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Bound solitons in coupled nonlinear Schrödinger equations

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Interaction of solitons belonging to different modes is analyzed in the framework of a system of nonlinear Schrödinger equations with incoherent and coherent nonlinear couplings and with different group velocities. It is demonstrated that the two solitons can form a strongly bound state with coinciding centers and several weakly bound states with far-separated centers. The bound states of the latter type can be produced only by the coherent nonlinear coupling, provided it is stronger than the incoherent one. The results obtained are employed to explain qualitatively recent experiments with interactions of solitary pulses in the subcritical traveling-wave convection.

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The coupled nonlinear Schrödinger (NS) equations describe copropagation of two independent modes in a quasi-one-dimensional nonlinear medium. As a paradigm, one can take the generalized system governing the copropagation of two orthogonally polarized electromagnetic waves in a nonlinear optical fiber (see, e.g., Refs. [1,2] and references therein):

$$iu_t+icu_x+u_{xx}+2(|u|^2+\mu|v|^2)u+2(1-\mu)v^2u^*=0\,, \tag{1a}$$

$$iv_t-icv_x+v_{xx}+2(|v|^2+\mu|u|^2)v+2(1-\mu)u^2v^*=0\,, \tag{1b}$$

where the asterisk stands for the complex conjugation, and μ is a real parameter taking values $0<\mu<1$. Usually, in the theory of the optical fibers $\mu=\frac{1}{3}$, which corresponds to the Kerr nonlinearity of the dielectric material with purely electronic response. However, the crucial point of the present work will be to regard μ as an arbitrary parameter (which is sometimes admitted in the optical fiber theory [2]). The variable t in Eqs. (1) will be treated as the evolution variable, although in the fibers it has the meaning of the propagation distance, while x is the so-called reduced time [1]. The parameter c measures the difference of the group velocities of the two modes. In the optical fibers, it is produced by the birefringence effect [1].

Each subsystem (1a) and (1b) supports the usual soliton solutions of the form

$$(u,v)=2i\eta\,\mathrm{sech}[2\eta(x-z_{u,v})]\exp(4i\eta^2t+i\phi_{u,v})\,, \tag{2}$$

where η is the amplitude, hereafter assumed equal for both solitons. If the solitons move at velocities c_u and c_v, the coordinates and phases of their centers in Eq. (2) are

$$z_{u,v}=c_{u,v}t,\ \ \phi_{u,v}=\tfrac{1}{2}(c_{u,v}\mp c)x-\tfrac{1}{4}(c_{u,v}^2-c^2)t+\phi_{u,v}^{(0)}\,, \tag{3}$$

$\phi_{u,v}^{(0)}$ being arbitrary phase constants. In the optical fiber theory, the solutions given by Eqs. (2) and (3) represent polarized solitons [1,2]. The coupled NS equations occur as well in some other physical applications, the purport of the corresponding soliton solutions being similar to that of the optical solitons in the nonlinear fibers. However, the coupled NS equations with additional dissipative terms, accounting for input and loss of energy, may also appear as coupled *Ginzburg-Landau* (GL) equations modeling a number of nonlinear nonequilibrium systems, a well-known example being the coupled GL equations for the right- and left-traveling waves in the oscillatory convection [3]. The GL equations of a certain form support *solitary-pulse* (SP) solutions, which may be interpreted, in terms of the perturbation theory, as the NS solitons with the amplitude and velocity uniquely selected by the balance of the energy input and loss [4,5].

The objective of the present work is to investigate bound states (BS's) of the solitons belonging to the different modes. This analysis is spurred by the recent experimental observation of the interaction between counterpropagating SP's in the subcritical oscillatory convection [6]. The experiment was carried out in a narrow annular channel filled with a binary fluid heated from below. The stable SP's in the subcritical state of this system (i.e., when the undisturbed state of the liquid is still stable) were first reported in Ref. [7]. In the experiments performed in Ref. [7], the SP's were immobile. However, in Ref. [6] it has been demonstrated that, in fact, they were stuck by microinhomogeneities of the channel. In a very homogeneous channel, the SP's are observed to be right or left traveling at small velocities. Both the amplitude and the velocity of the SP, except for the sign of the velocity, are uniquely selected in a well-controlled experiment. Next, collisions between the traveling SP's were investigated in detail in Ref. [6]. It has been demonstrated that two qualitatively different outcomes are possible: (i) If the relative velocity of the colliding pulses is moderately small, only one pulse survives the collision; and (ii) if the pulses approach each other at a very small velocity, they eventually form a persistent quiescent double-peaked structure, which may be interpreted as a stable BS of the two SP's [6]. In this work, I aim to put forward a qualitative interpretation of these challenging experimental facts in terms of the model (1) (it is postulated, as it was first suggested in Ref. [3], that the interaction of the right- and left-propagating disturbances in the convection layer is governed by the coupled GL equations). It will be shown that the model admits BS's of two different types: (i) when the centers of the bound solitons coincide and (ii) when they are far separated. In the former case, the solitons are strongly bound, the binding energy being equally contributed to by the incoherent and coherent couplings in Eqs. (1), i.e., by the nonlinear terms proportional, respectively, to μ and $1-\mu$. In the latter case, the binding is weak, and it is solely stipulated by the coherent coupling (jointly with the group-velocity difference). The BS's of type (ii) are multiple, the one with the minimum distance between the centers of the solitons being most stable.

