Computational approach to the quantum Zeno effect: Position measurements

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We present computational results of quantum-mechanical sequential measurements of a single dynamical variable (position) in a barrier penetration system. Our model is based on a well-rounded two-minima potential with a particle initially located on one side of the barrier. First we allow the particle wave function to propagate without further disturbance and calculate its barrier penetration time, then we perform measurements of the particle's position and contrast the barrier penetration time T with the results for the undisturbed system. The time-dependent Schrödinger equation for the problem was solved using fast-Fourier-transform techniques. The barrier penetration time was found to depend on (1) the type of measurement made, (2) the strength of the interaction of the system with the meter, (3) the rate at which the measurements were performed, and (4) the precision of the meter (which in our model is related to the width of the meter's wave function). We find no evidence in support of the Zeno effect. However, by using a caricature of quantum mechanics we can obtain results similar to the Zeno effect.

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I. INTRODUCTION

A discussion of the interpretation of quantum mechanics (QM) on any level, above the basic mathematical formalism, almost inevitably becomes rather vague. The difficulty which one faces is the concept of a quantum measurement. The simplest object of a measurement in QM is to determine the value of a single physical observable for a dynamical system with as much precision as possible. We take as self-evident that any measurement on a quantum system will disturb the system in some way. In this paper, we are considering a sequence of measurements, and their effect on the system.

The theory of sequential quantum measurements has applications in optical communications systems, in gravitational wave detection, and in the understanding of fundamental quantum phenomena, such as barrier penetration, decay processes, and the quantum Zeno effect.

Recently, Wineland and co-workers [1] have reported an observation of the quantum Zeno effect in action. They have seen an inhibition of ratio-frequency transitions between two hyperfine levels of the berillium ion caused by "sequential measurements" using short light pulses. Although Wineland and co-workers have undoubtedly seen an interesting quantum effect, we agree with the comment of Ballentine [2], in that it is most likely unrelated to the quantum Zeno effect and should be attributed to a form of quantum interference phenomenon. The original version of the quantum Zeno effect, as formulated by Misra and Sudarshan [3], referred to a spontaneous emission process of an unstable particle and stated that a continuously observed state will never decay. We believe it to be of the greatest importance to be able to describe the measuring procedure accurately and take account of all aspects of the system. We therefore choose an example where everything can be calculated and accounted for, before we delve into more complicated systems.

It has come to our attention that there are, sadly, very few simple relevant problems in the quantum theory of measurement that have been solved and discussed in the literature. This paper, together with papers published elsewhere [4–8], are an attempt to meet this need.

II. COMPUTER SIMULATION OF BARRIER PENETRATION

Our barrier penetration model is based on a well-rounded double-minimum potential with an elementary particle, described by its wave function, initially located on the right of the barrier. Figure 1 shows a graph of the potential, solid curve, and the eigenfunctions of the potential, various dotted and dashed curves. The energy eigenvalues are given in the caption. The wave function of the particle satisfies the time-dependent Schrödinger equation of the system, which can be solved by computational fast-Fourier-transform (FFT) techniques. We allow the particle to move freely and find its barrier penetration time T. We then carry out sequential particle position measurements [4–8], to be described later, and compare the resulting barrier penetration time with the free-propagation time T. This will enable us to observe what effect the measurements have on the system. This model usually leads to a discussion of the quantum Zeno paradox in terms of a particle decay. It is at first sight not obvious why we chose to consider a double-well potential rather than leakage through the barrier of a single well. The double well adequately describes tunneling through a barrier, which is the essential ingredient in some decay processes. It has the added advantage that the energy eigenfunctions and eigenvalues can be calculated with greater ease than the single-well case and used to define the initial wave function for the particle. Furthermore, knowing the energy eigenvalues gives a direct method of finding the undisturbed barrier penetration time, which we discuss below. The two-well model is im-

