Physical relation between quantum mechanics and solitons on a thin elastic rod

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The 2×2-matrix-valued first-order linear differential equation appears when we solve the modified Korteweg–de Vries (MKdV) equation. This linear differential equation is regarded as a fictitious quantum equation. On the other hand, it is known that the dynamics of an elastic rod is governed by the MKdV equation. In this paper, after we construct a Dirac equation on an elastic rod embedded into $(2+1)$-dimensional space-time, we show that this linear differential equation is naturally introduced through this Dirac equation. Then we can explain the reason why the classical nonlinear differential equation is associated physically with quantum mechanics. In other words, we prove that fictitious quantum mechanics related to the soliton is real quantum mechanics on the soliton as a base space. We also argue that the Berry phase of the Dirac particle is related to the Lax pair $L_\tau = i[L,B]$.

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I. INTRODUCTION

In the last three decades, soliton physics has evolved dynamically. The Korteweg–de Vries (KdV) equation has been especially studied in detail [1]. It is known that the modified Korteweg–de Vries (MKdV) equation is related to the KdV equation through the Miura transformation. Using this transformation, Gardner, Greene, Kruskal, and Miura (GGKM) found that a fictitious Schrödinger equation is associated with the solution of the KdV equation [2]. Wadati studied the MKdV equation in detail and showed the relation between the fictitious 2×2-matrix-valued first-order linear differential equation and the MKdV equation [3]. In other words, if we regard this linear differential equation as a Dirac equation, there seems to be a duality between the classical integrable nonlinear differential equation and quantum mechanics as a category. Why the classical equation is related to quantum mechanics is a natural question.

In this paper we explain this physical reason by considering quantum mechanics on an elastica or thin elastic rod. In other words, we show that fictitious quantum mechanics related to the soliton is real quantum mechanics on the soliton as a base space.

The shape and dynamics of the elastic rod is interesting. Tsuru and Wadati considered the shape of DNA as an elastic rod [4]. Konno, Ichikawa, and Wadati [5] proved that there are loop solitons on an elastic rod and Ishimori [6] showed that these solitons are governed by the MKdV equation. More general solutions of the equation of the rod are found by Tsuru [7]. An elastic rod embedded in a manifold was studied by Langer and Singer [8]. Recently, Goldstein and Petrich found a MKdV hierarchy by considering its dynamics [9].

On the other hand, quantum mechanics on one-dimensional (1D) or two-dimensional (2D) curved space embedded into three dimensions (3D) was studied by da Costa [10] using an operator formulation and by Matsutani [11] using path-integral formulation. We considered a curved quantum wire and found a shape of the wire that satisfies the reflectionless condition related to the KdV soliton [12]. In this paper, we will construct a Dirac equation on a curved elastic rod. This will give the Dirac equation some embedded effects. This Dirac equation becomes the da Costa's Schrödinger equation after we take the nonrelativistic limit. We also argue that the Berry phase of the Dirac particle is related to the Lax pair.

II. THIN ELASTIC ROD

In this section we will consider the dynamics of a thin elastic rod. The dynamics of the rod is studied in detail [5–9]. We will follow the argument of Watanabe [13]. First of all, we shall define a system: Let $\mathbf{r}$ be the position along the center axis of the rod C. For the sake of simplicity, we consider the flat surface F on which the curve C exists. In other words, since the problem we consider is essentially 2D, we shall reduce the 3D problem to 2D.

We introduce the orthonormal coordinate system along C; the tangent unit vector $\mathbf{n}_1$, the normal unit vector $\mathbf{n}_2$ of the curve C. Let s ($\equiv q^1$) be the arc length of C. If we point to a position $\mathbf{R}=(R^1,R^2)$ deviating from C but in the vicinity of C, we can express this position using the curved system,

$$\mathbf{R}=\mathbf{r}+\mathbf{n}_2 q^2 . \tag{1}$$

The curvature of the rod is denoted by $\mathbf{k}(s)=\partial_s \mathbf{n}_1$ and $\partial_s := \partial/\partial s$. Conveniently, we introduce a *zweibein* $e^i{}_\mu := \partial_\mu R^i$,

$$e^i{}_\mu = [1-k(q^1)q^2\delta_{\mu 1}]n^i{}_\mu , \tag{2}$$

not summed over μ. Here and hereafter, the Latin indices $(R^i, R^j, \ldots)$ indicate the Cartesian coordinate and the Greek indices $(q^\mu, q^\nu, \ldots)$ indicate the curved coordinate along C. Furthermore, we use the Einstein convention, in which we sum over the index if it appears twice in an equation unless we state.

