Quantum-statistical analysis of superfluorescence and amplified spontaneous emission in dense media

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We have studied the quantum-statistical properties of superfluorescence (SF) and amplified spontaneous emission (ASE) for dense media (media of densities corresponding to many atoms, on the average, within a cubic resonance wavelength). We use the Langevin equations of motion, obtained from a fully quantized model that takes into account induced near dipole-dipole interactions associated with dense media, together with the coupled Maxwell equation, to formulate a stochastic calculation ensemble. Quantum Monte Carlo methods are used in conjunction with the stochastic model, to conduct a statistical analysis of phase waves for this system. Results are presented for the phase-wave density, delay time, and peak-intensity fluctuations. The transition from SF to ASE is treated using the stochastic model.

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I. INTRODUCTION

In superfluorescence (SF), a collection of atoms or molecules prepared initially in a state of complete inversion undergoes relaxation by collective spontaneous decay [1–11]. In amplified spontaneous emission (ASE), the spontaneous emission from a single atom is amplified as it propagates [12–14]. Both of these processes are initiated by quantum noise, i.e., spontaneous emission. While SF is a cooperative process, and the radiating system acts as a collective coherent radiator, ASE, on the other hand, is completely incoherent with no phase relation among the radiating atoms. Several key experiments have been done to study various aspects of the SF and ASE phenomenon [15–17]. Recently, interest in SF and ASE has been revived by a very interesting experiment done by Boyd's group [18]. In this experiment, a transition from SF to ASE was observed in the emission from $KCl:O_2^-$, in the solid state. The parameter controlling this transition was found to be the dipole dephasing rate, $\gamma_\perp$, which was varied by controlled temperature variation (10–30 K) of the solid. The dependence of the dephasing rate on the temperature was the type $\gamma_\perp \alpha T^3$. This was the first experimental study (to our knowledge) of a transition in the cooperative emission from an extended dense system. A crucial parameter in their analysis was the number of atoms cooperating in the emission. Their analysis was based on the number of ions to be in the neighborhood of 10^9 cm^{-3}, while in reality this number could have been as high as 10^{10}–10^{13} cm^{-3} [19–21]. Since the density of atoms participating in the collective emission can be rather high, the role of dipole-dipole interaction among neighboring atoms becomes important. The dipole-dipole interactions in the context of SF have been previously investigated by several authors [22–29]. Friedberg, Hartmann, and Manassah [24] in their comprehensive report have shown that dipole-dipole interactions lead to frequency shifts in the emission and absorption of resonant radiation interacting with a system of two-level atoms. Steudal has also emphasized that when the densities become high, it becomes necessary to include the dipole-dipole interactions in the explanation of SF [26–29]. Bowden and co-workers have shown that near dipole-dipole (NDD) effects can lead to intrinsic optical bistability (IOB) due to NDD-induced dynamic frequency shifts [30,31], can lead to intrinsic self-phase modulation in self-induced transparency (SIT) [32], and can cause unique optical switching in a dense medium of two-level atoms [33].

In this paper we propose a model for the cooperative emission from dense optical systems with the inclusion of NDD interaction among neighboring atoms. It is well known that the effects of the NDD lead to modified Maxwell-Bloch equations (MBE's) [30–33]. We apply these MBE's to the understanding of the transient cooperative emission from a dense atomic system. In this approach, the effect of the NDD appears as an inversion-dependent renormalization of the atomic resonance frequency. The strength of the renormalization is proportional to the density of the atomic system and the square of the transition dipole moment. This model therefore is suitable for explaining cooperative emission from dense systems such as condensed matter. The statistical aspects of the emission are analyzed using the stochastic model discussed previously [34]. In this approach, the Langevin equations are integrated to perform a numerical analysis of the modified MBE's. The statistical analysis is carried out using a large ensemble of the numerical solutions [34] with stochastic initial conditions. It has been shown by Englund and Bowden in Ref. [34] that the ensemble averages obtained from the stochastic model are consistent with corresponding quantum

averages for the class of systems which includes SF, i.e., systems whose evolution stems from quantum initiation, and for which stochastic fluctuations and nonlinearity are not simultaneously important. We use the stochastic model approach developed in Ref. [34] in this analysis. For this ensemble, information regarding the delay time and peak intensity are obtained. Our results show good agreement with the experiments of Boyd's group. The phase-wave statistics are also discussed, in particular in terms of the transition from SF to ASE. Phase waves, which are stochastically induced macroscopic, sudden shifts in the signal amplitude phase, were discussed originally by Hopf [35] in connection with SF, and later by Hopf and Overman [36] in relation to swept-gain superfluorescence. More recently, phase waves were shown by Englnd and Bowden [34] to cause spontaneously induced solitons in stimulated Raman scattering (SRS). In the treatment presented here, we discuss the role of phase-wave yield in the transition from SF to ASE. The model proposed in this paper should have applications in a large class of dense systems interacting with radiation.

