\frac{\mathrm{Im}(\alpha_0)}{\mathrm{Re}(\alpha_0)}\ln\frac{r_+(l)}{r_+(0)}+kl=p\pi , \tag{4.15}$$

in which p is an arbitrary integer.

We note that Eq. (4.15) still involves the unknown amplitudes $r_+(0)$ and $r_+(l)$. These may be determined by solving Eq. (4.12a). This is accomplished by noting that the product $r_+(z)r_-(z)$ is spatially invariant, since it follows from Eq. (4.12a) that

$$\frac{d}{dz}[r_+(z)r_-(z)]=r_-(z)\frac{d}{dz}r_+(z)+r_+(z)\frac{d}{dz}r_-(z)=0 . \tag{4.16}$$

By writing

$$r_+(z)r_-(z)=C , \tag{4.17}$$

in which C is a constant to be determined later, we obtain from Eq. (4.12a) the following equation for r_+ alone:

$$\frac{d}{dz}r_+(z)=\frac{\mathrm{Re}(\alpha_0)r_+(z)}{1+[r_+^2(z)+C^2/r_+^2(z)]/I_s} , \tag{4.18}$$

which is easily integrated into the form

$$I_s\ln\frac{r_+(z)}{r_+(0)}+\tfrac{1}{2}[r_+^2(z)-r_+^2(0)]-\frac{C^2}{2}[r_+^{-2}(z)-r_+^{-2}(0)]=\mathrm{Re}(\alpha_0)I_s z , \tag{4.19a}$$

or equivalently, via (4.17), as

$$I_s\ln\frac{r_+(z)}{r_+(0)}+\tfrac{1}{2}[r_+^2(z)-r_-^2(z)-r_+^2(0)+r_-^2(0)]=\mathrm{Re}(\alpha_0)I_s z . \tag{4.19b}$$

Equation (4.19a) [or (4.19b)] represents the implicit solution of $r_+(z)$ in terms of z.

The constant C is specified uniquely by the boundary conditions (4.13). By putting $z=l$ in Eq. (4.19b) and using the amplitude part of Eq. (4.13b), we get the simple relation

$$I_s\ln\frac{r_+(l)}{r_+(0)}-\tfrac{1}{2}[r_+^2(0)-r_-^2(0)]=\mathrm{Re}(\alpha_0)I_s l . \tag{4.20}$$

Note that Eqs. (4.17) and (4.13) imply that

$$r_+^2(l)=r_+(l)r_-(l)=C \tag{4.21a}$$

and

$$r_+^2(0)=\tilde{r}^2r_1^2(0)=\tilde{r}C , \tag{4.21b}$$

and therefore from Eq. (4.20) it is now easy to determine C as follows:

$$C=\frac{\left[2\,\mathrm{Re}(\alpha_0)l-\ln\dfrac{1}{\tilde{r}}\right]}{(1/\tilde{r}-\tilde{r})}I_s . \tag{4.22}$$

In Figs. 2(a) and 2(b) we display the spatial dependence

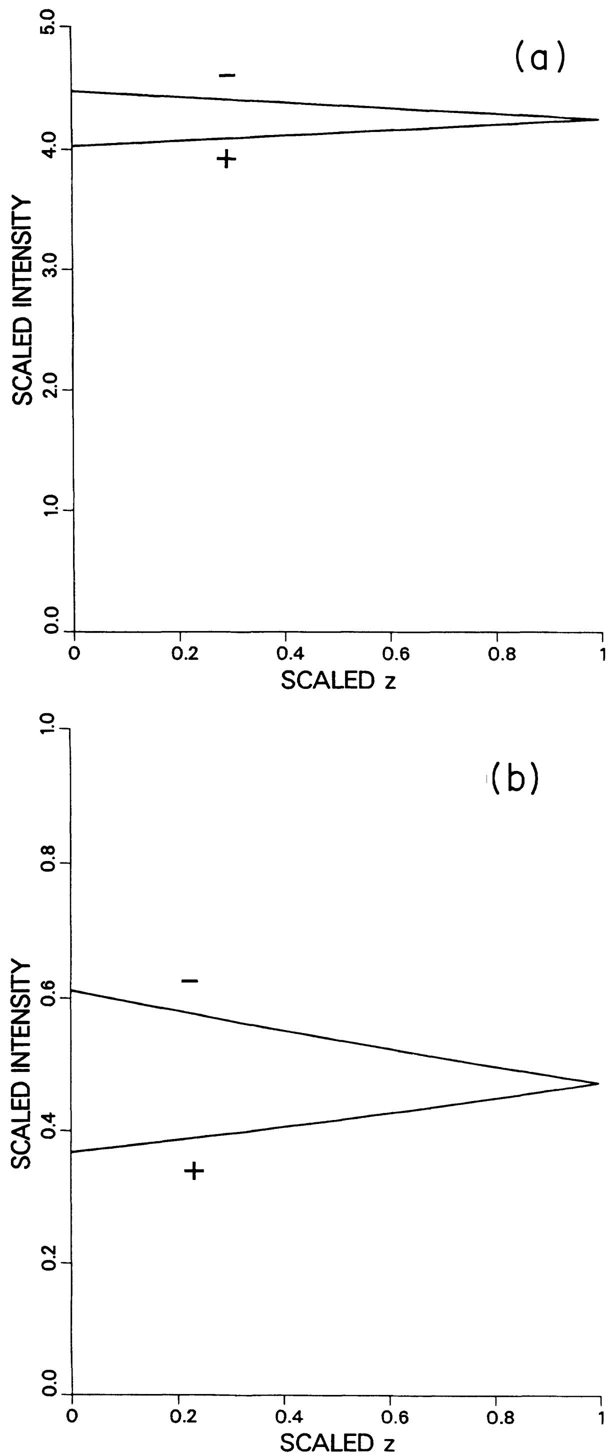

FIG. 2. The steady-state intensities of the right and left traveling pieces of the circulating coherent laser field. (a) $2\,\mathrm{Re}(\alpha_0)l=0.5$, $\tilde{r}^2=0.9$ and (b) $2\,\mathrm{Re}(\alpha_0)l=0.5$, $\tilde{r}^2=0.6$.

of the steady-state intensities $r_+^2(z)$ and $r_-^2(z)$ in the right- and left-traveling parts of the radiation field inside the lasing cavity. The intensities are shown scaled relative to the saturation parameter I_s and plotted for the round-trip linear gain coefficient $2\,\text{Re}(\alpha_0)l=0.5$ and two values of the mirror reflectance $\tilde{r}^2$. For values of $\tilde{r}$ that are not too small, the spatial dependence of the intensity can be well approximated as a simple linear growth.

Since $\text{Im}(\alpha_0)/\text{Re}(\alpha_0)=-\Delta/\gamma$ and since $r_+(l)/r_+(0)=1/(\tilde{r}^{1/2})$ from Eq. (4.21), we may express the frequency condition (4.15) in the simpler form

$$kl-\frac{\Delta}{2\gamma}\ln\frac{1}{\tilde{r}}=p\pi\ . \tag{4.23}$$

By writing $k=n\Omega/c$ and $\Delta=\omega_0-\Omega$, we derive the following result for Ω:

$$\Omega=\frac{\omega_0\gamma_c+\Omega_c\gamma}{\gamma_c+\gamma}\ , \tag{4.24}$$

in which $\Omega_c\equiv p\pi c/nl$ is an empty-cavity quasimode frequency and

$$\gamma_c\equiv\frac{c}{2nl}\ln\frac{1}{\tilde{r}} \tag{4.25}$$

reduces to the familiar cavity damping rate in the good-cavity limit. We note that in spite of its simple appearance, Eq. (4.24) is valid for arbitrary values of the saturating field intensity $r_+^2+r_-^2$ and reflectance $\tilde{r}^2$.