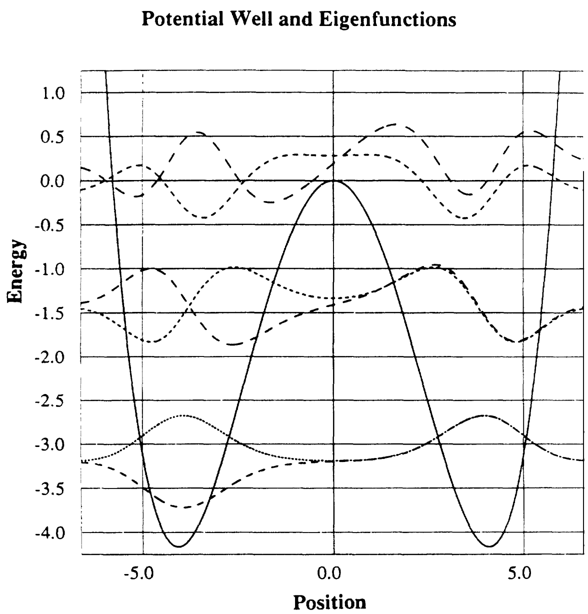

FIG. 1. Potential well and eigenfunctions: The graph shows the potential well, solid curve, and several bound eigenfunctions, various dotted and dashed curves. The x axis represents the spatial coordinate and the y axis represents the energy. The energy eigenvalues for these wave functions, from the lowest to the highest, are -3.19921, -3.19897, -1.43042, -1.40992, -0.134408, and 0.192680 respectively.

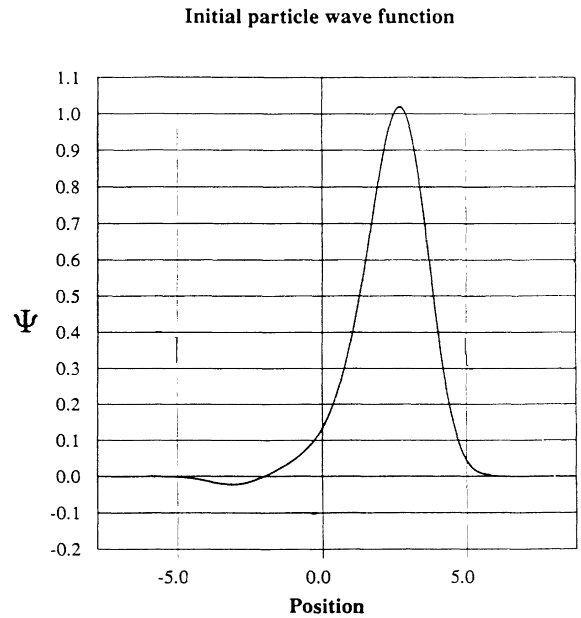

FIG. 2. Initial wave function: Superpose the lowest pair of wave functions in Fig. 1 to form the initial particle wave function. The plot shows the initial particle wave function as a function of position.

portant in its own right since it describes tunneling in systems of considerable importance in solid-state physics. Although we do not present the material here, we have found that the decay rate of a particle in a single well is disturbed by measurements in a similar way to the double-well case presented below.

The decay process can be described by the propagation of the particle wave function from the right-hand side of the well to the left-hand side. In this paper we consider particle position measurements. If the particle is found to be on the left of the barrier, we assume that it has decayed. It is necessary to choose an initial particle wave function in some way. We would like it to sit on the right of the barrier. Clearly, the initial wave function must be a linear superposition of the eigenstates of the problem. The time evolution of the particle density function will be dependent on differences in the energy eigenvalues of each eigenstate involved in its construction. Combination tones of higher frequencies are present and will cause peaks on the propagated wave function. The eigenvalue equation, for our potential, was computationally solved, and we choose as our initial particle wave function, a 50:50 linear superposition of the nearly degenerate ground-state pair of eigenfunctions. These two eigenfunctions represent the lowest-energy, even and odd solutions to the time-independent Schrödinger equation. This initial wave function resembles an eigenfunction for a single-well potential on the right of the barrier. See Fig. 2.