II. HAMILTONIAN AND EQUATIONS OF MOTION

Our system is composed of a large number of spatially distributed, two-level atoms. The Hamiltonian which describes the system in the rotating-wave and the electric-dipole approximation is given by

$$H = H_0 + H' , \tag{2.1}$$

$$H_0 = \tfrac{1}{2}\hbar\omega \sum_{i=1}^{N} \hat{\sigma}_z^{(i)} + \hbar \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k , \tag{2.2}$$

$$H' = -i\hbar \sum_k \sum_{i=1}^{N} g_k^{(i)} \hat{a}_k \sigma_+^{(i)} e^{i\mathbf{k}\cdot\mathbf{r}} + \text{H.c.} , \tag{2.3}$$

where H.c. denotes Hermitian conjugate and

$$g_k^{(i)} = \left[\frac{2\pi\omega_k}{\hbar V} \right]^{1/2} P \hat{\mathbf{p}}_i \cdot \hat{\mathbf{x}} . \tag{2.4}$$

In the above Hamiltonian H_0 includes the free atoms and the free field Hamiltonian. The $\hat{\sigma}_z^{(i)}$ represent the population inversion operators and $\hat{\sigma}_\pm^{(i)}$ are the raising and lowering operators for atom i with coordinate $\mathbf{r}_i$. V is the quantization volume for the field, P is the transition dipole moment matrix element and $\mathbf{p}_i$ is the unit vector for the dipole of atom i, and the summation runs over the total number of the atoms, N. The Heisenberg equations of motion and the field equation for propagation have been derived elsewhere [30]. In the "bad cavity limit," the field mode operators a_k are eliminated in terms of the atomic system variables by formal integration of the Heisenberg operator equations for each operator [11], a_k. After adiabatic elimination of the self-field terms to obtain the spontaneous decay and dephasing rates, the additive field components, which appear in the equations of motion, are the reaction field Ω_R^+ and the vacuum field, F^+, where the latter corresponds to the initial condition for the field and constitutes multiplicative Langevin noise source terms in the equations of motion [11,30,34]. The equations of motion for the slowly varying operators, where the atomic operators are averaged over a finite volume on the order of a cubic resonance wavelength, were obtained in Ref. [30], and are written here, in retarded time coordinates,

$$\frac{\partial\sigma_z}{\partial\tau} = -\gamma_\parallel(\sigma_z + N) - \sigma_+\Omega_R^{(+)} - \Omega_R^{(-)}\sigma_- - 2\sigma_+ F^{(+)} - 2F^{(-)}\sigma_- + G^z(z,\tau) , \tag{2.5}$$

$$\frac{\partial\sigma_+}{\partial\tau} = i(\Delta + i\gamma_\perp)\sigma_+ - i\epsilon\sigma_+\sigma_z + \tfrac{1}{2}\Omega_R^{(-)}\sigma_z + F^{(-)}\sigma_z + G^{(+)}(z,\tau) , \tag{2.6}$$

$$\frac{\partial\Omega_R^{(-)}}{\partial z} = -g\sigma_+ . \tag{2.7}$$

The above set of equations are our main working equations. The two damping rates $\gamma_\parallel$ and $\gamma_\perp$ describe the longitudinal and transverse decay rates for the atoms, respectively, for the homogeneously broadened atoms, g is the coupling constant, and Δ is the general frequency detuning between the field and the atoms. In the above equations, $F^{(\pm)}$ and $\Omega_R^\pm$ are the positive and negative frequency parts of the free and reaction field operators, respectively. For plane-wave propagation, which we assume, the free field operates are δ correlated,

$$\langle F^{(+)}(\tau)F^{(-)}(\tau')\rangle = \frac{1}{N\tau_R}\delta(\tau-\tau') , \tag{2.8}$$

and correspond to a bivariate Gaussian random process with zero mean [34].

Equations (2.5)–(2.7) differ from those derived in Ref. [30] by the last term in (2.5) and (2.6). We shall assume that the medium is homogeneously broadened; therefore we have added the non-Hamiltonian source terms, $G^z(z,\tau)$ and $G^+(z,\tau)$ to account for collisionally induced fluctuations.