Having established the steady-state operating point for the average intensity and oscillation frequency, we are now in a position to address questions concerning phase diffusion and linewidth that arise from quantum-mechanical fluctuations about this operating point. In the absence of any phase locking, the phase of the laser field is unconstrained and therefore diffuses freely on account of random kicks from spontaneous emissions into off-axis modes and on-axis longitudinal modes, the latter emissions being driven mostly by transmitted vacuum fluctuations.

V. INTRINSIC LINEWIDTH ABOVE THRESHOLD

As we have noted before, in a steady-state laser oscillation above threshold, the intensity is high on average and well stabilized by the equality of gain and loss. So, the intrinsic linewidth of the laser light is determined almost solely by phase diffusion [23].

To address phase diffusion, we go back to the field-evolution equation (2.11a) and to the atomic evolution equations (2.11b)–(2.11d) with the latter equations to be treated in our improved adiabatic approximation in which the atomic relaxation rates γ_a, γ_b, and γ are all much greater than the field damping and phase-diffusion rates. In this approximation, the atomic variables quickly adjust to the slower variations of the field. Mathematically, the approximation amounts to setting the time derivatives in Eqs. (2.11c) and (2.11d) to zero. Further, as we have noted before, we also ignore any fluctuations of the population density operators R_a and R_b. In effect, we may calculate $\langle R_a\rangle$ and $\langle R_b\rangle$ in steady state as done in the previous section. So we may take result (4.2) for the population inversion density and substitute it into Eq. (2.13) to obtain

$$p_\pm(z,t)\approx\frac{-i\mu^2}{\hbar(\gamma+i\Delta)}\times\frac{\mathcal{N}_0}{1+[\langle|e_+(z)|^2\rangle+\langle|e_-(z)|^2\rangle]/I_s}e_\pm(z,t)+\mu\int_0^\infty f_\pm(z,t-t')e^{-(\gamma+i\Delta)t'}dt'\ . \tag{5.1}$$

On combining Eq. (5.1) with Eq. (2.11a), we have the following evolution equation for the laser field:

$$\left[\frac{\partial}{\partial z}\pm\frac{n}{c}\frac{\partial}{\partial t}\right]e_\pm(z,t)=\pm\alpha_s(z)e_\pm(z,t)+\phi_\pm(z,t)\ , \tag{5.2}$$

in which $\alpha_s(z)$ is the spatially nonuniform saturated gain coefficient given by the relation

$$\alpha_s(z)=\frac{\alpha_0}{1+[\langle|e_+(z)|^2\rangle+\langle|e_-(z)|^2\rangle]/I_s}\ . \tag{5.3}$$

We note that Eq. (5.2) is formally identical to Eq. (3.2), except that the linear gain coefficient α_0 is replaced by the saturated gain function $\alpha_s(z)$, which depends on the intensity in both the rightward and leftward beams.

We now look at the imaginary part of Eq. (5.2), treating the electric field classically consistent with the approximation that the amplitudes are well stabilized and only the phases evolve in time. If $\theta_\pm(z,t)$ represents the phases of the fields $e_\pm(z,t)$, then the phases evolve according to the relation

$$\left[\frac{\partial}{\partial z}\pm\frac{n}{c}\frac{\partial}{\partial t}\right]\theta_\pm(z,t)=\pm\text{Im}[\alpha_s(z)]+F_\pm(z,t)\ , \tag{5.4}$$

where the noise forces driving the phase fluctuations are related to the $\theta_\pm$ quadrature of $\phi_\pm$ via the (steady-state) amplitudes $r_\pm(z)\equiv|e_\pm(z)|$

$$F_\pm(z,t)=\frac{1}{2ir_\pm(z)}[\phi_\pm(z,t)e^{-i\theta_\pm(z,t)}-\phi_\pm^\dagger(z,t)e^{i\theta_\pm(z,t)}]\ . \tag{5.5}$$

The solution of Eq. (5.4) is analogous to that of Eq. (3.2), when one uses the boundary conditions on the phases that follow from Eqs. (3.6). These boundary conditions on $\theta_\pm(z,t)$ are derived in Appendix D. Driven by the vacuum field and the atomic spontaneous emission noise, the phase $\theta_-(0,t)$ of the leftward field at the output mirror evolves via the difference equation

$$\begin{aligned}\theta_-(0,t+t_R)-\theta_-(0,t)&=\epsilon_\theta(t)+\int_0^l dz'F_+(z',t+nz'/c)\\&\quad-\int_0^l dz'F_-(z',t+n(2l-z')/c)\\&\quad-2\int_0^l\text{Im}[\alpha_s(z)]dz-2kl\ \text{mod}2\pi\ ,\end{aligned} \tag{5.6}$$

where $\epsilon_\theta(t)$ is related via Eq. (D4) to the θ_+ quadrature of the transmitted-in vacuum field at $z=0$

$$\epsilon_\theta(t)=\frac{(\tilde{t}/n)}{2ir_+(0)}[e_+^{(\mathrm{vac})}(0,t)e^{-i\theta_+(0,t)}-e_+^{(\mathrm{vac})\dagger}(0,t)e^{i\theta_+(0,t)}] . \quad (5.7)$$

Since one may show that (see Appendix E)

$$2\int_0^l \mathrm{Im}[\alpha_s(z)]dz+\frac{\Delta}{\gamma}\ln\frac{1}{\tilde{r}}=0 , \quad (5.8)$$

it follows that at the steady-state operating point given by Eq. (4.23),

$$2\int_0^l \mathrm{Im}[\alpha_s(z)]dz+2kl=0 \bmod 2\pi ,$$

so that Eq. (5.6) simplifies to

$$\theta_-(0,t+t_R)-\theta_-(0,t)=\epsilon_\theta(t)+\int_0^l dz'F_+(z',t+nz'/c)-\int_0^l dz'F_-(z',t+n(2l-z')/c) . \quad (5.9)$$

This expression is exact (and not indefinite up to an additive multiple of 2π) since the change in the field phase from one round trip to the next can only be small, which each one of the quantities on the right-hand side of Eq. (5.9) already is.

