General-relativistic hydrodynamics of a collisionless plasma in a strong magnetic field

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The closed set of general-relativistic hydrodynamical equations describing a strongly magnetized collisionless plasma with an anisotropic pressure tensor is derived. Consideration is based on the "3+1" formulation of magnetohydrodynamics and the orthonormal tetrad technique. The model is the further generalization of the theory of Chew, Goldberger, and Low [Proc. R. Soc. London Ser. A **236**, 1204 (1954)] for the general-relativistic case. In ultrarelativistic limit the equations of state are obtained, which differ noticeably from those known previously.

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I. INTRODUCTION

It is widely accepted that the study of the hydrodynamics of collisionless plasma in strong magnetic fields is important for a wide class of astrophysical objects, including relativistic pulsar winds and powerful jets in active galactic nuclei and quasars. Due to the high compactness of the central engines in these objects (rapidly rotating neutron stars and supermassive black holes), general-relativistic effects become important and must be taken into account. At the same time, it is known that in magnetohydrodynamics, consideration of relativistic effects becomes necessary not only due to the high velocities of plasma macroscopic motion, but also when the velocities of the plasma particles' microscopic motions are high enough, i.e., when the temperature of plasma becomes relativistic.

Relativistic magnetohydrodynamics for the medium with isotropic pressure was considered in Refs. [1,2]. In the most general form, the closed set of molecular hydrodynamic (MHD) equations for relativistic collisionless plasma may be derived on the basis of the relativistically invariant Vlasov kinetic equation. In Ref. [3] by means of the formalism developed in Ref. [2] from kinetic equations, the closed set of relativistic MHD equations was obtained.

When formulating the system of hydrodynamic equations, first will appear the problem of the definition of macroscopic parameters describing the state of the medium (such as particle number density, hydrodynamical velocity, the "temperature" corresponding to each plasma component, etc.). These definitions must be made in the rest frame of the given plasma component, i.e., macroscopic parameters must be introduced by means of the corresponding integration of the distribution function of plasma particles in the phase space of chaotic momenta. In nonrelativistic theory, this requirement is satisfied by means of the usual Galileo transformations for particle velocities. In the more general, special-relativistic case, it seems more correct to define the main macroscopic parameters of plasma on the basis of Lorentz transformations for particle energy and momentum. Such an approach was developed in Ref. [4], where the closed set of relativistic MHD equations was obtained for strongly magnetized collisionless plasma.

It is known that synchrotron-radiation losses considerably change the properties of the strongly magnetized collisionless plasmas of pulsar winds or jets in such a way that their temperatures become highly anisotropic. In this case, the plasma pressure is no longer a scalar and thus its consideration must be based on the hydrodynamical model of relativistic plasma with the anisotropic pressure tensor. Pressure anisotropy leads to the changes of the overall properties of such a medium. In particular, new types of magnetohydrodynamical instabilities may appear as well.

In the present paper, we formulate a hydrodynamical model for general-relativistic, strongly magnetized collisionless plasma. The model is the further generalization of the special-relativistic model outlined in Ref. [4], which in turn has generalized the Chew, Goldberger, and Low model [5] for a nonrelativistic plasma.

We present our model based on the "3+1" formalism of Thorne and Macdonald [6,7]. The plasma is assumed to exist in a space-time described by the standard metrics of a rotating body; we do not account for the self-gravity of the matter.

II. MAIN CONSIDERATION

Hereafter we shall use the following notations: (i) greek indices will range over t,r,θ,ϕ and represent space-time coordinates, components, etc., (ii) latin indices will range over r,θ,φ and represent coordinates in three-dimensional "absolute" space, and (iii) we shall use geometrical units, so that $G=c=1$.

The rotation of the central object (for example, a rapidly rotating neutron star or Kerr black hole) introduces off-diagonal terms $g_{t\varphi}$ in the metric so that the space-time generated by a rotating object is represented by the metric

$$ds^2 \equiv -d\tau^2 = g_{tt}dt^2 + 2g_{t\varphi}dt\,d\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 , \tag{2.1}$$

with the metric coefficients independent of t and φ. In $3+1$ notations, (2.1) may be rewritten as

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -\alpha^2 dt^2 + \gamma_{ik}(dx^i + \beta^i dt) \times (dx^k + \beta^k dt) , \tag{2.2}$$

where α is the so-called "lapse function" defined as

$$\alpha^2 \equiv \frac{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}{g_{\varphi\varphi}} , \tag{2.3}$$