Let us return to consider the rod problem. The Lagrangian density is given by

$$\mathcal{L}=\tfrac{1}{2}\rho\dot{\mathbf{r}}^2-\tfrac{1}{2}A\mathbf{k}^2 , \tag{3}$$

where ρ is the mass density and A is an elastic constant. The action becomes

$$S=\int dt\int_0^l ds\,\mathcal{L}[\mathbf{k},\mathbf{r}] , \tag{4}$$

where l is the length of the rod.

Using an infinitesimal value $\epsilon(s,t)$, we express the variation of curve C_ϵ from C as

$$\mathbf{r}_\epsilon(s_\epsilon)=\mathbf{r}(s)+\mathbf{n}_2\epsilon(s,t) . \tag{5}$$

Immediately we obtain the relations

$$\partial_s\mathbf{r}_\epsilon=(1-\epsilon k)\mathbf{n}_1+(\partial_s\epsilon)\mathbf{n}_2 \tag{6a}$$

and

$$ds_\epsilon^2=d\mathbf{r}_\epsilon d\mathbf{r}_\epsilon=(1-2\epsilon k)ds^2+O(\epsilon^2) . \tag{6b}$$

The curvature becomes

$$\mathbf{k}_\epsilon:=\frac{\partial^2}{\partial s_\epsilon^2}\mathbf{r}_\epsilon=\mathbf{n}_1[-(\partial_s\epsilon)k]+\mathbf{n}_2[k+(k^2+\partial_s^2)\epsilon]+O(\epsilon^2) , \tag{7a}$$

$$k_\epsilon=k+(\partial_s^2+k^2)\epsilon+O(\epsilon^2) , \tag{7b}$$

$$k_\epsilon^2 ds_\epsilon=[k^2+(2k\partial_s^2+k^3)\epsilon]ds+O(\epsilon^2) . \tag{7c}$$

When we define $\theta:=\angle(R^1,q^1)$, we can express $\mathbf{n}_1$ as

$$\mathbf{n}_1=(\sin\theta,\cos\theta) . \tag{8}$$

Then $k=\partial_s\theta$ and $\dot{\mathbf{n}}_1=\dot{\theta}\mathbf{n}_2$. In order to simplify the problem, we consider the case in which the shape of the rod does not change. Accordingly we assume that $\mathbf{n}_1(s,t)=\mathbf{n}_1(s-ut,0)$ or $\dot{\mathbf{r}}=:u\mathbf{n}_1$. We note that this assumption is corresponding to one soliton solution. When we assume that the local length of the rod does not change or $\partial_s\dot{\mathbf{r}}\equiv\dot{\mathbf{n}}_1$, we have the relations, $ku=\dot{\theta}$ and $\partial_s u=0$. The kinetic term becomes

$$\dot{\mathbf{r}}_\epsilon=(u-\epsilon\dot{\theta})\mathbf{n}_1+\dot{\epsilon}\mathbf{n}_2 , \tag{9a}$$

$$\dot{\mathbf{r}}_\epsilon^2 ds_\epsilon=(u^2-3u\epsilon\dot{\theta})ds+O(\epsilon^2) . \tag{9b}$$

The variation of $\mathcal{L}$ brings about the equation of motion,

$$-\tfrac{3}{2}\rho u\frac{\partial\theta}{\partial t}+A\frac{\partial^3\theta}{\partial s^3}+A\frac{1}{2}\left[\frac{\partial\theta}{\partial s}\right]^3=0 . \tag{10}$$

Redefining the time t by τ appropriately, (6a) becomes

$$\theta_\tau+\tfrac{1}{2}\theta_s^3+\theta_{sss}=0 . \tag{11}$$

In particular, when we write $v:=\frac{1}{2}k=\frac{1}{2}\theta_s$, we obtain the usual MKdV equation,

$$v_\tau+6v^2v_s+v_{sss}=0 . \tag{12}$$

This is in agreement with the results of Ishimori [6] and Goldstein and Petrich [9]. We note that they show that the rod is governed by the MKdV equation in general though we assumed one soliton situation to derive this equation. By the Miura transformation, $\omega:=v^2+v_x$, ω obeys the KdV equation.