From Eq. (5.9), which describes how the phase evolves over one round trip, one may compute how the phase evolves over N round trips by considering Eq. (5.9) N times, once for each round trip, and adding all such successive round-trip contributions

$$\Delta\theta(t)\equiv[\theta_-(0,t=Nt_R)-\theta_-(0,0)]=\sum_{p=0}^{N-1}\left[\epsilon_\theta(t+pt_R)+\int_0^l dz'F_+(z',t+pt_R+nz'/c)-\int_0^l dz'F_-(z',t+pt_R+n(2l-z')/c\right] . \quad (5.10)$$

The variance of $\Delta\theta(t)$, which determines the linewidth, is then obtained in terms of bilinear moments of ϵ_θ and of $F_\pm$. From definition (5.7) of ϵ_θ, the bilinear two-time moments of $\epsilon_\theta(t)$ are obtained in terms of such moments of the positive- and negative-frequency parts of the vacuum field. These moments are of course zero unless the two times are one and the same [see Eqs. (B3) and (B5) in Appendix B]. It is evident from Eqs. (3.4) and (5.5) that the bilinear moments of $F_\pm$ are obtained in terms of the diffusion coefficients for $f_\pm$ noise operators. These moments have been computed in Appendices B and D. From the form of Eq. (5.10), it is clear that the variance of $\Delta\theta$ involves single-time second-order moments of $\epsilon_\theta(t)$ and of $F_\pm(z,t)$, as well as their two-time moments. The latter are all either exactly zero (for ϵ_θ) or are close to zero (for $F_\pm$, whose correlation time of order γ^{-1} is very short compared with the round-trip time t_R). Since all single-time moments of ϵ_θ are equal, as they are separately also for F_+ and for F_-, it follows that $\langle[\Delta\theta(t)]^2\rangle$ is proportional to N, or equivalently, to t. Thus, we may write

$$\langle[\Delta\theta(t)]^2\rangle=2Dt .$$

The proportionality constant $2D$ is the FWHM linewidth $\Delta\Omega$ of the laser field arising from a Gaussian random process like phase diffusion. Clearly, $2D$ is the sum of a vacuum-fluctuation piece $2D_{\mathrm{vac}}$ and a spontaneous-emission piece $2D_{\mathrm{sp}}$

$$\Delta\Omega=2D=2D_{\mathrm{vac}}+2D_{\mathrm{sp}} . \quad (5.11)$$

From the preceding arguments, it may be seen that

$$2D_{\mathrm{vac}}=\frac{1}{t_R}\langle\epsilon_\theta^2\rangle , \quad (5.12)$$

which, with the aid of result (D5) of Appendix D and the relation

$$P_1=2A\epsilon_0 nc(1-\tilde{r}^2)[r_+^2(0)/\tilde{r}^2]$$

between the output power P_1 and the field amplitude $r_+(0)$, may be written as

$$2D_{\mathrm{vac}}=\frac{\hbar\Omega n}{4P_1t_R^2}\frac{(1-\tilde{r}^2)^2}{\tilde{r}^2} . \quad (5.13)$$

On the other hand, D_{sp} involves a double integral of the single-time moments of $F_\pm(z,t)$ over the cavity length [see Eq. (5.9)]. These moments, computed in Eqs. (B21) of Appendix B, depend on the diffusion constants $D_{\Sigma\Sigma^\dagger}(z)$ and $D_{\Sigma^\dagger\Sigma}(z)$ of $f_\pm$, which are in turn expressed in terms of spatially nonuniform fields and locally saturated populations via Eqs. (B10′) and (B11′). Thus, $2D_{\mathrm{sp}}$ may be shown to be equal to

$$2D_{\mathrm{sp}}=\frac{1}{2A\gamma t_R}\left[\frac{\mu\Omega^2}{4\epsilon_0kc^2}\right]^2\int_0^l dz\left[\frac{1}{r_+^2(z)}+\frac{1}{r_-^2(z)}\right]\times\{[\Lambda_a-\gamma_a\langle R_a(z)\rangle]+[\Lambda_b-\gamma_b\langle R_b(z)\rangle]+2\gamma[\langle R_a(z)\rangle+\langle R_b(z)\rangle]\} . \quad (5.14)$$

The integral in Eq. (5.14) is quite involved, but may be carried out analytically as follows. One first notes from Eqs. (4.1c) and (4.1d) that the sum of the quantities in the first two pairs of square brackets in Eq. (5.14) vanishes exactly. The explicit expression for the sum of the saturated populations $\langle R_a(z)\rangle$ and $\langle R_b(z)\rangle$, which is all that survives in the integrand, is obtained easily by solving Eqs. (4.1b)—(4.1d) with the help of Eq. (4.2).

Before we state the explicit expressions for $\langle R_a(z)\rangle$ and $\langle R_b(z)\rangle$, we should like to point out that in Ujihara's treatment [17] the aforementioned sum is forced to become independent of the field strength (and therefore of z) and in fact equal to the total number density n_0 of atoms. This is a result of his inconsistent assumption, insofar as his expressions for the noise moments $D_{\Sigma\Sigma^\dagger}$ and $D_{\Sigma^\dagger\Sigma}$ (G_{mm} and $G_{mm'}$ in his notation) are concerned, that each laser atom has exactly two levels. Nevertheless, the final form of this linewidth expression is correct if saturation can be ignored, as one might expect on the basis of this assumption.

The explicit expressions for the saturated population densities are

$$\langle R_a(z)\rangle=\langle R_a^{(0)}\rangle-\frac{\mathcal{N}_0}{\gamma_a/\gamma_b+1}\frac{[r_+^2(z)+r_-^2(z)]}{[I_s+r_+^2(z)+r_-^2(z)]} \tag{5.15a}$$

and

$$\langle R_b(z)\rangle=\langle R_b^{(0)}\rangle+\frac{\mathcal{N}_0}{\gamma_b/\gamma_a+1}\frac{[r_+^2(z)+r_-^2(z)]}{[I_s+r_+^2(z)+r_-^2(z)]} . \tag{5.15b}$$

Via the use of these expressions in Eq. (5.14), the expression for $D_{\rm sp}$ becomes

$$2D_{\rm sp}=\frac{1}{At_R}\left[\frac{\mu\Omega^2}{4\epsilon_0 kc^2}\right]^2 \times\left[(\langle R_a^{(0)}\rangle+\langle R_b^{(0)}\rangle)I+\mathcal{N}_0\frac{(\gamma_a-\gamma_b)}{(\gamma_a+\gamma_b)}J\right] , \tag{5.16}$$

where I and J are the following two integrals:

$$I=\int_0^l dz\left[\frac{1}{r_+^2(z)}+\frac{1}{r_-^2(z)}\right] \tag{5.17}$$

and

$$J=\int_0^l dz\frac{[r_+^2(z)+r_-^2(z)]^2}{r_+^2(z)r_-^2(z)[I_s+r_+^2(z)+r_-^2(z)]} . \tag{5.18}$$

The calculation of these two integrals is carried out in detail in Appendix E.