γ_{ik} is three-dimensional "absolute" space metric tensor (with nonzero components g_{rr}, $g_{\theta\theta}$, and $g_{\varphi\varphi}$), and

$$\beta^i \equiv \left\{ 0, 0, \frac{g_{t\varphi}}{g_{\varphi\varphi}} \right\} , \quad \beta_i = \gamma_{ik}\beta^k . \tag{2.4}$$

The microscopic state of the collisionless electron-ion (or electron-positron) plasma can be described by means of the relativistic distribution function of plasma particles $\Phi(x^\alpha, p^\beta)$. The function satisfies the kinetic equation

$$\frac{\partial\Phi}{\partial x^\alpha}\frac{dx^\alpha}{d\tau} + \frac{\partial\Phi}{\partial p^\alpha}\frac{dp^\alpha}{d\tau} = 0 , \tag{2.5}$$

where p^α are contravariant components of the plasma particles' four-momentum defined as

$$p^\alpha \equiv m u^\alpha = \frac{m dx^\alpha}{d\tau} . \tag{2.6}$$

In $3+1$ notations, components of 4-velocity u^α may be expressed through 3-velocity $\mathbf{v}$ components in absolute three-dimensional space in the following way:

$$u^\alpha \equiv \left\{ \frac{\gamma}{\alpha} ; \frac{\gamma}{\alpha}(\alpha\mathbf{v} - \boldsymbol{\beta}) \right\} , \tag{2.7a}$$

$$u_\alpha \equiv \gamma\{ -\alpha + (\boldsymbol{\beta}, \mathbf{v}) ; \mathbf{v} \} , \tag{2.7b}$$

where $\gamma \equiv (1 - \gamma_{ik}v^i v^k)^{-1/2}$ is the Lorentz factor corresponding to 3-velocity $\mathbf{v}$. The equation of motion for plasma individual particles may be written as

$$\frac{mdp^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma + qF^{\alpha\beta}p_\beta . \tag{2.8}$$

Here $F^{\alpha\beta}$ is a general-relativistic tensor of the electromagnetic field. In $3+1$ formalism the tensor may be expressed by means of spatial vectors of electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields in the following way:

$$F^{\alpha\beta} = e^\alpha_{[t]}E^\beta - e^\beta_{[t]}E^\alpha + \epsilon^{\alpha\beta\gamma\delta}e_{\gamma[t]}B_\delta , \tag{2.9}$$

where $E^0 = E_0 = B^0 = B_0 = 0$, $e_{\gamma[t]} = g_{\gamma\alpha}e^\alpha_{[t]}$, and

$$e^\alpha_{[t]} = \{ 1/\alpha ; -\boldsymbol{\beta}/\alpha \} . \tag{2.10}$$

Note that $e^\alpha_{[t]}$ in general are the components of orthonormal tetrads known in the mathematical theory of black holes as locally nonrotating frames (LNRF's) [8], zero-angular-momentum observers (ZAMO's) [7], or simply Bardeen tetrads. Other nonzero components of $e^\alpha_{[\beta]}$ are $e^i_{[k]} = (g_{ii})^{-1/2}$. Note that $e^\alpha_{[\beta]}e^{[\gamma]}_\alpha = \delta^{[\gamma]}_{[\beta]}$ and

$$e_t^{[t]} = \alpha , \tag{2.11a}$$

$$e_t^{[\varphi]} = \beta_\varphi (g_{\varphi\varphi})^{-1/2} , \tag{2.11b}$$

$$e_i^{[i]} = \sqrt{g_{ii}} . \tag{2.11c}$$

Inserting (2.8) into (2.5), we get a kinetic equation in the covariant form

$$P^\alpha \tilde{\nabla}_\alpha \Phi = -qF^{\alpha\beta}p_\beta \frac{\partial\Phi}{\partial p^\alpha} , \tag{2.12}$$

where $\tilde{\nabla}_\alpha$ is Cartan's covariant derivative operator, defined as

$$\tilde{\nabla}_\alpha \equiv \frac{\partial}{\partial x^\alpha} - \Gamma^\gamma_{\alpha\beta}P^\beta \frac{\partial}{\partial p^\gamma} .$$