III. DIRAC EQUATION ON AN ELASTIC ROD

In this section we consider a Dirac equation on the elastic rod. This Dirac equation will govern the behavior of an electron and a hole along the rod in the vicinity of the Fermi surface. First of all, we extend 2D space to (2+1)-dimensional space-time. Let μ run from 0 to 2; $q^0\equiv R^0:=\tau$, $e^0{}_\mu:=\delta^0{}_\mu$ and $e^i{}_0:=\delta^i{}_0$. The field operator is $\Psi=:(\Psi^1,\Psi^2)^T$. We shall quickly review the curved Dirac equation [14] and confine it into the elastic rod. The original Lagrangian is given by

$$\mathcal{L}=ie\overline{\Psi}\gamma^\mu D_\mu\Psi-e\overline{\Psi}V\gamma_0\Psi , \tag{13}$$

where the (2+1)-dimensional γ matrix is $\gamma^\mu:=\gamma^i e_i{}^\mu$, $e:=\det(e^i{}_\mu)=(1-kq^2)$, and V is a confinement potential, $V:=\gamma_0(q^2)^2/2\delta$ with $\delta\to 0$. Furthermore, D_μ denotes the spin connection

$$D_\mu:=(\partial_\mu+\Omega_\mu) , \tag{14a}$$

$$\Omega_\mu:=\tfrac{1}{2}\Sigma^{ij}e_i{}^\nu(\nabla_\mu e_{j\nu}) , \tag{14b}$$

where $\partial_\mu:=\partial/\partial q^\mu$. The spin matrix is $\Sigma^{ij}:=\frac{1}{4}[\gamma^i,\gamma^j]$. The covariant derivative ∇_μ is $\nabla_\mu b_\nu:=\partial_\mu b_\nu-\Gamma^\lambda_{\mu\nu}b_\lambda$ for a covariant vector b_μ where the Christoffel symbols are calculated; $\Gamma^2_{11}=k(1-kq^2)$, $\Gamma^1_{21}=-k/(1-kq^2)$, $\Gamma^1_{11}=-\partial_1 kq^2(1-kq^2)$, and the others vanish. After a straightforward calculation, the spin connections become

$$D_0=\partial_\tau,\quad D_1=\partial_s+\kappa\Sigma^{12},\quad D_2=\partial_2 , \tag{15}$$

where $\kappa\equiv 0$. Let us follow the argument of da Costa [10]. Since the measure on the curved system is $d^3R=ed^3q$, we redefine the field operator as $\psi=(1-kq^2)^{1/2}\Psi$. Due to the confinement potential V, q^2 vanishes and then the Lagrangian density along the elastic rod becomes

$$\mathcal{L}_{\text{eff}}=i\overline{\psi}(\gamma^0\partial_0+\gamma^1\partial_1+\tfrac{1}{2}k\gamma^2)\psi , \tag{16}$$

where we rewrite the field operator as $\psi(q^0,q^1):=\psi(q^0,q^1,q^2\equiv 0)$. We notice that κ vanishes, while the effect from the Jacobian e remains. This implies that the spin connection is not essential. This is natural because this (1+1)-dimensional space-time surface is flat as a manifold. However, if we rewrite $\gamma^1\partial_1+\frac{1}{2}k\gamma^2=\gamma^1(\partial_1+i\frac{1}{2}k\gamma^0)$, we can regard the $k/2$ term (the effect from the Jacobian) as a new spin connection. From (16), the equation of motion, or the Dirac equation along the elastic rod, becomes

$$(\sigma^3\partial_0-\sigma^2\partial_1-\sigma^1\tfrac{1}{2}k)\psi=0 , \tag{17}$$

where (1+1)-dimensional γ matrices are denoted by Pauli matrices; $\gamma^0:=\sigma^3$, $\gamma^1:=\sigma^2$, and $\gamma^2:=i\gamma^0\gamma^1=\sigma^1$. We notice that ψ satisfying this equation of motion is classical in the meaning of the second quantization. In other words, this equation is that of the quantum mechanics. If we take the nonrelativistic limit, it agrees with the Schrödinger equation of da Costa [10–12] (see the Appendix). The energy of the electron and hole is opposite sign. When the energy eigenvalue is written as E, we can express $\psi^a(q^0,q^1)=\phi^a(q^1)\exp[(-1)^a iE\tau]$for $a=1,2$. Then we can make this equation easy,