The characteristic superradiance time τ_R is given by $1/N\gamma_\perp$. In Eqs. (2.5)–(2.7), ϵ arises due to the NDD which produces an inversion-dependent renormalization in the Bloch equations. It depends upon the density of the medium, and appears in Eq. (2.6), due to the fact that on taking the local volume averages to obtain the operator density variables, one must account for the fact that atoms within the volume over which one averages, in fact, interact with one another by NDD [30]. The value for ϵ in terms of system parameters [37] is

$$\epsilon = \frac{4\pi}{3}\frac{\mu^2 n}{\hbar} , \tag{2.9}$$

where γ_0 is the spontaneous decay rate for a single atom. The effect of ϵ has been shown to cause a dynamic chirp in the pulse propagation [30,32,33]. We find that the parameter ϵ also plays an important role in the pulse build-up for SF and ASE.

We have included the effects of the collisional dephasing by introducing the non-Hamiltonian Langevin noise source terms $G^\pm$and G^z which depend upon τ and z. The decay rates for the non-Hamiltonian collisional dipole de-

phasing are connected with $G(z,\tau)$ through the fluctuation-dissipation relation as shown below. The statistical properties of the source terms in the G's are given by the correlation function,

$$\langle G^{(+)}(z',\tau')G^{(-)}(z,\tau)\rangle_R=\gamma\delta(z-z')\delta(\tau-\tau') \tag{2.10}$$

and $\langle\ \rangle_r$ denotes reservoir average.

Equations (2.5)–(2.7), (2.8), and (2.10) describe the dynamics of our system completely. Unfortunately, analytical solutions of the full system of equations are not possible, except in the linearized regime of quantum initiation [34]. Several approximations can be made to study the dynamics of the system. In Sec. III, we linearize these equations about the initial condition. This is the stochastic regime of quantum initiation. Under this approximation, analytical solutions are possible. In Sec. IV, we use the stochastic model [34], which treats Eqs. (2.5)–(2.7) as classical stochastic equations, and solve them numerically by techniques of integration known for stochastic differential equations. In Sec. V we derive valuable information about phase waves. Section VI is devoted to a discussion of the role of NDD, and the transition from SF to ASE on the basis of our model. The final section is devoted to conclusions and summary of our work.

III. LINEAR REGIME OF QUANTUM INITIATION

The dynamics of SF can be divided into three regimes [34]. These are as follows: (a) the linear regime of stochastic quantum initiation; (b) the linear regime of amplification and coherence buildup, where fluctuations are no longer important; and (c) the nonlinear regime of amplification. In the linear regime, which results from linearization for the equations of motion about the initial condition [34,38], the quantum mechanical MBE's become manifestly linear and can be solved analytically by Laplace transform methods. All moments can be derived for the linear regime and can be used to validate the numerical ensemble [34]. Furthermore, analytical results can also be derived for the density of phase waves and phase-wave statistics [34]. Although we are primarily interested in the nonlinear regime, which requires numerical simulation, and where SF pulse evolution occurs, we can gain insight by interpreting ensemble averages in terms of quantum averages in the linear regime, and certain effects can be extrapolated into the nonlinear dynamical evolution.

In the linear regime of SF the inversion is nearly constant and the linearized equations from (2.5)–(2.7), linearized about the initial condition, are

$$\frac{\partial\sigma_+}{\partial\tau}=i(\Delta+i\gamma_\perp-\epsilon N)\sigma_+ +\frac{N}{2}\Omega_R^{(-)}+NF^{(-)}(\tau) +G^{(+)}(z,\tau)\ , \tag{3.1}$$

$$\frac{\partial\Omega_R^{(-)}}{\partial z}=g\sigma_+\ . \tag{3.2}$$

Here, both of the atomic variables $\sigma_\pm$ and $\Omega_R^{(-)}$ are functions of space and time coordinates. To solve these two equations analytically, we take the Laplace transform with respect to the z coordinates ($z\rightarrow s$):

$$\frac{\partial\tilde\sigma_+(s,\tau)}{\partial\tau}=-\alpha\tilde\sigma_+(s,\tau)+\frac{N}{2}\tilde\Omega_R^{(-)}(s,\tau)+\frac{N}{s}F^{(-)}(\tau) +\tilde G^{(+)}(s,\tau)\ , \tag{3.3}$$

$$\tilde\Omega_R^{(-)}=\frac{g}{s}\tilde\sigma_+(s,\tau)\ , \tag{3.4}$$

where the generalized detuning parameter $\alpha=-i(\Delta+i\epsilon-\epsilon N)$, N is the total number of atoms, and $G(z,t)$ is the stochastic collisional contribution. In the above equations, $\tilde G$ denotes the Laplace transform of G with respect to the space variable. Now we can time-integrate these equations. The time integral of the polarization is