If we introduce the distribution function $f(x^\alpha, p^i)$ defined in seven-dimensional phase space according to the expression

$$\Phi(x^\alpha, p^\beta) = f(x^\alpha, p^i)\delta((-g_{\alpha\beta}p^\alpha p^\beta)^{1/2} - m)\theta(p^t) , \tag{2.13}$$

where $\theta(p^t) = 1$ when $p^t \geq 0$ and is equal to zero when $p^t < 0$ when using the fact that

$$F^{t\beta}u_\beta = \gamma(\mathbf{E}, \mathbf{v})/\alpha , \tag{2.14a}$$

$$F^{i\beta}u_\beta = \gamma[\mathbf{E} + (\mathbf{v}\times\mathbf{B}) - \boldsymbol{\beta}(\mathbf{E}, \mathbf{v})/\alpha] , \tag{2.14b}$$

we can write down the kinetic equation for the function $f(x, \mathbf{p})$ in the following fashion:

$$P^\alpha \tilde{\nabla}_\alpha f = -\alpha q[\mathbf{E} + \mathbf{v}\times\mathbf{B} - (\boldsymbol{\beta}/\alpha)(\mathbf{E}, \mathbf{v})]\nabla_{\mathbf{p}} f . \tag{2.15}$$

It is well known that from kinetic equations (2.12) or (2.15), one can obtain the system of transport equations for the macroscopic parameters of plasma (such as proper particle number density n, or the mean thermal energy mW). But it should be emphasized that all these quantities must be defined in the rest frame of the medium—in the reference frames co-moving with the fluid. For the metric (2.2), it is possible to introduce such orthonormal tetrads in each point of space-time. Hereafter, we shall call them general co-moving frames (GCMF). Below we write down all nonzero components of these tetrads:

$$e^t_{(t)} = \frac{\gamma}{\alpha} , \tag{2.16a}$$

$$e^t_{(i)} = \frac{\gamma}{\alpha}v^{[i]} , \tag{2.16b}$$

$$e^i_{(t)} = \gamma(v^{[i]} - \beta^{[i]}/\alpha)(g_{ii})^{-1/2} , \tag{2.16c}$$

$$e^i_{(k)} = \left[\eta^{[i][k]} + (\gamma - 1)\frac{v^{[i]}v^{[k]}}{v^2} - \frac{\gamma}{\alpha}\beta^{[i]}v^{[k]} \right](g_{ii})^{-1/2} , \tag{2.16d}$$

$$e_t^{(t)} = \gamma[\alpha - (\boldsymbol{\beta}, \mathbf{v})] , \tag{2.16e}$$

$$e_t^{(i)} = -\alpha\gamma v^{[i]} + \beta^{[i]} + (\gamma - 1)\frac{(\boldsymbol{\beta}, \mathbf{v})}{v^2}v^{[i]} , \tag{2.16f}$$

$$e_i^{(t)} = -\gamma\sqrt{g_{ii}}\, v^{[i]} , \tag{2.16g}$$

$$e_i^{(k)} = \sqrt{g_{ii}} \left[\eta^{[i][k]} + (\gamma - 1)\frac{v^{[i]}v^{[k]}}{v^2} \right] . \tag{2.16h}$$

Note that for GCMF tetrad indices we use parentheses, while for LNRF tetrad indices we use square brackets.

In the case of the Kerr metric, if $v^{[i]}$ has only one azimuthal nonzero component, (2.16) tetrads reduce to so-called "orbiting systems" [9]. When all $v^{[i]}=0$ it, in turn, reduces [i] to Bardeen tetrads. It must also be noted that in the absence of gravitation, these tetrads reduce to plain Lorentz transformations which, in turn, further reduce to usual Galileo transformations in nonrelativistic limit (as it should be). As far as at each point of space-time by means of (2.11) and (2.16), we can define two instant local Lorentz reference frames, it is obvious that transition between them occurs due to ordinary Lorentz transformations with the velocity $v^{[i]}$. It may easily be proved by simple algebraic calculations.

GCMF appear to be useful tools for the derivation of the transport equations that we mentioned earlier. In particular, they may be used to generalize the method used in Ref. [4] (see also Ref. [10], where the analogous problem is solved in the three-formalism approach). The transport equations may be obtained directly similar to the approach used in Ref. [4]. But these equations may be simply written if we apply general covariancy principles to special-relativistic transport equations presented in Ref. [4]. According to the principle (see Ref. [11]), correct general-relativistic equations may be obtained by replacement of ordinary derivatives by covariant ones and of Lorentz metric tensors $\eta^{[\alpha][\beta]}$ by the metric tensor of curved space-time $g^{\alpha\beta}$. This means that we get the following set of equations:

$$J^{\alpha}{}_{;\alpha} = 0 , \tag{2.17}$$

$$T^{\alpha\beta}{}_{;\beta} = qF^{\alpha\beta}J_{\beta} , \tag{2.18}$$

$$M^{\alpha\beta\gamma}{}_{;\gamma} = q(F^{\alpha}_{\gamma}T^{\gamma\beta} - F^{\beta}_{\gamma}T^{\alpha\gamma}) . \tag{2.19}$$