By making use of expressions (E2) and (E3) from Appendix E and the fact that the output power P_1 in the central surviving quasimode is down by the factor [24] $\gamma t_R/2$ from the power $P_{\rm out}$ related to $r_-^2(0)=C/\tilde{r}$ [see Eq. (4.21b)] via

$$P_{\rm out}=2A\epsilon_0 ncr_-^2(0)(1-\tilde{r}^2) , \tag{5.19}$$

we may finally express $D_{\rm sp}$ in the form

$$2D_{\rm sp}=\frac{\hbar\Omega}{4t_R^2}\left[\frac{(\langle R_a^{(0)}\rangle+\langle R_b^{(0)}\rangle)}{(\langle R_a^{(0)}\rangle-\langle R_b^{(0)}\rangle)}\frac{(1-\tilde{r}^2)^2}{P_1\tilde{r}^2} +\frac{1}{P_{\rm sat}}\left[\ln\frac{1}{\tilde{r}^2}+\frac{1}{2\tilde{r}^2}(1-\tilde{r}^4)\right]\right] . \tag{5.20}$$

The power-independent piece in Eq. (5.20) is given in terms of a saturation power

$$P_{\rm sat}=nc\epsilon_0 A\left[\frac{\hbar^2\gamma\gamma_a\gamma_b}{2\mu^2}\right]\left[\frac{\langle R_a^{(0)}\rangle-\langle R_b^{(0)}\rangle}{\Lambda_a+\Lambda_b}\right] .$$

The presence of the power-independent term in Eq. (5.20) is a key result of this paper. It is related to the saturation of population densities and its spatial nonuniformity for laser operation well above threshold. We wish to note that a careful treatment of the laser linewidth above threshold even in the good-cavity limit should contain an analogous term, as also noted by Yariv and Vahala [25], since indeed such a treatment must express the linewidth in terms of saturated population densities, not the unsaturated ones. However, as we have demonstrated in the present work, the dependence of this term on the cavity outcoupling becomes more and more nontrivial the larger the outcoupling, since with increasing outcoupling the spatial nonuniformity of gain saturation becomes important. The relation of this power-independent term to an observed power-independent contribution to the linewidth of semiconductor lasers [10] is worthy of further investigation.

The dependence of the linewidth on the mirror transmission coefficient $T\equiv(1-\tilde{r}^2)$ is quite different above threshold from that below threshold [Eq. (3.20)]. This difference arises mainly in the power-independent term of Eq. (5.20), which results from nonuniform gain saturation. This is in contrast with the results of Ujihara [17], Henry [11], and Hamel and Woerdman [18], who for fixed output power predict a dependence on mirror reflectivity that is in fact identical to Eq. (3.20). Above threshold, Ujihara's results, as we have discussed above, differ from our results, applying correctly only when saturation is negligible. Since Henry and Hamel and Woerdman do not include nonuniform gain saturation, their results are naturally quite different from ours.

We plot in Figs. 3(a) and 3(b) the intrinsic linewidth versus the output-mirror power transmittance $T\equiv(1-\tilde{r}^2)$ for a fixed output power in a gas laser ($n\approx 1$) for high inversion and for two values of $P_1/P_{\rm sat}$.

For $P_1/P_{\rm sat}=0.2$, which may not be unreasonable for a diode laser too, the two upper curves do not differ by more than a few percent over much of the range of T. The distributed-loss approximation to the linewidth just above threshold [6] for high inversion

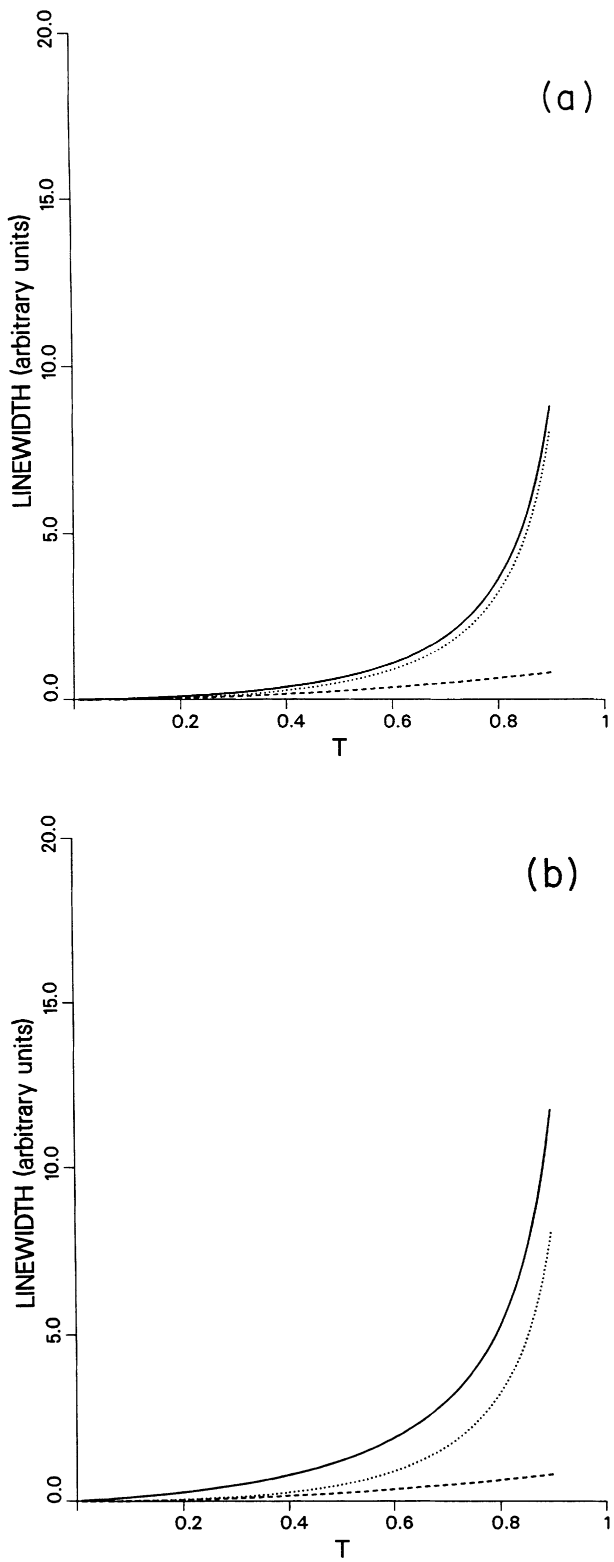

FIG. 3. Intrinsic linewidth (in arbitrary units) vs output-mirror transmittance T well above threshold for fixed output power and with $\Lambda_b = 0$. The dashed curve is the distributed-loss approximation; the dotted curve is the result of Ujihara, Henry, and Hamel and Woerdman; the solid curve is the result of the present work, for (a) $P_1/P_{sat} = 0.2$ and for (b) $P_1/P_{sat} = 1.0$.

$$\Delta\Omega = \frac{\hbar\Omega}{2P_1 t_R^2}(1-\tilde{r}^2)^2$$

fails badly even for moderate values of T. On the other hand, when the laser power is in the highly saturated domain $P_1/P_{sat} = 1.0$, the distinction between the solid curve representing the result of the present paper [given by Eqs. (5.11), (5.13), and (5.20)] and the other two curves is dramatic for all values of T.

VI. CONCLUSIONS

In this paper, we have introduced an approach based on boundary conditions to problems involving interaction of light with matter residing in cavities whose output coupling can be arbitrary. In particular, we have analyzed in great detail the steady-state operating point and the intrinsic linewidth, both below and above threshold, of a homogeneously broadened laser whose output mirror has arbitrary transmission. The nature of the two contributions to the intrinsic linewidth, namely, spontaneous emission into the on-axis longitudinal modes and into the off-axis modes, was clarified, and it was shown that the two linewidth contributions are affected differently as the cavity Q value is reduced. In fact, the off-axis spontaneous-emission contribution to phase diffusion of the (on-axis) coherent field has a power-independent piece above threshold, in contrast with most previous theories of the laser linewidth. Below threshold, the intrinsic linewidth (of a single quasimode of the field) increases with increasing cavity output transmission to the point where it becomes comparable to the natural linewidth of emission from free-space atoms. At that point, the concept of single-quasimode linewidth loses its meaning, as it must.