$$\tilde\sigma_+(s,\tau)=\tilde\sigma_+(s,0)e^{-\tilde\lambda(s)\tau}+\int_0^\tau d\tau' e^{-\tilde\lambda(s)(\tau-\tau')}\left[\frac{N}{s}F^{(-)}(\tau')+\tilde G^{(+)}(s,\tau')\right]\ , \tag{3.5}$$

where

$$\tilde\lambda(s)\equiv-\frac{Ng}{2s}+\alpha\ . \tag{3.6}$$

A complete solution for $\sigma_+(z,\tau)$ and $\Omega_R^{(-)}$ is now easily obtained by Laplace inversion as

$$\begin{aligned}\sigma_+(z,\tau)=&\sigma_+(z,0)e^{-\alpha\tau}+\int_0^z dz' e^{-\alpha\tau}\sigma_+(z',0)\left[\frac{Ng\tau}{2(z-z')}\right]^{1/2}I_1\left[2\left[\frac{Ng\tau}{2}(z-z')\right]^{1/2}\right]\\ &+\int_0^\tau d\tau' e^{-\alpha(\tau-\tau')}\left[NF^{(-)}(\tau')I_0\left[2\left[\frac{Ngz}{2}(\tau-\tau')\right]^{1/2}\right]+G^{(+)}(z,\tau')\right.\\ &\left.+\int_0^z dz' G^{(+)}(z',\tau')\left[\frac{Ng(\tau-\tau')}{2(z-z')}\right]^{1/2}I_1\left[2\left[\frac{Ng}{2}(\tau-\tau')(z-z')\right]^{1/2}\right]\right]\ ,\end{aligned} \tag{3.7}$$

and the field is given by

$$\Omega_R^{(-)}(z,\tau)=-ge^{-\alpha\tau}\int_0^z dz'\sigma_+(z',0)I_0\left[2\left[\frac{Ng\tau}{2}(z-z')\right]^{1/2}\right]$$
$$-g\int_0^\tau d\tau' e^{-\alpha(\tau-\tau')}\left[NF^{(-)}(\tau')\left[\frac{z}{Ng(\tau-\tau')/2}\right]^{1/2}I_1\left[2\left[\frac{Ngz}{2}(\tau-\tau')\right]^{1/2}\right]\right.$$
$$\left.+\int_0^z dz' G^{(+)}(z',\tau')I_0\left[2\left[\frac{Ng}{2}(\tau-\tau')(z-z')\right]^{1/2}\right]\right], \tag{3.8}$$

where the various correlations occurring in the above equations are

$$\langle F^{(-)}(\tau)F^{(+)}(\tau')\rangle=0\ ,\quad \langle F^{(+)}(\tau)F^{(-)}(\tau')\rangle=\frac{1}{N\tau_R}\delta(\tau-\tau')\ , \tag{3.9}$$

$$\langle G^z(z,\tau)G^{(\pm)}(z',\tau')\rangle_R=0\ ,\quad \langle G^{(+)}(z',\tau')G^{(-)}(z,\tau)\rangle_R=\gamma_\perp\delta(z-z')\delta(\tau-\tau')\ . \tag{3.10}$$

The results for the polarization and the field obtained in the above equations are used in Sec. V to analyze the statistical properties of phase waves for the system.

IV. NUMERICAL SOLUTIONS OF THE LANGEVIN EQUATIONS

We have solved the set of Langevin equations for our system, Eqs. (2.5)–(2.7), (2.8), and (2.10), by numerical techniques based on the quantum Monte Carlo method [38]. The quantum noise is simulated by random numbers generated by the Box-Muller method [39]. These are stochastic differential equations and great care is needed in their numerical solution. The step size requirements involved in the integration method are different from those of the deterministic equations [36]. We have obtained a large number of numerical solutions for a given set of parameters (of the order of 500), each corresponding to a particular stochastic realization according to Eqs. (2.8) and (2.10). These constitute an ensemble from which we infer the statistical properties of our system. This procedure is repeated for new values of the parameters for our system.

The dynamics of the system can be studied on the basis of phase-wave analysis. Phase waves were first proposed by Hopf [35,36], and recently they have been investigated in a variety of phenomena such as stimulated Raman [34,40,41] and Brillouin [42] scattering. Hopf had originally discussed them in the context of SF [35]. In this paper we identify phase waves to be crucial in the transition from SF to ASE. Using the Langevin approach we find that the density of phase waves governs the nature of the cooperative emission. For dense system they are relatively easier to observe. The dephasing rate identified in the experiment of Ref. [18] is found to be related to the density of phase waves. For the sake of completeness we give below a brief description of phase waves.