All quantities appearing in these equations may be split on their components in GCMF's. For example, for current four-vector J^{α} we have

$$J^{\alpha} = e^{\alpha}_{(\nu)} I^{(\nu)} , \tag{2.20a}$$

$$I^{(\nu)} \equiv \int p^{(\nu)}\Phi' d\Omega'_4 = -n\eta^{(\nu)(t)} , \tag{2.20b}$$

$$J^{\alpha} = nu^{\alpha} . \tag{2.20c}$$

For energy-momentum tensor $T^{\alpha\beta}$, similarly we have

$$T^{\alpha\beta} = e^{\alpha}_{(\nu)} e^{\beta}_{(\mu)} \Pi^{(\nu)(\mu)} , \tag{2.21a}$$

$$\Pi^{(t)(t)} \equiv mn(W+1) = \int p^{(t)^2}\Phi' d\Omega'_4 , \tag{2.21b}$$

$$\Pi^{(i)(t)} = \Pi^{(t)(i)} \equiv q^{(i)} = \int p^{(i)}p^{(t)}\Phi' d\Omega'_4 , \tag{2.21c}$$

$$\Pi^{(i)(k)} = \int p^{(i)}p^{(k)}\Phi' d\Omega'_4 . \tag{2.21d}$$

As for $M^{\alpha\beta\gamma}$, we can write

$$M^{\alpha\beta\gamma} = e^{\alpha}_{(\mu)} e^{\beta}_{(\nu)} e^{\gamma}_{(\eta)} N^{(\mu)(\nu)(\eta)} , \tag{2.22a}$$

$$N^{(t)(t)(t)} \equiv m^2 nV = \int p^{(t)^3}\Phi' d\Omega'_4 , \tag{2.22b}$$

$$N^{(i)(t)(t)} \equiv 2mg^{(i)} = \int p^{(t)^2}p^{(i)}\Phi' d\Omega'_4 , \tag{2.22c}$$

$$N^{(i)(k)(t)} \equiv m\mu^{(i)(k)} = \int p^{(t)}p^{(i)}p^{(k)}\Phi' d\Omega'_4 , \tag{2.22d}$$

$$N^{(i)(k)(l)} \equiv m\delta^{(i)(k)(l)} = \int p^{(i)}p^{(k)}p^{(l)}\Phi' d\Omega'_4 . \tag{2.22e}$$

In Eqs. (2.20)–(2.22) all quantities with primes are defined in GCMF's or in the rest frames of plasma particles. All quantities defined in (2.20)–(2.22), n, W, V, $\Pi^{(i)(k)}$, $\mu^{(i)(k)}$, $g^{(i)}$, $q^{(i)}$, and $\eta^{(i)(k)(l)}$ may be rewritten more conveniently by means of a three-dimensional distribution function in the following way (see for comparison Refs. [5,10]):

$$n \equiv \int f' d\Omega'_3 , \tag{2.23a}$$

$$W \equiv \frac{1}{mn}\int \epsilon' f' d\Omega'_3 - 1 , \tag{2.23b}$$

$$V \equiv (nm^2)^{-1}\int \epsilon'^2 f' d\Omega'_3 , \tag{2.23c}$$

$$q^{(i)} \equiv \int p^{(i)} f' d\Omega'_3 , \tag{2.23d}$$

$$g^{(i)} \equiv \frac{1}{2m}\int \epsilon' p^{(i)} f' d\Omega'_3 , \tag{2.23e}$$

$$\Pi^{(i)(k)} \equiv \int \frac{p^{(i)}p^{(k)}}{\epsilon'} f' d\Omega'_3 , \tag{2.23f}$$

$$\mu^{(i)(k)} \equiv \frac{1}{m}\int p^{(i)}p^{(k)} f' d\Omega'_3 , \tag{2.23g}$$

$$\eta^{(i)(k)(l)} \equiv \frac{1}{m}\int \frac{p^{(i)}p^{(k)}p^{(l)}}{\epsilon'} f' d\Omega'_3 . \tag{2.23h}$$

Here all integrals are also taken in GCMF's. Introduced tensor and vector quantities have concrete physical meanings of their own. In particular, $\Pi^{(i)(k)}$ is the viscous stress tensor, $q^{(i)}$ the heat flux density. It can be seen that in the nonrelativistic limit ($\epsilon' \simeq m$), $\mu^{(i)(k)} \to \Pi^{(i)(k)}$, and the vector $g^{(i)}$ reduces to $q^{(i)}$. Therefore $\mu^{(i)(k)}$ may be called the modified stress tensor and $g^{(i)}$ may be called the modified heat flux tensor. The $\eta^{(i)(k)(l)}$ are the third-order moments. They, in turn, may be expressed via fourth-order moments, and in such a way we arrive at the infinite set of interlinked equations. To make the system solvable it is necessary to close the system somehow. For this purpose one can, for example, neglect the third-order moments $\eta^{(i)(k)(l)}$. This can be justified by the fact that the higher-order moments arise due to the higher-order deviation of the macroscopic system from equilibrium than that of the lower-order moments.