To attack the problem of the BS's analytically, it will be assumed that the group-velocity parameter c in Eqs. (1) is small, $c \ll \eta$. To analyze the stability of the strongly overlapped BS, in the lowest approximation one may completely neglect c. Then, just setting $u=v$, one immediately finds the bound two-soliton state in the form [cf. Eq. (2)]

$$u=v=2\sqrt{2}i\eta\,\mathrm{sech}(4\eta x)\exp(16i\eta^2 t)\,. \qquad (4)$$

The simplest way to analyze the stability of the BS (4) against a decay into free u and v solitons is to compare energies of the initial and final states. The full Hamiltonian of the underlying system (1) is $H=\int_{-\infty}^{+\infty}\mathcal{H}\,dx$, where the Hamiltonian density is

$$\mathcal{H}=\tfrac{1}{2}ic(uu_x^*-u^*u_x)-\tfrac{1}{2}ic(vv_x^*-v^*v_x)+|u_x|^2+|v_x|^2-|u|^4-|v|^4$$
$$-2\mu|u|^2|v|^2-(1-\mu)[u^2(v^*)^2+(u^*)^2v^2]\,. \qquad (5)$$

In addition to the energy (Hamiltonian), Eqs. (1) also conserve the momentum and the total wave action

$$N=\int_{-\infty}^{+\infty}(|u|^2+|v|^2)dx\,. \qquad (6)$$

It is straightforward to find the energy of the BS (4):

$$E=-\tfrac{128}{3}\eta^3+O(c)\,. \qquad (7)$$

If the BS is unstable, it may decay into the symmetric u and v solitons plus some radiation (quasilinear dispersive waves). Note that, if one neglects the group-velocity term, the Hamiltonian density (5) is always positive for the radiative component of the wave field. Thus, a sufficient condition for the stability of the BS (4) can be formulated as follows: If one assumes that the BS completely decays into a pair of the free solitons, their net energy must exceed the initial energy given by Eq. (7). Taking the eventual free solitons in the form of Eq. (2), it is straightforward to see that the conservation of the wave action (6) is satisfied. At last, the net energy E' of the assumed final state, which is twice the energy of the free soliton (2), is

$$E'=2[-\tfrac{16}{3}\eta^3+\eta(c')^2-\eta c^2]\,, \qquad (8)$$

$\pm c'$ being the final velocities of the free solitons. Thus, the minimum of the final energy (8) is attained at $c'=0$:

$$(E')_{\min}=-\tfrac{32}{3}\eta^3+O(c^2)\,. \qquad (9)$$

Comparing Eqs. (9) and (7), one concludes that, at least as long as c remains small, the sufficient stability condition $E<(E')_{\min}$ holds. Note that the BS (4) is strongly bound in the sense that its binding energy

$$E_b\equiv(E')_{\min}-E=32\eta^3+O(c) \qquad (10)$$

is of the same order as $|E|$.

Let us proceed to interaction of the weakly overlapping solitons. In this case, the two-soliton state may be approximated just by a linear superposition of the solitons (2). The interaction is governed by the effective potential

$$V\equiv-2\mu\int_{-\infty}^{+\infty}|u(x)|^2|v(x)|^2dx$$
$$-(1-\mu)\int_{-\infty}^{+\infty}[u^2(v^*)^2+(u^*)^2v^2]dx\,. \qquad (11)$$

Inserting Eqs. (2) and (3) with $c_u=c_v=0$ into Eq. (11), one can readily find

$$V(z,\phi)=-512\mu\eta^4e^{-4\eta z}z-512(1-\mu)c^{-1}\eta^4e^{-4\eta z}$$
$$\times\sin(2cz)\cos(2\phi)\,, \qquad (12)$$

where $z\equiv z_u-z_v$, $\phi\equiv\phi_u^{(0)}-\phi_v^{(0)}$, and, according to what was said above, it is assumed that $\eta z\gg 1$ and $c\ll\eta$.