In spite of its appearance, the present treatment of the intrinsic linewidth is quantum mechanically rigorous. We have, however, not treated diffractive losses from the laser cavity, which is another kind of output coupling with a contribution to the intrinsic linewidth. A recent paper of Deutsch and Garrison [26] espouses a quantum mechanically consistent paraxial expansion to describe diffraction of a traveling electromagnetic field, but the usefulness of a similar expansion for properly treating the linewidth arising from diffractive losses from a standing-wave laser cavity remains unclear.

The author recently came across two papers, one by Goldberg, Milonni, and Sundaram [27] and another by van Exter, Hamel, and Woerdman [28], on the subject of the intrinsic linewidth for arbitrary outcoupling. The former paper adopts an approach that is quite similar to the approach of the present paper. That work, however, addresses somewhat different aspects of the linewidth problem. It focuses on the inclusion of spatial hole burning and proper interpretations of the intrinsic linewidth and the excess-spontaneous-emission noise factor, the so-called Petermann K factor [29]. The authors arrive at these interpretations by reference to the detailed results they derive for laser operation below threshold in the unsaturated regime, although they justify the validity of these interpretations for above-threshold saturated-regime operation as well. In contrast, the present work establishes two different but equally important results, namely, the manner in which the natural atomic linewidth serves to limit the laser linewidth below threshold and the detailed treatment of the power-independent linewidth above threshold and its complete dependence on the cavity outcoupling. The second paper continues the work of Woerdman and co-workers to calculate the K factor, focusing on a traveling-wave alternative to their previous calculation in terms of nonorthogonal eigenmodes of a leaky cavity. They also estimate the effects of saturation on linewidth but do not provide a comprehensive treatment.

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APPENDIX A

Projection of left- and right-traveling pieces from the dipole fluctuation operator $\mu F^-(z,t)$

On substituting Eqs. (2.10a) and (2.10b) into Eq. (2.8a) we obtain

$$\left[\frac{\partial}{\partial t}+\gamma\right][p_+(z,t)e^{ikz}+p_-(z,t)e^{-ikz}] = \frac{\mu^2}{i\hbar}[R_a(z,t)-R_b(z,t)] \times[e_+(z,t)e^{ikz}+e_-(z,t)e^{-ikz}]+\mu F^-(z,t)e^{i\Omega t} . \tag{A1}$$

We may "project out" the $p_+(z,t)$ [or $p_-(z,t)$] dynamics from this equation by multiplying both sides by e^{-ikz} (or e^{ikz}), integrating the resulting equation over the range $(z-\lambda/2,z+\lambda/2)$, and by noting that since $p_\pm(z,t)$ and $e_\pm(z,t)$ are slowly varying in z over the scale of a wavelength, integrations of the form $\int p_\mp(z,t)\exp(\mp 2ikz)dz$ and $\int e_\mp(z,t)\exp(\mp 2ikz)dz$ over the stated range are negligible. This procedure then yields

$$\left[\frac{\partial}{\partial t}+\gamma\right]p_+(z,t)=\frac{\mu^2}{i\hbar}[R_a(z,t)-R_b(z,t)]e_+(z,t) +\mu f_+(z,t) , \tag{A2}$$

where

$$f_+(z,t)=\frac{e^{i\Omega t}}{\lambda}\int_{z-\lambda/2}^{z+\lambda/2}F^-(z',t)e^{-ikz'}dz' . \tag{A3}$$

To simplify Eq. (A3), we split the integration domain into N subintervals, each of width Δz, the smoothing interval introduced earlier. Thus,

$$f_+(z,t)\cong e^{i\Omega t}\frac{1}{\lambda}\sum_{i=1}^{N}\frac{1}{A\,\Delta z}\left[\sum_{i_z}^{\Delta z}f_{i_z}^- e^{-ikz_{i_z}}\right]\Delta z = e^{i\Omega t}\frac{1}{\lambda A}\sum_{i_z}^{\lambda}f_{i_z}^- e^{-ikz_{i_z}} , \tag{A4}$$

in which the superscript λ over the summation sign represents a sum over all atoms i_z belonging to the interval $(z-\lambda/2,z+\lambda/2)$.

APPENDIX B

1. Two-time moments of the input field $e_+(0,t)$ in a thermal state

In Ref. [16], the general expansion of the electromagnetic field operator in terms of the modes of the universe was written down. The rightward-traveling component of the electromagnetic field just outside the cavity at $z=0^-$ is given by the expression

$$e_+(0^-,t)=\frac{1}{2i}\sum_k\left[\frac{\hbar\Omega_k}{\epsilon_0 AL}\right]^{1/2}\xi_k a_k e^{ikL}e^{-i\delta\Omega_k t} , \tag{B1}$$

where ξ_k alternates between $+1$ and -1 from one universe mode to the next, L is the length of the one-dimensional universe, and $\delta\Omega_k=\Omega_k-\Omega$. From this expression, the normally ordered, two-time moment of the electromagnetic field in the input state, which we take to be the thermal blackbody field, is the following:

$$\begin{aligned}\langle e_+^{(\mathrm{in})\dagger}(0,t)e_+^{(\mathrm{in})}(0,t')\rangle &\cong\frac{1}{4}\frac{\hbar\Omega}{\epsilon_0 AL}\sum_k|\xi_k|^2\bar{n}_k e^{i\delta\Omega_k(t-t')}\\ &\simeq\frac{\hbar\Omega}{4\epsilon_0 AL}\frac{L}{c\pi}\bar{n}\int_{-\infty}^{+\infty}d(\delta\Omega_k)e^{i\delta\Omega_k(t-t')}\\ &=\frac{\hbar\Omega\bar{n}}{2\epsilon_0 Ac}\delta(t-t') ,\end{aligned} \tag{B2}$$

in which $\bar{n}_k\simeq\bar{n}$ represents the average photon number in one mode of the broad-bandwidth blackbody field. In the vacuum state, $\bar{n}=0$ and so,

$$\langle e_+^{(\mathrm{vac})\dagger}(0,t)e_+^{(\mathrm{vac})}(0,t')\rangle=0 . \tag{B3}$$

Similarly, the antinormally ordered moment is

$$\langle e_+^{(\mathrm{in})}(0,t)e_+^{(\mathrm{in})\dagger}(0,t')\rangle\cong\frac{\hbar\Omega}{2\epsilon_0 Ac}(\bar{n}+1)\delta(t-t') , \tag{B4}$$

which reduces, in the vacuum state, to

$$\langle e_+^{(\mathrm{vac})}(0,t)e_+^{(\mathrm{vac})\dagger}(0,t')\rangle\simeq\frac{\hbar\Omega}{2\epsilon_0 Ac}\delta(t-t') . \tag{B5}$$

2. Single-time second-order moments of the input field in a thermal state

Note that

$$\langle e_+^{(\mathrm{in})\dagger}(0,t)e_+^{(\mathrm{in})}(0,t)\rangle=\frac{\hbar\Omega}{4\epsilon_0 AL}\sum_k\bar{n}_k .$$