Let us assume that in GCMF's the electric field is equal to zero. General connections between the $\mathbf{B} \equiv B^{[i]}$, $\mathbf{E} \equiv E^{[i]}$ and $\mathbf{B}' \equiv B^{[i]}$, $\mathbf{E}' \equiv E^{[i]}$ vectors may be written in the following way:

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - (\gamma - 1)\frac{(\mathbf{v},\mathbf{E})}{v^2}\mathbf{v} , \tag{2.24a}$$

$$\mathbf{B}' = \gamma(\mathbf{B} - \mathbf{v}\times\mathbf{E}) - (\gamma - 1)\frac{(\mathbf{v},\mathbf{B})}{v^2}\mathbf{v} , \tag{2.24b}$$

$$\mathbf{E} = \gamma(\mathbf{E}' - \mathbf{v}\times\mathbf{B}') - (\gamma - 1)\frac{(\mathbf{v},\mathbf{E}')}{v^2}\mathbf{v} , \tag{2.24c}$$

$$\mathbf{B}=\gamma(\mathbf{B}'+\mathbf{v}\times\mathbf{E}')-(\gamma-1)\frac{(\mathbf{v},\mathbf{B}')}{v^2}\mathbf{v}\ . \tag{2.24d}$$

Based on the assumption $\mathbf{E}'=0$, one gets from (2.24)

$$\mathbf{E}=-\gamma\mathbf{v}\times\mathbf{B}'=-\mathbf{v}\times\mathbf{B}\ . \tag{2.25}$$

Obviously, the assumption demands that $\mathbf{E}\perp\mathbf{B}$ and $|\mathbf{B}|>|\mathbf{E}|$. Using (2.24) and (2.25) once again, we come to the following connection between LNRF and GCMF components of the magnetic field

$$B^{(i)}\equiv\frac{1}{\gamma}e^{(i)}_{[k]}B^{[k]}\ . \tag{2.26}$$

Here $e^{(i)}_{[k]}$ are the components of the tensor that describes the Lorentz transformation between the LNRF and GCMF tetrads in the same point of space-time. The components of this matrix are

$$e^{(t)}_{[t]}=\gamma\ , \tag{2.27a}$$

$$e^{(i)}_{[t]}=e^{(t)}_{[i]}=-\gamma v^{[i]}\ , \tag{2.27b}$$

$$e^{(i)}_{[k]}=e^{(k)}_{[i]}=\delta^i_k+(\gamma-1)\frac{v^{[i]}v^{[k]}}{v^2}\ . \tag{2.27c}$$

The combination B^2-E^2 is the relativistic invariant and hence $B'^2=B^2-E^2$. It leads [together with (2.26)] to the expression that connects B' with B,

$$|\mathbf{B}'|=\frac{|\mathbf{B}|}{\gamma}[1+\gamma^2(\mathbf{v},\mathbf{b})^2]^{1/2}\ , \tag{2.28}$$

where $\mathbf{b}\equiv\mathbf{B}/|\mathbf{B}|$ is the unit vector in the direction of the magnetic-field vector.

In the forthcoming analysis, we shall assume that the three-dimensional distribution function may be expressed as

$$f'\equiv f'(t,x^i,p'_\perp 2,(\mathbf{p}',\mathbf{B}'))\ , \tag{2.29}$$

where $p_\perp$ is the transversal in respect to the $\mathbf{B}'$ component of particle momentum. In Ref. [4] it is argued that when the distribution function is of the form (2.29) and $\mathbf{q}=\mathbf{g}=0$, the tensors $\Pi^{(i)(k)}$ and $\mu^{(i)(k)}$ may be written in the following way:

$$\Pi^{(i)(k)}\equiv P_\parallel b^{(i)}b^{(k)}+P_\perp(\eta^{(i)(k)}-b^{(i)}b^{(k)})\ , \tag{2.30}$$

$$\mu^{(i)(k)}\equiv\mu_\parallel b^{(i)}b^{(k)}+\mu_\perp(\eta^{(i)(k)}-b^{(i)}b^{(k)})\ . \tag{2.31}$$