In a macroscopic system of inverted atoms, each atom begins to radiate independently, and initially (i.e., in the linear regime of stochastic initiation) the phases of the fields are stochastic. Eventually, the cooperation among neighboring atoms increases and the region of cooperation grows. But the phases were independent initially, hence an apparent confrontation is encountered at the boundary of two neighboring regions. If a phase change of the field amplitude occurs, the polarization will echo this change. In a way, this is analogous to the dislocations encountered at the grain boundaries as materials are crystallizing. In our case, the phase is a well-defined quantity for the emitted field, hence we can qualitatively define a phase angle for the radiating system. In general, this phase angle can take any value between 0 and π. A phase wave is a flip in the relative phase between the field amplitude and the polarization, within a time less than the depashing time in the medium. The density of phase waves as a function of the relative phase angle constitutes an important statistical property of the system and we have studied this in detail by numerical techniques similar to the quantum Monte Carlo method. Phase waves are macroscopic effects arising due to microscopic quantum fluctuations. For sufficiently large phase fluctuations, the output pulses lose energy during encounters with phase waves and the atomic medium momentarily experiences gain rather than loss [35]. This has drastic effects on the radiated pulse since it depletes a part of the pulse and can cause the appearance of post- as well as prepulses. Therefore the density of phase waves can play an important role in determining the pulse shape. We determine the phase-wave statistics in the linear and nonlinear regimes. The numerical solutions are obtained as a function of the spatial coordinate. The phase jumps are

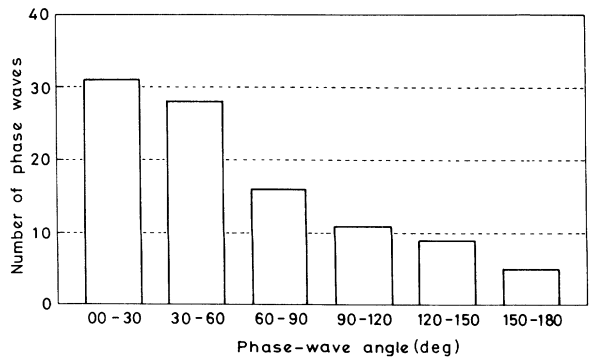

FIG. 1. Phase-wave density for $\gamma_\perp=10^9$, $\epsilon=1.0$, corresponding to the case of SF.

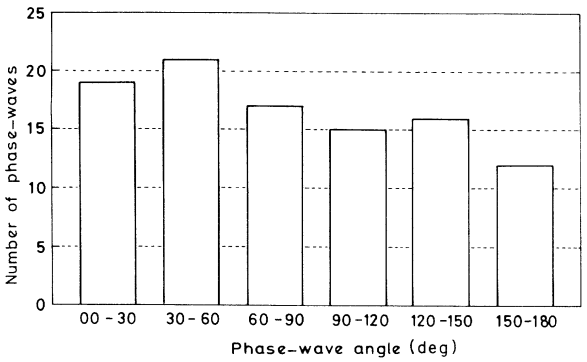

FIG. 2. Phase-wave density for $\gamma_\perp = 10^{10}$, $\epsilon = 1.0$, corresponding to the case of SF-ASE transition.

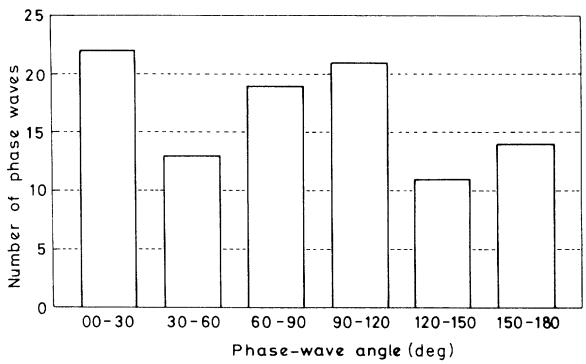

FIG. 3. Phase-wave density for $\gamma_\perp = 10^{11}$, $\epsilon = 1.0$, corresponding to the case of ASE.