A stable BS of the two weakly overlapping solitons corresponds to a local minimum of the potential (12). Straightforward analysis demonstrates that the local minima, satisfying the underlying condition $\eta z\gg 1$, exist only in the case when μ is a small parameter ($\mu\ll 1$). The minima lie at the points

$$z_n=\frac{\pi}{2c}n+(-1)^n(2c)^{-1}\delta_n,\quad \cos\phi_n=(-1)^n\,, \qquad (13)$$

where $n=1,2,3,\ldots$, and the small correction $\delta_n>0$ is determined by the equation

$$\pi n\mu+2\delta_n=c/\eta \qquad (14)$$

(for the sake of definiteness, c is assumed positive). From Eq. (14) it follows that the minima exist at

$$n\le n_{\max}\equiv c/\pi\mu\eta\,. \qquad (15)$$

In turn, it is necessary to have $n_{\max}>1$ in Eq. (15), which implies

$$\mu<\mu_{\max}\equiv c/\pi\eta\,. \qquad (16)$$

Recall that the assumption $c\ll\eta$ underlies the analysis developed, so that the inequality (16) tells us once again that μ must be small.

Inserting Eqs. (13) and (14) into Eq. (12), it is straightforward to find the binding energies [cf. Eq. (10)]

$$E_b^{(n)}\equiv-V(z_n,\phi_n)\approx 256\eta^3e^{-2\pi n\eta/c}\,. \qquad (17)$$

As it follows from Eq. (17), even the binding energy corresponding to the most stable BS ($n=1$) is exponentially small. This exponential smallness, as well as the very mechanism to create the BS's of the weakly overlapping solitons, are similar to those found in Ref. [8] for the strongly separated solitons governed by one equation (not a coupled system) of the NS-GL type. In that case, a term in the soliton's phase linear in x and proportional to a small wave number was produced by dissipative terms in the NS-GL equation, and then it gave rise to a set of the bound states through the coherent nonlinear soliton-soliton interaction. In the situation considered here, the linear terms in the solitons' phases were directly produced by the group-velocity terms in Eqs. (1). If the dissipative terms are added to Eqs. (1) to transform them into a full GL system, they will induce a correction to the wave numbers $\pm \frac{1}{2}c$ [see Eq. (3)] as in Ref. [8].

The full GL system can model the two above-mentioned basic types of the inelastic collisions between the SP's discovered in Ref. [6], viz., (i) their fusion into one SP if the collision velocity is not very small, and (ii) the formation of a double-peaked BS if the velocity is very small. Both inelastic processes are characterized by their threshold velocities, i.e., a maximum collision velocity which makes the process possible. Although the process (i) cannot be directly modeled analytically, at the qualitative level it seems akin to a fusion of the soliton-antisoliton pair into the strongly overlapping BS; the process (ii) can be modeled directly as the fusion into the weakly overlapping BS. The latter process can be accurately analyzed in terms of the perturbation theory (to be presented elsewhere), while the former one cannot. However, it has been demonstrated above that the binding energy of the BS in case (i) is much larger than that in case (ii). This implies that the threshold velocity should also be much larger in case (i) than in case (ii). This inference is in full agreement with the experimental observations of Ref. [6].

As has been shown, the analysis developed is applicable if μ is sufficiently small. With the growth of μ, the weakly overlapping BS's disappear, and the last one disappears when μ attains the critical value given by Eq. (16). If, however, the ratio c/η is not a small parameter, $\mu_{\max}$ may become nonsmall as well. One can therefore expect that in the case one or several BS's (apart from the obvious BS with the coinciding centers of the solitons) still exist at μ not very small (in particular, at the physical value $\mu = \frac{1}{3}$). To find the critical value $\mu_{\max}$ for this case could be a pertinent problem for numerical simulations.

Finally, it seems worthy to note that in the limit opposite to that employed here, i.e., when $1 - \mu \ll 1$, the system (1) can be attacked in another way, if one takes into account that it is close to the exactly integrable Manakov's system. This approach was developed in detail in Ref. [2]. It has been demonstrated, in particular, that weakly overlapped bound states are possible between the so-called vector solitons, i.e., the ones of the type (4). However, the mechanism of formation of these BS's is different from that investigated in the present work.

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