$$L:=\sigma^2\partial_s+\sigma^1 v \ , \qquad L\psi=\Lambda\psi, \quad \Lambda:=-iE \ , \tag{18}$$

where $v\equiv k/2$ as we defined. This operator L agrees with the Lax operator of the MKdV equation [3]. Accordingly, we find the physical meaning of the relation between the classical nonlinear equation and quantum mechanics, or the GGKM method. We notice that this relation is deduced naturally and then the inverse problem actually has a physical meaning.

What physical meaning the other operator B of the Lax pair, $L_\tau=i[L,B]$, has is another question. For the MKdV equation, B has the form [3]

$$B=-4i\partial_s^3-3i\partial_s[v^2-i(\partial_s v)\sigma_3]-3i[v^2-i(\partial_s v)\sigma_3]\partial_s \ . \tag{19}$$

The equation $i\psi_\tau=B\psi$ indicates $\partial_\tau E\equiv 0$ for the MKdV equation. This time evolution of the wave function implies that the base space (rod) adiabatically changes its shape without changing the eigenstate of the particles. In other words, this change of ψ is the same as the Berry phase [15].

From this point of view, $\partial_\tau E\equiv 0$ is an adiabatic condition for a particle and a hole. When we change the form of the base space in order to satisfy this condition, we obtain the time evolution of the soliton. This problem can be solved using the Lax pair [3]. We notice that the time evolution $i\psi_\tau=B\psi$ is independent of the time-development equation (17).

We note that if we consider the dynamics of the rod, it generates the curvature along the time direction on the $(1+1)$-dimensional space-time surface. However, we have neglected this effect because it slowly changes compared with k.

IV. CONCLUSION

In this paper we have shown the physical meaning of the Lax pair [3] of the MKdV equation by considering the Dirac equation along the elastic rod. Then one of the Lax operators L is regarded as the Dirac equation, while the other B is as the Berry phase operator. In other words, we found the fact that a fictitious quantum mechanics related to the soliton is a real quantum mechanics on the soliton as a base space.

We have also shown that the Dirac equation on a curved 1D space embedded in 3D has an embedded effect.

It is known that the number of the loop solitons on an elastic rod is the same as the nontrivial crossing points of the rod. When we write the right-hand side of (17) $\lambda\psi$ instead of 0, 0 means the zero eigenvalue of the field operator. Let us restrict ψ to being the rapidly decreasing function [16]. Since the number of the solitons agrees with the number of zero eigenvalues, this fact reminds us of the index theorem or the anomaly in the quantum field theory [15,16].

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APPENDIX

Add the mass term to the Dirac equation (17),

$$(m+i\hbar\sigma^3\partial_0)\psi=i\hbar(\sigma^2\partial_1+\sigma^1\tfrac{1}{2}k)\psi \ . \tag{A1}$$

We make the operators on both sides act on them again and we assume $E\ll m$,

$$(m^2+2mi\hbar\sigma^3\partial_0)\psi\simeq\left[-\hbar^2\partial_1^2-\frac{\hbar^2}{4}k^2+i\frac{\hbar^2}{2}\sigma^3(\partial_1 k)\right]\psi \ . \tag{A2}$$

After the energy level changes, we obtain

$$i\sigma^3\left[\hbar\partial_0-\frac{\hbar^2}{2}(\partial_1 k)\right]\psi=\left[-\frac{\hbar^2}{2m}\partial_1^2-\frac{\hbar^2}{8m}k^2\right]\psi \ . \tag{A3}$$

If we take the Hermitian part, the particle equation becomes

$$i\hbar\partial_0\psi_1=\left[-\frac{\hbar^2}{2m}\partial_1^2-\frac{\hbar^2}{8m}k^2\right]\psi_1 \ . \tag{A4}$$

This agrees with da Costa's result [10–12]. We notice that we consider the case $\frac{1}{2}(P^*P+PP^*)$ instead of (P^*P) or (PP^*) for an operator P here, which corresponds to the fact that we take the Hermitian part. We note that this Schrödinger equation is related to the KdV equation [12,1].

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