recorded in the electric field from Maxwell's equation. The number and the angle of the phase jumps are counted. In this way we have obtained the density of phase waves, pulse shape, delay time, and peak intensity, by the integration of the stochastic differential equations. The density of phase waves obtained in this way, for important cases, is shown in Figs. 1–3. In Fig. 1, we have shown the relative numbers of phase waves as a function of phase angle, for low dephasing rate. We observe that the maximum number of phase waves is between 0° and 30°, and their number decreases as the phase angle increases. This is very close to an exponential decay, but we have not been able to establish this analytically. The effect of such small-angle phase waves on the pulse is not very devastating. Thus, this corresponds to the cooperative regime of SF. As the dephasing rate increases, a systematic increase in the number of phase waves is observed for larger phase angles. Thus the effect on the pulse is great and a trend toward a lower degree of cooperation is observed. This is shown in Fig. 2. It is interesting to note that the phase-wave density is slightly higher now. We recall that a signature of a phase-wave–phase collision is to cause depletion of the SF pulse. For a phase flip of $\pi/2$ or greater, the pulse amplitude must go through zero, causing a null in the intensity. Thus phase waves corresponding to such phase flips are particularly devastating to smooth pulse evolution. This can be considered the transition region to ASE. For even larger values of dephasing rate, a somewhat random trend is observed, and on the average all phase angles occur with equal density. This is the case shown in Fig. 3. These results are for an ensemble of 500, and we anticipate some smoothing for larger ensembles. This is the case corresponding to ASE. Figures 1–3 thus show a systematic increase in the density of phase waves with increase in phase angle. The dephasing rate $\gamma_\perp$ was varied on the scale of 1 to 10 to 100 here (all scalings here are normalized to $\lambda_\perp = 1$).

V. PHASE-WAVE STATISTICS

In the linear regime, the solutions for the polarization and the field have been obtained in Sec. III. The normalized correlation coefficient between the polarization and the field was introduced by Hopf [35] as a measure of the stochastic properties of phase waves, and related to the phase-wave probability density using standard methods for bivariate Gaussian processes [41]. This correlation coefficient $C(z,\tau)$ is given by [35,34]

$$C(z,\tau)=\frac{|\langle \Omega_R^{(-)}(z,\tau)\sigma_+(z,\tau)\rangle|}{[\langle \Omega_R^{(-)}(z,\tau)\Omega_R^{(+)}(z,\tau)\rangle\langle \sigma_+(z,\tau)\sigma_-(z,\tau)\rangle]^{1/2}} . \tag{5.1}$$

The correlation function is very important for our discussion here. The deviation of the correlation coefficient $C(z,\tau)$ from unity reflects the influence of quantum fluctuations. For the purely classical case, when field and polarization are uncorrelated, $C=1$. For the quantum case, $C<1$. Substituting from Eq. (3.7) and Eq. (3.8), we get

$$C(z,\tau)=\frac{1}{N}\frac{\int_0^\tau d\tau' e^{-2\alpha(\tau-\tau')}\left[\frac{2z}{Ng(\tau-\tau')}\right]^{1/2} I_1(\sqrt{Ng(\tau-\tau')z})I_0(\sqrt{Ng(\tau-\tau')z})}{\left[\int_0^\tau d\tau' e^{-2\alpha(\tau-\tau')}I_0^2(\sqrt{Ng(\tau-\tau')z})\right]\left[\int_0^\tau d\tau' e^{-2\alpha(\tau-\tau')}I_1^2(\sqrt{2Ng(\tau-\tau')z})\right]} . \tag{5.2}$$

This can be reduced to a somewhat simpler form by making the substitution

$$x=\sqrt{2Ng(\tau-\tau')z} , \tag{5.3}$$

$$C(\zeta,\tau)=\frac{\int_0^\zeta e^{-(\alpha/Ng)x^2}2zI_1(x)I_0(x)\frac{dx}{Ng}}{\left[\int_0^\zeta e^{-\alpha x^2/Ng}I_0^2(x)\frac{x\,dx}{Ng}\right]\left[\int_0^\zeta e^{-\alpha x^2/Ng}\frac{4z^2}{x^2}I_1^2(x)\frac{x\,dx}{Ng}\right]} .$$

The probability density of the phase waves is obtained using the standard techniques of multivariate theory [41]. We treat σ_+ and $\Omega_R^{(-)}$ as complex processes with covariance matrix,

$$V(z,\tau)=\begin{bmatrix} \langle|\Omega_R^{(+)}|^2\rangle & \langle\Omega_R^{(+)}\sigma_+\rangle \\ \langle\sigma_-\Omega_R^{(-)}\rangle & \langle\sigma_+\sigma_-\rangle \end{bmatrix} . \tag{5.4}$$