Clearly, the k sum diverges for $\bar{n}_k=(e^{\hbar\omega/kT}-1)^{-1}$ since $\int_0^\infty d\omega(e^{\hbar\omega/kT}-1)^{-1}$ has a logarithmic singularity coming form the lower integration limit. The singularity is of course "cured" in three dimensions, since the density of states provides a regularizing factor there. However, we are interested in the on-axis field only. So, we cure this problem differently: we assume that there is only one quasimode with wave vectors in the range $[-\pi/2l+k,\pi/2l+k]$ involved, so that

$$\langle e_+^{(\mathrm{in})\dagger}(0,t)e_+^{(\mathrm{in})}(0,t)\rangle \cong \frac{\hbar\Omega\bar{n}}{4\epsilon_0 AL}\frac{L}{l}=\frac{\hbar\Omega\bar{n}}{4\epsilon_0 Al} \tag{B6}$$

and

$$\langle e_+^{(\mathrm{in})}(0,t)e_+^{(\mathrm{in})\dagger}(0,t)\rangle \simeq \frac{\hbar\Omega(\bar{n}+1)}{4\epsilon_0 Al} . \tag{B7}$$

For Q quasimodes, each of the expressions (B6) and (B7) gets multiplied by Q. For the special case of the vacuum state,

$$\langle e_+^{(\mathrm{vac})\dagger}(0,t)e_+^{(\mathrm{vac})}(0,t)\rangle=0 \tag{B8}$$

and

$$\langle e_+^{(\mathrm{vac})}(0,t)e_+^{(\mathrm{vac})\dagger}(0,t)\rangle=\frac{\hbar\Omega}{4\epsilon_0 Al} . \tag{B9}$$

3. Second-order moments of the noise operators $f_\pm(z,t)$

Since $f_\pm(z,t)$ are composed of the individual atomic dipole fluctuation operators f_i^- via Eq. (A4), we first need the second-order moments of f_i^-, which we find, for example, in Ref. [6], Chap. 20,

$$\langle f_i^+(t)f_j^-(t')\rangle=2D_{\Sigma^\dagger\Sigma}\delta_{ij}\delta(t-t') \tag{B10}$$

and

$$\langle f_i^-(t)f_j^+(t')\rangle=2D_{\Sigma\Sigma^\dagger}\delta_{ij}\delta(t-t') , \tag{B11}$$

in which, in terms of the number density n_0 of atoms,

$$2n_0D_{\Sigma^\dagger\Sigma}=\Lambda_a+(2\gamma-\gamma_a)\langle R_a(z,t)\rangle , \tag{B10$'$}$$

$$2n_0D_{\Sigma\Sigma^\dagger}=\Lambda_b+(2\gamma-\gamma_b)\langle R_b(z,t)\rangle . \tag{B11$'$}$$

Use of these relations in Eq. (A4) implies

$$\begin{aligned}\langle f_+^\dagger(z,t)f_+(z',t')\rangle &= \frac{e^{-i\Omega(t-t')}}{A^2\lambda^2}\sum_i\sum_{i'}e^{ik(z_i-z_{i'})}2D_{\Sigma^\dagger\Sigma}\delta_{ii'}\delta(t-t')\\ &=\frac{1}{A^2\lambda^2}2D_{\Sigma^\dagger\Sigma}\delta(t-t')\left[\sum_i^\lambda 1\right]\delta_{z,z'} ,\end{aligned}$$

where the symbol $\delta_{z,z'}$ represents 1 if z and z' refer to the same slice (of thickness λ) and 0 otherwise. Note that in the limit $\lambda\to0$, $\delta_{z,z'}/\lambda\to\delta(z-z')$, so that

$$\begin{aligned}\langle f_+^\dagger(z,t)f_+(z',t')\rangle &=\frac{1}{A^2\lambda^2}2D_{\Sigma^\dagger\Sigma}\delta(t-t')n_0A\lambda^2\delta(z-z')\\ &=\frac{2n_0D_{\Sigma^\dagger\Sigma}}{A}\delta(t-t')\delta(z-z')\end{aligned} \tag{B12}$$

on distance scales much greater than λ. Similarly,

$$\langle f_+(z,t)f_+^\dagger(z',t')\rangle=\frac{2n_0D_{\Sigma\Sigma^\dagger}}{A}\delta(t-t')\delta(z-z') , \tag{B13}$$

while the other two left-right-mixed moments

$$\langle f_+^\dagger(z,t)f_-(z',t')\rangle,\quad \langle f_+(z,t)f_-^\dagger(z',t')\rangle$$

involve the sums $\sum_i^\lambda e^{\pm 2ikz_i}$, which are identically zero, since

$$\begin{aligned}\sum_i^\lambda e^{\pm 2ikz_i} &= n_0A\int_{z-\lambda/2}^{z+\lambda/2}dz'e^{(4\pi i/\lambda)z'}\\ &=n_0Ae^{\pm 2ikz}\frac{1}{2k}\int_{-2\pi}^{2\pi}e^{i\theta}d\theta\\ &=0 .\end{aligned}$$

4. Normally ordered two-time moments of the noise operator $G(t)$

We assume that the incoming external field is in the vacuum state, so that Eqs. (B3) and (B5) apply. Then, from Eqs. (3.8) and (B12) it follows that

$$\begin{aligned}\langle G^\dagger(t_1)G(t_2)\rangle &= \left[\frac{\mu\Omega^2}{2\epsilon_0kc^2}\right]^2\frac{2n_0}{A}D_{\Sigma^\dagger\Sigma}\\ &\quad\times\int_0^l dz'\int_0^\infty dt'\int_0^\infty dt''e^{-\gamma(t'+t'')+i\Delta(t'+t'')}\delta(t_1-t_2-t'+t'')[e^{2\,\mathrm{Re}(\alpha_0)(2l-z')}+e^{2\,\mathrm{Re}(\alpha_0)z'})\\ &=\left[\frac{\mu\Omega^2}{2\epsilon_0kc^2}\right]^2\frac{2n_0}{A}D_{\Sigma^\dagger\Sigma}\frac{(e^{2\,\mathrm{Re}(\alpha_0)l}+1)(e^{2\,\mathrm{Re}(\alpha_0)l}-1)}{2\,\mathrm{Re}(\alpha_0)}\\ &\quad\times e^{i\Delta(t_1-t_2)}\int_0^\infty dt'\int_0^\infty dt''e^{-\gamma(t'+t'')}\delta(t_1-t_2-t'+t'') .\end{aligned} \tag{B14}$$

The t' and t'' integrations may be done by first noting that the result must be invariant under the interchange of t_1 and t_2. Thus, we may assume, for the moment, that $t_1 > t_2$. Then, since

$$\int_0^\infty dt' \int_0^\infty dt'' e^{-\gamma(t'+t'')} \delta(t_1 - t_2 - t' - t'')$$

$$= \int_{t_1-t_2}^\infty dt' e^{-\gamma(2t'-t_1+t_2)} = \frac{1}{2\gamma} e^{-\gamma(t_1-t_2)} ,$$

it follows that in general

$$\int_0^\infty dt' \int_0^\infty dt'' e^{-\gamma(t'+t'')} \delta(t_1 - t_2 - t' + t'')$$

$$= \frac{1}{2\gamma} e^{-\gamma|t_1-t_2|} .$$

Use of this relation in Eq. (B14) gives us the desired moment of $G(t)$

$$\langle G^\dagger(t_1) G(t_2) \rangle = \left[\frac{\mu\Omega^2}{2\epsilon_0 k c^2} \right]^2 \frac{2 n_0 D_{\Sigma^\dagger \Sigma}}{A} \times \left[\frac{e^{4\,\mathrm{Re}(\alpha_0) l} - 1}{2\,\mathrm{Re}(\alpha_0)} \right] \times \frac{1}{2\gamma} e^{-\gamma|t_1-t_2| + i\Delta(t_1-t_2)} . \tag{B15}$$

For finite γ, $G(t)$ is indeed *not* δ correlated.