Here $b^{(i)}$ are defined as

$$b^{(i)}\equiv B^{(i)}/|\mathbf{B}'|=e^{(i)}_k b^k[1+\gamma^2(\mathbf{v},\mathbf{b})^2]^{-1/2}\ , \tag{2.32}$$

and $P_\parallel$, $\mu_\parallel$ and $P_\perp$, $\mu_\perp$ are parallel and transverse pressure and modified pressure, respectively, defined as

$$p_\perp\equiv\tfrac{1}{2}\int\frac{p'^2_\perp}{\epsilon'}f'\,d\Omega'_3\ , \tag{2.33a}$$

$$P_\parallel\equiv\int\frac{p'^2_\parallel}{\epsilon'}f'd\Omega'_3\ , \tag{2.33b}$$

$$\mu_\perp\equiv\frac{1}{2m}\int p'^2_\perp f'd\Omega'_3\ , \tag{2.33c}$$

$$\mu_\parallel\equiv\frac{1}{m}\int p'^2_\parallel f'd\Omega'_3\ . \tag{2.33d}$$

Based on these definitions and using the fact that

$$e^\alpha_{(i)}e^{(i)}_m\equiv\delta^\alpha_m+u^\alpha u_m\ , \tag{2.34}$$

we get for the energy-momentum tensor the following important expression:

$$T^{\alpha\beta}\equiv[mn(W+1)+P_\perp]u^\alpha u^\beta+P_\perp g^{\alpha\beta}+(P_\parallel-P_\perp)\Lambda^\alpha_m\Lambda^\beta_n b^m b^n\ , \tag{2.35}$$

where

$$\Lambda^\alpha_m\equiv\frac{\delta^\alpha_m+u^\alpha u_m}{[1+\tilde\gamma^2(\mathbf{v},\mathbf{b})^2]^{1/2}}\ . \tag{2.36}$$

The last term in (2.35) may be rewritten more conveniently if we introduce the following kind of electromagnetic tensor:

$$\mathsf{F}_{\alpha\beta}\equiv\tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}\ , \tag{2.37}$$

and define a new four-vector h^α via the expression

$$h^\alpha\equiv\mathsf{F}^{\alpha\beta}u_\beta\ . \tag{2.38}$$

It is worthwhile to note here that Maxwell equations

$$F_{\alpha\beta;\gamma}+F_{\beta\gamma;\alpha}+F_{\gamma\alpha;\beta}=0$$

may be written by means of $\mathsf{F}^{\alpha\beta}$ and h^α as [3]

$$\mathsf{F}^{\alpha\beta}{}_{;\beta}=0\ , \tag{2.39a}$$

$$(u^\alpha h^\beta-u^\beta h^\alpha)_{;\beta}=0\ . \tag{2.39b}$$

The four-vector h^α is orthogonal to the vector of 4-velocity $h^\alpha u_\alpha=0$. It is easily expressed through the three-dimensional vector of magnetic field $\mathbf{B}$ as

$$h^\alpha=\frac{B^m}{\gamma}(\delta^\alpha_m+u^\alpha u_m)\ , \tag{2.40a}$$

$$|h|=(B/\gamma)^2+(\mathbf{v},\mathbf{B})^2\ , \tag{2.40b}$$

and hence we arrive at the following important formula:

$$\Lambda^\alpha_m b^m=h^\alpha/|h|\ . \tag{2.41}$$

Neglecting third-order, purely spatial moments $\eta^{(i)(k)(l)}$ from the definitions (2.22) and taking into account (2.41), we get for $M^{\alpha\beta\gamma}$

$$M^{\alpha\beta\gamma}\equiv m\left[(mnV+3\mu_\perp)u^\alpha u^\beta u^\gamma+\frac{\mu_\parallel-\mu_\perp}{|h|^2}(u^\alpha h^\beta h^\gamma+u^\beta h^\gamma h^\alpha+u^\gamma h^\alpha h^\beta)+\mu_\perp(u^\alpha g^{\beta\gamma}+u^\beta g^{\alpha\gamma}+u^\gamma g^{\alpha\beta})\right]\ . \tag{2.42}$$

We see that for the hydrodynamical description of the relativistic collisionless plasma in the strong magnetic field the set of macroscopic parameters n, W, V, $P_\parallel$, $P_\perp$, $\mu_\parallel$, and $\mu_\perp$ should be introduced. In the next section we try to write down general transport equations (2.17)–(2.19), taking into account the concrete features of their ingredients.