The Hamiltonian of the system can be written as the sum of the kinetic and potential energies of the particle,

$$H=T(p)+V(x)\ , \tag{1}$$

where $p=\hbar k$, $T(p)=\hbar^2k^2/(2m)$, and the potential is given by

$$V(x)=-ax^2+bx^4\ , \tag{2}$$

where the constants $a=1.6725$ and $b=0.0408$ were chosen to give only four energy eigenstates lower than the barrier. This helped to quicken the undisturbed barrier penetration time to an acceptable duration.

Now we will take a moment to describe the units used. Consider the time-independent Schrödinger equation,

$$\partial^2\psi(x)/\partial x^2=\frac{2m}{\hbar^2}[V(x)-E]\psi(x)\ . \tag{3a}$$

To each parameter (x, m, E, and V) assign a dimensionless amplitude and unit magnitude parameter which carries the dimension. (For example, $x\rightarrow xx_1$ where x is now a dimensionless number and x_1 is unit magnitude parameter with dimensions of length.) The above equation can be rewritten as

$$\frac{\partial^2\psi(x)}{\partial x^2}=2m\left[\frac{m_1E_1x_1^2}{\hbar^2}\right][V(x)-E]\psi(x)\ , \tag{3b}$$

where the factor in the first square brackets is a dimensionless number. We take $m=\hbar=1$, $E_1=\hbar\omega/2$, and ω is determined by the curvature of the well at one of the minima. By assuming that the wells are nearly quadratic in nature we find that

$$\frac{\partial^2V}{\partial x^2}=m\omega^2=4a\ . \tag{4}$$

This leads to a value of $\omega=2.5865$ which is our inverse time unit.

Our choice of initial particle wave function had the advantage that it was easy to calculate the undisturbed propagation barrier penetration time T for this wave function. It is well known that the time dependence of the probability amplitude of a wave function composed of a superposition of two states with different energies is given by

$$\Delta E\tau=\hbar\ , \tag{5}$$

where ΔE is the energy difference between the even and odd ground-state eigenfunctions, and the period of oscillation is twice the barrier penetration time, $\tau=2T$.

For our system the even eigenfunction had a bound-state energy E, of 3.199 21 n.u. (natural or dimensionless units for the problem), and the corresponding energy for the odd function was 3.198 97 n.u. with an error tolerance 1×10^{-5} n.u. Hence, the barrier penetration time can be given as

$$T=805.5\omega^{-1}\ , \tag{6}$$

where the time is measured in inverse natural frequency units. This value was found to be in good agreement with the computationally observed value, as we shall see later.

Solving the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=H\psi(x,t) \tag{7}$$

yields

$$\psi(x,t)=\exp[-i(T(p)+V(x))t/\hbar]\psi(x,0)\ . \tag{8}$$

Generally speaking, $T(p)$ and $V(x)$ are noncommuting operators in QM. However, following the successes of beam propagation techniques, we apply the symmetrized split operator [9–11]

$$\exp[-iT(p)\Delta t/2\hbar]\exp[-iV(x)\Delta t/\hbar]\times\exp[-iT(p)\Delta t/2\hbar]\ , \tag{9}$$

where the $V(x)$ exponent acts on $\psi(x,t)$ and the $T(p)$ exponent acts on the Fourier-transformed wave function $\Psi(x,k)$. The time integration is carried out by iteration using many small time steps Δt. Thus, we have an effective way of numerically solving the time-dependent Schrödinger equation for the barrier penetration problem.