We may express Eq. (B15) in terms of more familiar quantities by taking the real part of expression (3.3a) and substituting it and expression (B10′) into Eq. (B15). One may show after some straightforward algebra and the use of Eq. (3.1) that *below threshold*

$$\langle G^\dagger(t_1) G(t_2) \rangle = \frac{\hbar\Omega}{2\epsilon_0 c n A} \left[\frac{\langle R_a^{(0)} \rangle}{\langle R_a^{(0)} - \langle R_b^{(0)} \rangle} \right] \times (e^{4\,\mathrm{Re}(\alpha_0) l} - 1) \left[\frac{\gamma^2 + \Delta^2}{2\gamma} \right] \times e^{-\gamma|t_1-t_2| + i\Delta(t_1-t_2)} . \tag{B16}$$

In terms of $i_-(0)$ defined by Eq. (3.10), we may rewrite Eq. (B16) as

$$\langle G^\dagger(t_1) G(t_2) \rangle = i_-(0)(1 - \bar{r}^2 e^{4\,\mathrm{Re}(\alpha_0) l}) \times e^{-\gamma|t_1-t_2| + i\Delta(t_1-t_2)} . \tag{B17}$$

5. Second-order moments of phase-noise operators $F_\pm(z,t)$

From Eqs. (5.5) and (3.4), we find that

$$F_\pm(z,t) = \pm \frac{\mu\Omega^2}{4 r_\pm(z) \epsilon_0 k c^2} \left[e^{-i\theta_\pm(z,t)} \int_0^\infty f_\pm(z,t-t') e^{-(\gamma+i\Delta)t'} dt' + e^{i\theta_\pm(z,t)} \int_0^\infty f_\pm^\dagger(z,t-t') e^{-(\gamma-i\Delta)t'} dt' \right] . \tag{B18}$$

Therefore, the moments of $F_\pm(z,t)$ are simply related to the moments of $f_\pm(z,t)$ and $f_\pm^\dagger(z,t)$, which we have already calculated earlier in this appendix. Note that since mixed moments of type $\langle f_+^\dagger f_- \rangle$ and $\langle f_+ f_-^\dagger \rangle$, etc., vanish, mixed moments of type $\langle F_+ F_- \rangle$ and $\langle F_+ F_- \rangle$ are also zero

$$\langle F_+(z,t) F_-(z',t) \rangle = \langle F_+(z,t) F_-(z',t') \rangle = \cdots = 0 . \tag{B19}$$

The only nonzero moments of F_+ and F_- are the following

$$\langle F_+(z,t) F_+(z',t) \rangle = \frac{1}{16 r_+(z) r_+(z')} \left[\frac{\mu\Omega^2}{\epsilon_0 k c^2} \right]^2 \times \int_0^\infty dt_1 \int_0^\infty dt_2 [\langle f_+(z,t-t_1) f_+^\dagger(z',t'-t_2) \rangle e^{-(\gamma+i\Delta)t_1} e^{-(\gamma+i\Delta)t_2} \times e^{-i[\theta_+(z,t) - \theta_+(z',t')]} + \langle f_+^\dagger(z,t-t_1) f_+(z',t'-t_2) \rangle \times e^{-(\gamma-i\Delta)t_1} e^{-(\gamma+i\Delta)t_2} e^{i[\theta_+(z,t) - \theta_+(z',t')]}] .$$

The use of Eqs. (B12) and (B13) and the relation

$$\int_0^\infty dt_1 \int_0^\infty dt_2 e^{-\gamma|t_1+t_2| + i\Delta(t_1-t_2)} \delta(t - t' - t_1 + t_2) = \begin{cases} \int_0^\infty dt_2 e^{-\gamma(t-t'+2t_2) + i\Delta(t-t')} & \text{if } t > t' \\ \int_0^\infty dt_1 e^{-\gamma(t'-t+2t_1) + i\Delta(t-t')} & \text{if } t < t' \end{cases}$$

$$= \frac{1}{2\gamma} e^{-\gamma|t-t'| + i\Delta(t-t')}$$

yields the following result:

$$\langle F_+(z,t)F_+(z',t')\rangle \cong \frac{n_0}{16r_+^2(z)A\gamma}\left[\frac{\mu\Omega^2}{\epsilon_0 kc^2}\right]^2 \delta(z-z')e^{-\gamma|t-t'|} \times [D_{\Sigma\Sigma^\dagger}(z)e^{-i\Delta(t-t')-i[\theta_+(z,t)-\theta_+(z,t')]}+D_{\Sigma^\dagger\Sigma}(z)e^{i\Delta(t-t')+i[\theta_+(z,t)-\theta_+(z,t')]}] . \quad \text{(B20)}$$

The derivation of the only other nonzero moment, viz., $\langle F_-(z,t)F_-(z',t')\rangle$, is similar and is, in fact, given by Eq. (B20) in which all + subscripts are replaced by −. Note that for $t=t'$, we have the following expressions, which we shall need:

$$\langle F_+(z,t)F_+(z',t)\rangle = \frac{n_0}{16r_+^2(z)A\gamma}\left[\frac{\mu\Omega^2}{\epsilon_0 kc^2}\right]^2 \delta(z-z') \times [D_{\Sigma\Sigma^\dagger}(z)+D_{\Sigma^\dagger\Sigma}(z)] \quad \text{(B21a)}$$

and

$$\langle F_-(z,t)F_-(z',t)\rangle = \frac{n_0}{16r_-^2(z)A\gamma}\left[\frac{\mu\Omega^2}{\epsilon_0 kc^2}\right]^2 \delta(z-z') \times [D_{\Sigma\Sigma^\dagger}(z)+D_{\Sigma^\dagger\Sigma}(z)] . \quad \text{(B21b)}$$

APPENDIX C

Relation between P_{out} and P_1

From the Lorentzian atomic envelope for the power spectrum, it is clear that if P_1 is the output power from the central quasimode, the output power from the pth quasimode is

$$P_1^{(p)} \cong \frac{\gamma^2}{\gamma^2+p^2\left[\frac{\pi c}{nl}\right]^2}P_1 .$$

Thus,

$$P_{\text{out}} = \sum_{p=-\infty}^{\infty} P_1^{(p)} \cong P_1 \sum_{p=-\infty}^{\infty} \frac{1}{1+p^2\left[\frac{2\pi}{\gamma t_R}\right]^2} .$$

But

$$\frac{1}{\pi}\sum_{p=-\infty}^{\infty}\frac{1}{1+p^2/p_0^2} = p_0\coth(\pi p_0) ,$$

which may be established by a contour integration, so that

$$P_{\text{out}} = P_1 \cdot \frac{\gamma t_R}{2}\coth\left[\frac{\gamma t_R}{2}\right] .$$