This covariance matrix contains all the statistical information about our system. To determine the joint probability density of phase waves, we break $\Omega_R^{(+)}$ and σ_+ into a phase and amplitude part as

$$\Omega_R^{(+)}=r_1\exp(i\phi_1) , \tag{5.5}$$

$$\sigma_+=r_2\exp(i\phi_2) . \tag{5.6}$$

The inverse of the covariance matrix ($W=V^{-1}$) is used to define the joint probability density as [34,43]

$$P(r_1,r_2,\phi_1,\phi_2;z,\tau)=\frac{r_1r_2\det w}{\pi^2}\exp(-\{w_{11}r_1^2+w_{22}r_2^2+2[w_{12}r_1r_2\cos(\phi_1-\phi_2)]\}) , \tag{5.7}$$

where the relative phase angle

$$\phi\equiv\arg(\Omega_R^{(-)}\sigma_+)\equiv\phi_1-\phi_2 , \tag{5.8}$$

and the joint probability density can be simplified to

$$P(r_1,r_2,\phi;z,\tau)=\frac{2r_1r_2\det w}{\pi^2}\exp\{-w_{11}r_1^2-w_{22}r_2^2-2w_{12}r_1r_2\cos\phi\} , \quad 0\le\phi\le2\pi . \tag{5.9}$$

One can also consider the marginal of the above generalized probability density by integrating the full probability density determined above.

From the above analysis, it is possible to obtain analytic expressions for the density of the phase waves and other features discussed in Sec. IV. These plots have been previously obtained in the numerical works of Englund and Bowden for a very similar set of equations in the context of stimulated Raman scattering [34,40]. In general, this can be obtained as a function of z and τ. Our results indicate good agreement between the analytic results and the numerical results of the preceding section.

VI. DIPOLE-DIPOLE INTERACTION EFFECTS IN SF AND ASE

Recent experiments done for the solid-state system with $KCl:O_2^-$ [18] involve densities which are high enough so that it is necessary to include the dipole-dipole interaction among the atoms. This question has been addressed in a large number of papers [19–29]. In an earlier paper, we have taken the strength of the dipole-dipole interaction as a free parameter and analyzed the effects on SF emission [32]. For sufficiently large dipole-dipole interaction strength, our results showed the existence of

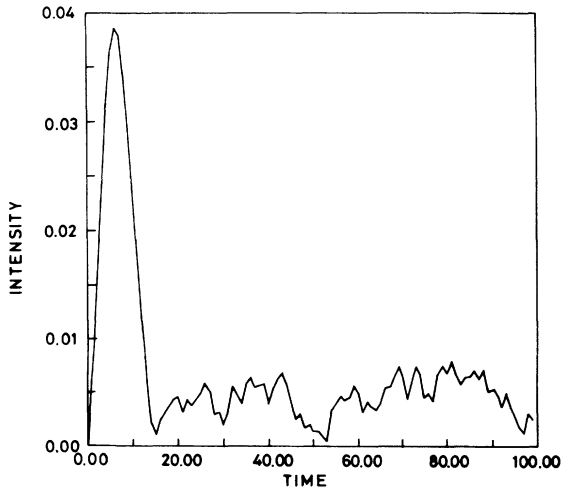

FIG. 4. Pulse buildup from SF to ASE for the values in Fig.1.

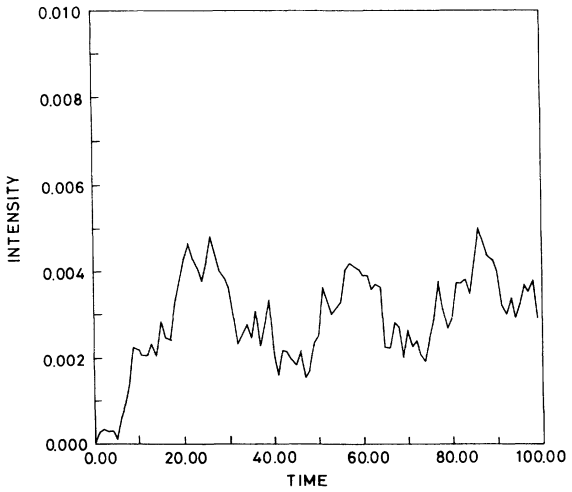

FIG. 5. Pulse buildup from SF to ASE for the values in Fig. 2.