III. QUANTUM THEORY OF MEASUREMENT

Let us briefly consider the evolution of measurement theory; Dirac [12] stated that a measurement of an observable will give an eigenvalue of that observable and that a very fast repetition of the measurement would give the same result. This is equivalent to the collapse of the wave function (or reduction) hypothesis of von Neumann. Later, von Neumann introduced a measuring device (meter) into the picture. In the last few pages of his book [13], he gave an account of the dynamics of the interaction of the meter with the system. Take $\psi(x)$ as the wave function for the system and $M(X)$ as the wave function for the meter. The interaction Hamiltonian between the meter and the system is taken in the form κxP, where P is the momentum of the meter and in terms of the meter coordinate $P=-i\hbar\partial/\partial X$. We have added the κ as a variable interaction strength. During the brief time of the interaction t_0 we assume that we may ignore the kinetic energies of the system and meter since these energies are overwhelmed by the interaction strength. By integrating the time-dependent Schrödinger equation through the small time step t_0,

$$\left[\frac{\partial}{\partial t}+\kappa x\frac{\partial}{\partial X}\right]\phi(x,X)=0\ , \tag{10}$$

we obtain

$$\phi(x,X)=f(x,X-t_0\kappa x)\ . \tag{11}$$

Hence, the wave function of the combined system, taking $t_0=1/\kappa$, becomes

$$\phi(x,X)=\psi(x)M(X-x)\ . \tag{12}$$

Taking this value of t_0 essentially leads to a minimum uncertainty in the system wave function, or one could say it gives the best meter performance [14]. (von Neumann would have taken $\kappa=1$ and hence $t_0=1$.) At this point von Neumann's book ends and we are never told what to do with the combined x,X system. The first steps towards a more useful technique were made in 1965 by Arthurs and Kelly [15], in connection with simultaneous measurement of conjugate observables. They considered a single simultaneous measurement of position and momentum, and were the first to introduce the variable interaction strength κ which was used in the way described above.

In recent years, one of us [4–8] has outlined a way in which QM can describe the effect of a sequence of observations to be made on an otherwise carefully isolated system. This approach was not to find the best method of practical measurement, but to examine methods which can be carefully and clearly explained. We assume, as in [4–8], that when the interaction takes place, and we have obtained the combined state $\phi(x,X)$, a measurement of X gives a reading X_m. The system is assumed to be restored to a pure case wave function $\phi(x)$ which is given by

$$\phi(x)=\phi(x,X_m)\ . \tag{13}$$

The following steps are involved in the measurement of X. First, we form the probability distribution for the meter coordinate X, regardless of what the value of x might be,

$$D(X)=\int|\phi(x,X)|^2dx\ . \tag{14}$$

The position X_m can be chosen by using a random number generator to select one of the X values from the weighted $D(X)$ distribution. In practice, one can easily achieve this by chopping the $D(X)$ distribution into n equal regions of probability $1/n$. Each of the n regions should be associated with one X coordinate, possibly the central coordinate of the segment. Then a random number generator chooses one of the n,X coordinates. Thus, we obtain an X_m which can be used as the measured value of X. Substituting $X = X_m$, into Eq. (12), we regain a pure case wave function in x for the system. In practice, one would use a meter only once and then throw it away and use freshly initialized meters for the subsequent measurements. (In our work we took $n = 100$.) One should note that no wave-packet reduction or quantum-mechanical collapse of the system wave function has occurred. (In Refs. [4–8] one is led to believe that this measurement procedure is equivalent to collapsing the meter wave function. The author no longer believes this to be the case and will publish new material on this subject elsewhere.)

IV. RESULTS

Our results have been split into four sections in order to cover four important features of the Zeno effect, as mentioned in the abstract. The first point mentioned was that the type of measurement performed would greatly effect the outcome of a sequential measurement procedure. We will delay this discussion until the concluding section. The second point involved the strength of the interaction between the system and the meter, namely the value of κ. The third point considered the rate at which the measurements were performed and the fourth point involved the precision of the meter.

A. Strength of the interaction between the system and meter

In Sec. III we introduced the variable interaction strength κ into the system-meter interaction formalism of von Neumann. There are two ways to think about the effect of κ, an intuitive picture and a more mathematical approach. Intuitively, Eq. (11) suggests that if κt_0 were not unity the meter wave function would not point accurately to the system variable x resulting in a less-precise measurement. Mathematically, one may calculate the Heisenberg uncertainty relation for x and p of the system [14]. (The calculation is quite involved and we do not give it here.) The uncertainty relation is minimized when $\kappa t_0 = 1$. This is therefore the optimum criterion for the meter and the value that we have used throughout this paper.