III. DISCUSSION

First, let us write the continuity equation. It may directly be rewritten in $3+1$ notations so we may obtain [12]

$$\partial(mn\gamma)+\mathrm{div}[mn\gamma(\alpha\mathbf{v}-\boldsymbol{\beta})]=0 , \tag{3.1}$$

where the divergence operator is defined as

$$\mathrm{div}\,\mathbf{A}\equiv(\tilde{\gamma})^{-1/2}[(\tilde{\gamma})^{1/2}A^i]_{,i} , \tag{3.2}$$

and $\tilde{\gamma}\equiv\det(\gamma_{ik})$ is the determinant of three-dimensional spatial metric γ_{ik} from (2.2). Let us write now the equation of energy conservation. Generally, it may be written as

$$u^\beta T^\alpha{}_{\beta;\alpha}=0 . \tag{3.3}$$

Taking into account (2.35) and (2.41), we can write it as

$$mnu^\alpha(W+p_\perp/mn)_{,\alpha}-u^\beta p_{\perp,\beta}+\frac{P_\parallel-P_\perp}{|h|^2}h^\alpha h^\beta u_{\alpha;\beta} . \tag{3.4}$$

The last term in (3.4) may be rewritten in a more convenient form. In particular, if we use the continuity equation in the form (2.17) and Maxwell equations in the form (2.39b), we can get the following important formula:

$$\frac{1}{|h|^2}h^\alpha h^\beta u_{\alpha;\beta}=u^\alpha[\ln(|h|/n)]_{,\alpha} , \tag{3.5}$$

and hence Eq. (3.3) may be written as

$$mnu^\alpha(W+p_\perp/mn)_{,\alpha}-u^\beta p_{\perp,\beta}+(P_\parallel-P_\perp)u^\alpha[\ln(|h|/n)]_{,\alpha}=0 . \tag{3.6}$$

Let us now turn our attention to Eq. (2.19). Using various methods of projection, we can obtain the following three independent scalar equations:

$$g_{\alpha\beta}M^{\alpha\beta\gamma}{}_{;\gamma}=0 , \tag{3.7a}$$

$$u_\alpha u_\beta M^{\alpha\beta\gamma}{}_{;\gamma}=0 , \tag{3.7b}$$

$$h_\alpha h_\beta M^{\alpha\beta\gamma}{}_{;\gamma}=0 . \tag{3.7c}$$

Equation (3.7a) leads to the following simple one:

$$u^\alpha[V-(\mu_\parallel+2\mu_\perp)/mn]_{,\alpha}=0 . \tag{3.8}$$

Equations (3.7b) and (3.7c) lead to the following ones:

$$u^\alpha\mu_{\parallel,\alpha}-(\mu_\parallel/n)u^\alpha n_{,\alpha}+2\mu_\parallel u^\alpha[\ln(|h|/n)]_{,\alpha}=0 , \tag{3.9}$$

$$u^\alpha\mu_{\perp,\alpha}-(2\mu_\perp/n)u^\alpha n_{,\alpha}-\mu_\perp u^\alpha[\ln(|h|/n)]_{,\alpha}=0 . \tag{3.10}$$

Deriving (3.8)–(3.10) we once again use (3.5) for making necessary simplifications. Equations (3.6) and (3.8)–(3.10) may easily be written in $3+1$ notations if we introduce the notion of the so-called hydrodynamical derivative, defined as

$$\frac{D}{D\tau}\equiv(u^\alpha/\gamma)\frac{\partial}{\partial x^\alpha}=(1/\alpha)[\partial_t+(\alpha\mathbf{v}-\boldsymbol{\beta})\nabla] . \tag{3.11}$$

Equation (3.8) says that V is not the independent characteristic parameter of the macroscopic state of the medium, but may be expressed through $\mu_\parallel$ and $\mu_\perp$ in the following way:

$$V=\frac{\mu_\parallel+2\mu_\perp}{mn}+\text{const.} \tag{3.12}$$

From (3.9) and (3.10), we can also find one interesting expression for $\mu_\parallel$ and $\mu_\perp$. If we combine these equations in such a way that the terms containing $|h|$ are neglected, then we get

$$\frac{\mu_\perp^2\mu_\parallel}{n^5}=\text{const.} , \tag{3.13}$$

together with the equation of motion

$$(\delta^\alpha_\beta+u^\alpha u_\beta)T^{\beta\gamma}{}_{;\gamma}=0 , \tag{3.14}$$

and the modified equation for energy momentum

$$u_\beta M^{\alpha\beta\gamma}{}_{;\gamma}=q(F^{\alpha\beta}T^\gamma{}_\beta-F^\gamma{}_\beta T^{\alpha\beta})u_\beta , \tag{3.15}$$

which we will not write here in an extended form, then Eqs. (3.1), (3.6), and (3.8)–(3.10) constitute the closed set of hydrodynamical equations for the general-relativistic collisionless plasma in the strong magnetic field. Note that (3.15) connects ordinary ($P_\parallel$ and $P_\perp$) and "modified" ($\mu_\parallel$ and $\mu_\perp$) pressure. If the Chew, Goldberger, and Low theory [5] is written for the nonrelativistic, strongly magnetized plasma and the model [4] is valid for the special relativity, then the equations obtained in the present paper may be viewed as the further generalization of the latter theory, since it may be attributed to the various kinds of astrophysical flows where general-relativistic effects must be taken into account.