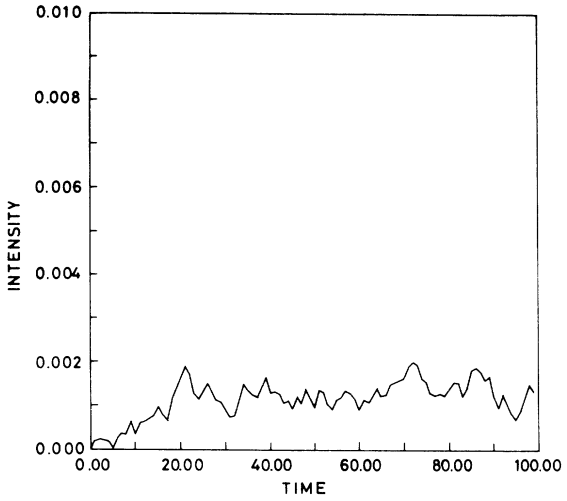

FIG. 6. Pulse buildup from SF to ASE for the values in Fig. 3.

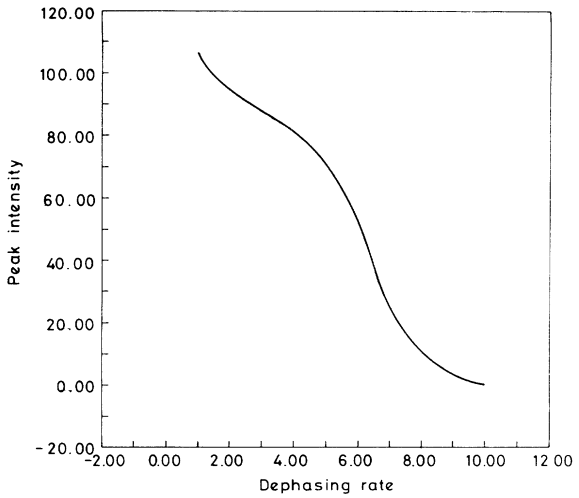

FIG. 8. Peak-intensity distribution as a function of dephasing rate.

radiation trapping and subradiance in SF. To actually estimate the strength of dipole-dipole interaction for the experiments is a rather difficult task, since the estimate for the number of atoms cooperating depends very strongly on the model used for the SF theories. In our model the dipole-dipole interaction introduces a frequency renormalization of the atomic resonance frequency. With this we have integrated MBE's to obtain the solutions numerically. For our calculations, the value of ϵ was determined from the data of the experiment of Ref. [18]. This corresponds to the scaled value $\epsilon \approx 1$. If the dipole-dipole interaction is arbitrarily turned off, the dependence of the results on the dephasing rate is much weaker. This establishes the relevance of the dipole-dipole interaction for the experiment of Ref. [18]. We also find that the number of atoms cooperating is larger than taken in their analysis.

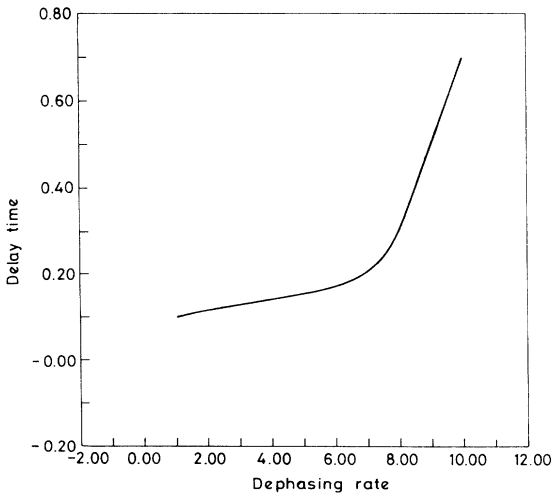

FIG. 7. Delay-time distribution as a function of dephasing rate.

The evolution for the pulse as the dephasing rate increases is shown in Figs. 4–6. From the complete cooperation, i.e., SF to the ASE, we have once again observed a systematic trend as the dephasing rate is increased. The delay time and peak intensity data also constitute a valuable characterization of the system. These are also found to vary in a systematic way. The delay time is found to increase as the dephasing rate increases (Fig. 7). It is important to note that as the shape of the pulse becomes more erratic, the definition of delay time has meaning only in the time integral, i.e, the energy. The peak intensity, on the other hand, is found to decrease as the dephasing rate increases (Fig. 8). These results are in good agreement with the experiment of Ref. [18]. Thus a close connection has been established between our theoretical model, which is based on a fully quantum-mechanical approach, and their experiment. The need for this was pointed out in Ref. [18].

In conclusion, we have provided a microscopic basis for SF-ASE transition based upon quantum initiation. The number of cooperating atoms is found to be much larger than taken for the cases without dipole-dipole interaction. Quantum Monte Carlo methods for the Langevin equations for our system provide valuable statistical information pertaining to the system. Phase waves are found to play a key role in governing the transition rom SF to ASE. The procedure used here should be applicable to other systems such as ASE and four-wave mixing, and stimulated Brillouin scattering.

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