B. Rate of the measurements

Applying the methods of Sec. II, we find the undisturbed particle propagation time T to be roughly $800\omega^{-1}$; see Fig. 3. This is in good agreement with our calculation, in Eq. (6). Furthermore, when the particle probability density function is plotted at various times we see a smooth transition of the particle probability density on the right moving over to the left and back again as expected.

After finding the undisturbed barrier penetration time T, we were able to introduce Gaussian meters into the problem. Applying the methods of Sec. III, we took the Gaussian meters to have a width of 3 n.u. During a set time interval t of more than $2T$ ($t = 2000\omega^{-1}$), we carried out 5, 10, and 100 measurements of the particle position. We plotted the resulting probability densities versus time in Figs. 4–6. It is quite clear that 100 measurements, which is equivalent to a measurement every $T/40$ time units, causes a vigorous flipping of the particle from one side of the barrier to the other. This example shows that the Zeno effect is completely nonexistent. We may go further and say that quite the reverse of the Zeno effect occurs, the measurements appear to help the particle to surmount the barrier. We would expect measurements of the type described in Sec. III (especially those performed in quick succession) to cause the particle to decay more rapidly than it would have done without the disturbance.

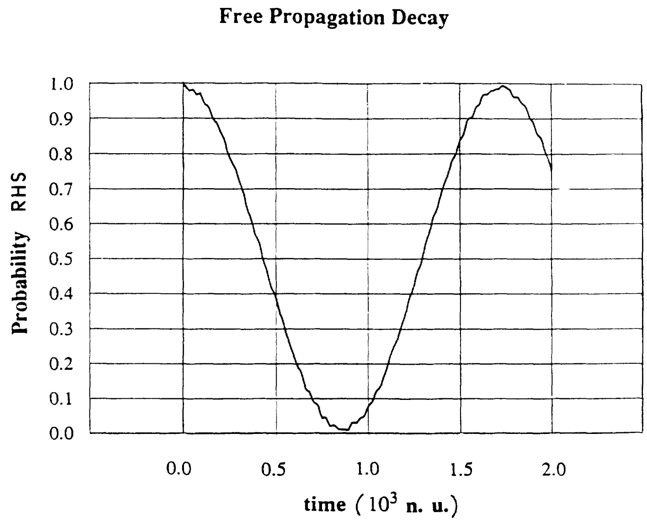

FIG. 3. Propagation of the undisturbed particle: Plot of the probability of finding the particle on the right-hand side (RHS) of the barrier vs time.

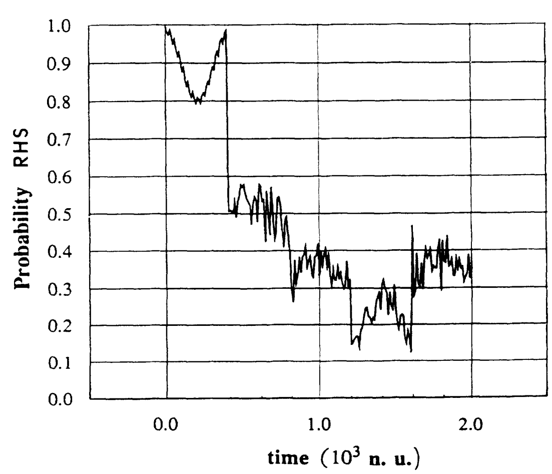

FIG. 4. Effect of the number of measurements: Plot of the probability of finding the particle on the RHS of the barrier vs time. Taking five measurements in a time of $2000\omega^{-1}$, the graph represents a meter width of 3 n.u.