Since we have assumed $\gamma t_R \gg 1$ in this paper, we may use the fact that for $\gamma t_R \gg 1$

$$\coth\left[\frac{\gamma t_r}{2}\right] \simeq 1 ,$$

to obtain

$$P_{\text{out}} \simeq P_1\left[\frac{\gamma t_R}{2}\right] .$$

APPENDIX D

1. Boundary conditions on $r_\pm(z)$ and $\theta_\pm(z,t)$

On writing $e_\pm(z,t)=r_\pm(z)e^{i\theta_\pm(z,t)}$ it follows from the boundary condition (3.6a) at the perfect mirror $z=l$ that

$$r_+(l)=r_-(l) , \qquad \theta_+(l,t)=\theta_-(l,t)-2kl+\pi \bmod 2\pi . \quad \text{(D1)}$$

The relations between $r_\pm$ and $\theta_\pm$ are more complicated at the partially transmitting mirror $z=0$. We assume that above threshold the amplitudes of the $e_\pm$ fields are large, so that the vacuum field $e_+^{(\text{vac})}$ contributes a very small phase shift $\epsilon_\theta(t)$ to the circulating coherent field. Thus, if we let

$$r_+(0)=\tilde{r}r_-(0)+\epsilon_r$$

and

$$\theta_+(0,t)=\theta_-(0,t)+\pi+\epsilon_\theta(t)\bmod 2\pi , \quad \text{(D2)}$$

then from Eq. (3.6b) we obtain

$$r_+(0)=[r_+(0)-\epsilon_r]e^{-i\epsilon_\theta}+\frac{\tilde{t}}{n}e^{-i\theta_+(0,t)}e_+^{(\text{vac})}(0,t)$$

so that for $|\epsilon_r|,|\epsilon_\theta| \ll 1$, we get, up to the lowest order in ϵ_r and ϵ_θ,

$$r_+(0)i\epsilon_\theta \cong -\epsilon_r+\frac{\tilde{t}}{n}e^{-i\theta_+(0,t)}e_+^{(\text{vac})}(0,t) . \quad \text{(D3)}$$

Taking the imaginary part of Eq. (D3) gives us the desired ϵ_θ

$$\epsilon_\theta=\frac{(\tilde{t}/n)}{2ir_+(0)}[e_+^{(\text{vac})}(0,t)e^{-i\theta_+(0,t)}-e_+^{(\text{vac})\dagger}(0,t)e^{i\theta_+(0,t)}] . \quad \text{(D4)}$$

2. Variance of $\epsilon_\theta(t)$

In the vacuum state, from Eq. (D4),

$$\langle \epsilon_\theta^2(t)\rangle = +\frac{\tilde{t}^2}{4n^2 r_+^2(0)} \times [\langle e_+^{(\mathrm{vac})}(0,t) e_+^{(\mathrm{vac})\dagger}(0,t)\rangle + \langle e_+^{(\mathrm{vac})\dagger}(0,t) e_+^{(\mathrm{vac})}(0,t)\rangle],$$

so that by use of Eqs. (B8) and (B9), we get

$$\langle \epsilon_\theta^2(t)\rangle = \frac{\hbar\Omega\tilde{t}^2}{16\epsilon_0 A l n^2 r_+^2(0)}, \tag{D5}$$

which like Eq. (B9) is valid only for the vacuum field transmitted into a *single* quasimode.

APPENDIX E

1. Establishing result (5.8)

Note that

$$\int_0^l \mathrm{Im}[\alpha_s(z)]dz = -\frac{\Delta}{\gamma}\mathrm{Re}(\alpha_0)\int_0^l \frac{dz}{1+[r_+^2(z)+r_-^2(z)]/I_s}$$

$$= -\frac{\Delta}{\gamma}\int_{z=0}^{z=l}\frac{dr_+(z)}{r_+(z)} \quad \text{(from Eq. (4.12a)}$$

$$= -\frac{\Delta}{\gamma}\ln\frac{r_+(l)}{r_+(0)} = -\frac{\Delta}{2\gamma}\ln\frac{1}{\tilde{r}},$$

from which Eq. (5.8) follows. To perform the last step of the above derivation, we used Eqs. (3.22a) and (3.22b).

2. Evaluation of integrals I and J

With the help of Eqs. (4.17) and (4.18), I may be recast solely in terms of $r_+(z)$ as

$$I = \int_{z=0}^{z=l}\frac{dr_+^2(z)[1/r_+^2(z)+r_+^2(z)/C^2]}{dr_+^2(z)/dz} = \frac{1}{2\,\mathrm{Re}(\alpha_0)I_s}\int_{r_+^2(0)}^{r_+^2(l)} du\,(1/u^2+1/C^2)(I_s+u+C^2/u)$$

$$= \frac{1}{2\,\mathrm{Re}(\alpha_0)I_s}\int_{r_+^2(0)}^{r_+^2(l)}\left[\frac{C^2}{u^3}+\frac{I_s}{u^2}+\frac{2}{u}+\frac{I_s}{C^2}+\frac{u}{C^2}\right]$$

$$= \frac{1}{2\,\mathrm{Re}(\alpha_0)I_s}\left[\frac{C^2}{2}\left[\frac{1}{r_+^4(0)}-\frac{1}{r_+^4(l)}\right]+I_s\left[\frac{1}{r_+^2(0)}-\frac{1}{r_+^2(l)}\right]+2\ln\frac{r_+^2(l)}{r_+^2(0)}+\frac{I_s}{C^2}[r_+^2(l)-r_+^2(0)] + \frac{1}{2C^2}[r_+^4(l)-r_+^4(0)]\right]. \tag{E1}$$

We now use Eqs. (4.21) to rewrite Eq. (E1) entirely in terms of C, whose explicitly expression is given by Eq. (4.22)

$$I = \frac{1}{2\,\mathrm{Re}(\alpha_0)I_s}\left[\frac{1}{2}\left[\frac{1}{\tilde{r}^2}-\tilde{r}^2\right]+\frac{I_s}{C}\left[\frac{1}{\tilde{r}}-\tilde{r}\right]+2\ln\frac{1}{\tilde{r}}\right], \tag{E2}$$

in which C is determined by Eq. (4.22) and related to $r_\pm^2(0)$ via Eq. (4.21b).

We may evaluate J similarly with the help of Eqs. (4.17) and (4.18)

$$J = \frac{1}{2\,\mathrm{Re}(\alpha_0)I_s}\int_{r_+^2(0)}^{r_+^2(l)} du\left[\frac{C^2}{u}+u\right]^2\frac{1}{C^2u} = \frac{1}{2\,\mathrm{Re}(\alpha_0)I_s}\int_{r_+^2(0)}^{r_+^2(l)} du\left[\frac{C^2}{u^3}+\frac{2}{u}+\frac{u}{C^2}\right]$$

$$= \frac{1}{2\,\mathrm{Re}(\alpha_0)I_s}\left[\frac{C^2}{2}\left[\frac{1}{r_+^4(0)}-\frac{1}{4_+^4(l)}\right]+2\ln\frac{r_+^2(l)}{r_+^2(0)} + \frac{1}{2C^2}[r_+^4(l)-r_+^4(0)]\right].$$

With the aid of Eqs. (4.21) and (4.22) we finally obtain

$$J = \frac{1}{2\,\mathrm{Re}(\alpha_0)I_s}\left[2\ln\frac{1}{\tilde{r}}+\frac{1}{2}\left[\frac{1}{\tilde{r}^2}-\tilde{r}^2\right]\right]. \tag{E3}$$

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