At the end of the paper it will be worthwhile to examine various concrete consequences of general equations (3.6) and (3.9)–(3.10). First of all, let us consider the nonrelativistic case, when $\epsilon'\simeq m$. From the definition of W (2.23b), it is easy to find that

$$W=\frac{P_\perp}{mn}+\frac{P_\parallel}{2mn} , \tag{3.16a}$$

and noting that relativistic enthalpy is defined as $\sigma\equiv[P_\perp+mn(W+1)]/n$, we immediately get the following result:

$$\sigma=m+\frac{2P_\perp}{mn}+\frac{P_\parallel}{2mn} , \tag{3.16b}$$

which coincides with the result contained in the literature [13]. Note that in this limit $\mu_\perp\rightarrow P_\perp$ and $\mu_\parallel\rightarrow P_\parallel$, so that from (3.9) and (3.10) we obtain two state equations,

$$\frac{P_\perp}{n|h|}=\text{const.}\ , \tag{3.17a}$$

$$\frac{P_\parallel |h|^2}{n^3}=\text{const.}\ , \tag{3.17b}$$

which also coincide with the well-known equations of state for nonrelativistic collisionless plasma in a strong magnetic field [13]. On the other hand, if plasma pressure is isotropic ($P_\parallel = P_\perp$), then from (3.13) we get also the well-known result

$$\frac{P}{n^{5/3}}\equiv\text{const.} \tag{3.18}$$

Now let us consider the ultrarelativistic case (when $\epsilon' \simeq |p|$). Taking into account definitions of $P_\perp$ and $P_\parallel$ and Eq. (2.23b), we get for relativistic enthalpy

$$\sigma=\frac{P_\parallel + 3P_\perp}{mn}\ , \tag{3.19}$$

and for W

$$W=\frac{P_\parallel + 2P_\perp}{mn}-1\ . \tag{3.20}$$

Using (3.19) and the general equation (3.6) we can consider various interesting cases. For example, if plasma pressure is isotropic then we immediately get a well-known result for ultrarelativistic fluids,

$$\frac{P}{n^{4/3}}=\text{const.} \tag{3.21}$$

Now, let us consider the case $P_\parallel \gg mn \gg P_\perp$. Such a situation is, as a rule, realized in pulsar winds where, because of radiative losses in the strong magnetic field, the transverse particle momentum $p_\perp$ is noticeably reduced [13]. For such a case, the same equations lead to the following equation of state:

$$\frac{P_\parallel |h|}{n^2}=\text{const.} \tag{3.22}$$

Finally, we can examine the opposite case, when $P_\perp \gg mn \gg P_\parallel$. Such an assumption seems to be reasonable for hypothetical cosmic pinches, the formation of which should probably be preceded by the state of gradual plasma plunging into a cylinder [14,15]. Thus for the latter case we get

$$\frac{P_\perp^2}{n^2|h|}=\text{const.} \tag{3.23}$$

Equations (3.22) and (3.23) noticeably differ from the ones known earlier [5,13]. This circumstance may play an active role in the explanation of actual processes taking place in relativistic MHD flows with the anisotropic pressure tensor. It must be emphasized once again that all results obtained in this section are valid for the general-relativistic collisionless plasma that is strongly magnetized. It is obvious that they may easily be applied to various kinds of astrophysical flows where the presence of a medium with similar properties is proved in one way or another.

We did not take into account the influence of radiation or pair-production processes on the flow dynamics. However, we believe that under proper conditions in astrophysical problems mentioned in the Introduction, these aspects may be relatively unimportant. For example, in relativistic pulsar winds, pressure anisotropy due to synchrotron-radiation losses is established on characteristic time scales $t_0 = 3m^3c^5/2e^4B^2$ [13]. On larger time scales ($t \geq t_0$) influence of radiation becomes negligible and relativistic, collisionless, magnetized plasma may be described by the MHD theory discussed above. It was not our aim to discuss the reasons of pressure anisotropy. In particular, inclusion of radiative effects needs separate consideration and is beyond the scope of this paper.

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