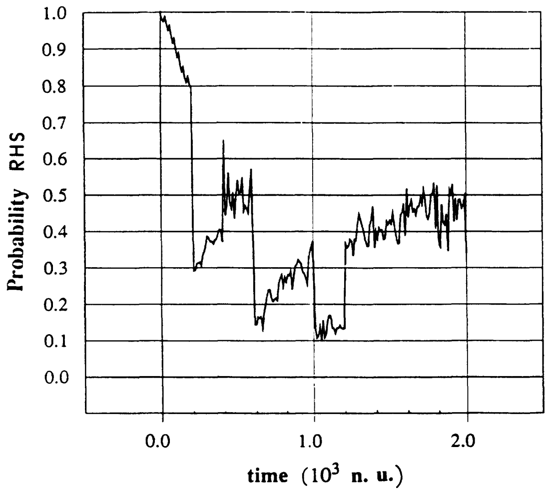

FIG. 5. Effect of the number of measurements: Plot of the probability of finding the particle on the RHS of the barrier vs time. Taking 10 measurements in a time of $2000\omega^{-1}$, the graph represents a meter width of 3 n.u.

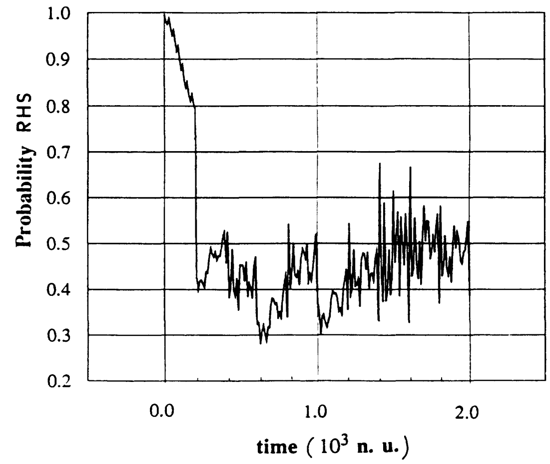

FIG. 7. Effect of meter precision: Plot of the probability of finding the particle on the RHS of the barrier vs time. Taking 10 measurements in a time of $2000\omega^{-1}$, the graph represents a meter width of 5 n.u.

C. Precision of the meter

In this section we consider the effect of the width of the Gaussian meter wave function on the outcome of the sequential measurements. Taking 10 measurements, and setting a time limit of $2000\omega^{-1}$, as above, we plotted the particle density against time for various meter widths. Figures 5, 7, 8, and 9 show the results of this procedure for meter widths of 3, 5, 8, and 12 n.u. The smallest meter width represents the greatest precision. It is found that the greater the precision of the meter the more disruptive is the measurement, which agrees with Peres [16]. For a sharp meter, we expect that a high momentum would be transferred to the particle and that as a consequence the particle would be more likely to penetrate the barrier. Without access to a great number of such graphs it is difficult to understand what is going on. It should be remembered that a random number generator is being used and so the graphs shown here represent single examples of possible outcomes of such measurements.

V. CONCLUSIONS

In general, we have found that measurements involve a transfer of momentum to the particle which has a tendency to enhance the probability of barrier penetration, and therefore speed up the decay process.

Returning to the first point mentioned in the abstract, we would like to reconsider the form of measurement adopted, in an attempt to revive the Zeno paradox. After close examination of the paper by Misra and Sudarshan,

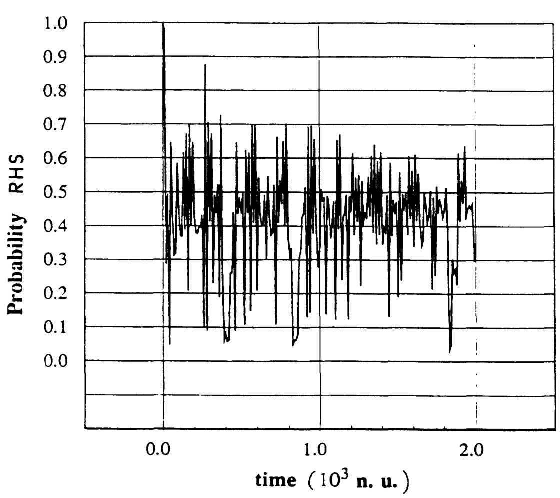

FIG. 6. Effect of the number of measurements: Plot of the probability of finding the particle on the RHS of the barrier vs time. Taking 100 measurements in a time of $2000\omega^{-1}$, the graph represents a meter width of 3 n.u.

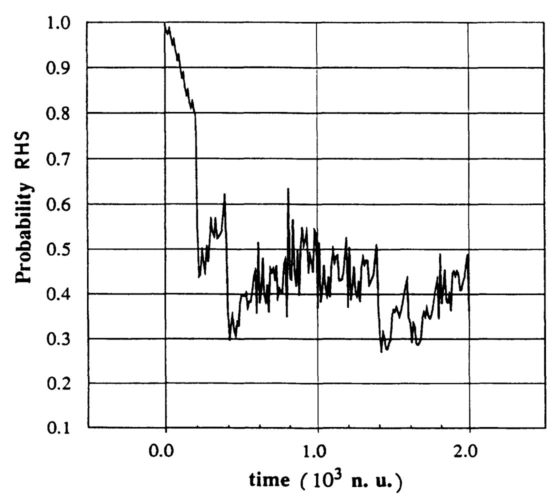

FIG. 8. Effect of meter precision: Plot of the probability of finding the particle on the RHS of the barrier vs time. Taking 10 measurements in a time of $2000\omega^{-1}$, the graph represents a meter width of 8 n.u.

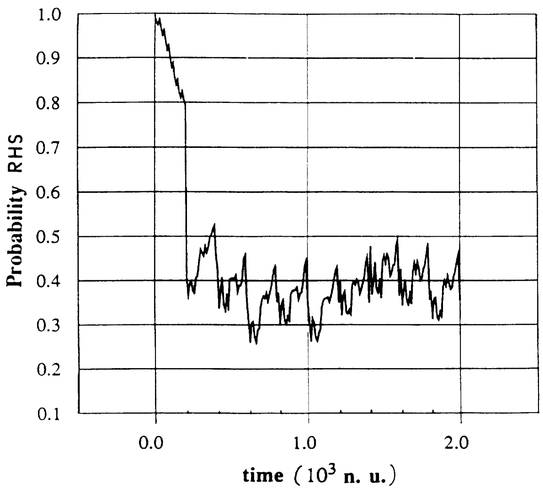

FIG. 9. Effect of meter precision: Plot of the probability of finding the particle on the RHS of the barrier vs time. Taking 10 measurements in a time of $2000\omega^{-1}$, the graph represents a meter width of 12 n.u.

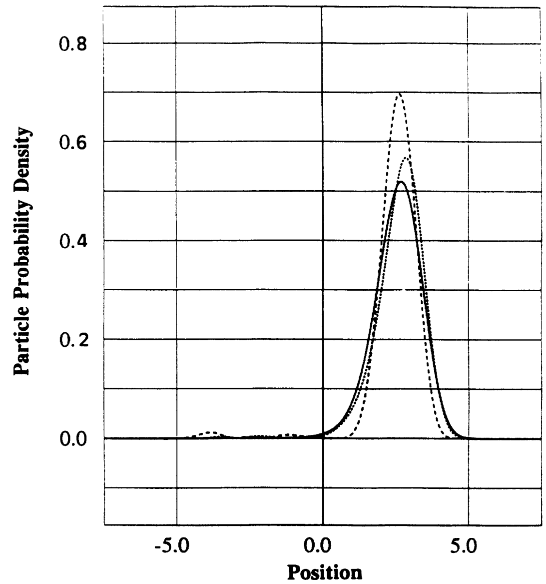

FIG. 10. An attempt to get the Zeno effect as described by Misra and Sudarshan. Plot of the particle probability density functions, as a function of position. The time of each plot is related to the barrier penetration time of the undisturbed particle, T. The solid line represents the initial probability density function, the dotted line shows the probability density function at a time $0.025T$ later and the dashed curve shows the probability density at a time $0.175T$. These times were chosen to show the full range of motion of the curve during the entire run. 100 measurements were made during a time duration, $2000\omega^{-1}$.

we were unable to find any mention of a meter to obtain the measurements. The paradoxical results reported in their paper appear *not* to require a meter. The best we could do to imitate their procedure was to assume that we knew the probability density of the particle at all times, and chose the measured position to always be at the peak of the probability density function. Our normalized Gaussian meter was given a measured X of X_m equal to the position of the maximum in the particle density. The meter then has the effect of somewhat collapsing the wave function of the particle into a "measured state" (in the Dirac–von Neumann sense). Misra and Sudarshan simply stated that a measurement would result in a reduction of the wave packet, without providing any physical means of achieving this. Our imitation of their method actually tells the meter what value to give as a measurement, and thus robs the meter of its purpose. However, this procedure closely resembles the intended method as stated by Misra and Sudarshan.

The results we obtained by this method are shown in Figs. 10 and 11. We found that this unrealistic assumption would show a very clear Zeno effect. Figure 10 shows the initial particle density and two sample densities plotted at later times. The particle probability density functions are seen to stay on the right of the barrier. The samples were chosen to reveal the full extent of the oscillation in density height and width. It appears that the density function was trying to adapt itself to an eigenfunction of a single well potential, situated on the right of the barrier. The oscillations which occurred are more clearly seen in Fig. 11, which shows a plot of probability for finding the particle on the right-hand side of the barrier versus time. Although the time of the run was much longer than that shown in the graph, it was necessary to cut the time at $350\omega^{-1}$ in order to show the initial oscillation clearly on the scale chosen. The oscillations continue in the same manner, well past the maximum time shown (at least until $t = 2000\omega^{-1}$).

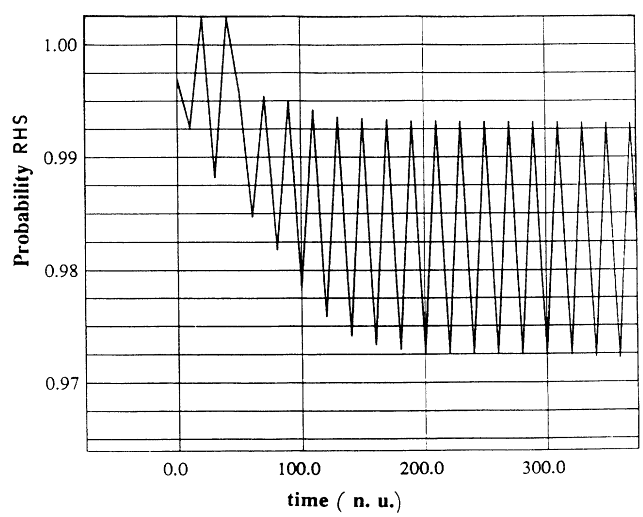

FIG. 11. An attempt to get the Zeno effect as described by Misra and Sudarshan. Plot of the probability of finding the particle on the right, as a function of time. Times up to $300\omega^{-1}$ are plotted to show the initial oscillations and the general trend of the curve thereafter. Clearly, there is some numerical error since we show one point with a probability greater than one. However, this is a small error on the scale used and does not detract from the physics portrayed by the figure.

According to our calculations using a meter measurement, we find no quantum Zeno effect. In this paper we have not tried to model "continuous measurements." In fact, we would say that the concept is not well defined. The obvious meaning would be to take the limit when the consecutive measurements become so close that a measurement is made immediately after the previous one. We have shown that precise sequential quantum measurements lead to a rapid flipping of the particle from one side of the barrier to the other. For example, see Fig. 6, which corresponds to a narrow meter width and measurements made in rapid succession. Finally, it is clear that one should be very careful about the theoretical (or experimental) conditions before claiming to observe quantum Zeno-type results.

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