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Pinning by directed percolation

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We propose directed percolation as a generic mechanism for interface pinning in a two-dimensional medium with quenched point disorder. The roughness of the interface just below a critical depinning transition is found to obey a power law with an exponent $\zeta_c = \nu_\perp/\nu_\parallel \simeq 0.63$. A moving interface near the transition is not self-affine, but may nevertheless be assigned an effective roughness exponent comparable to the values determined in fluid displacement experiments in porous media.

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Recent experiments [1–3] and simulations [4] on fluid displacement in a two-dimensional porous medium have produced rough interfaces with novel scaling properties. Macroscopically planar moving interfaces were found to be stable at low flow rate when the invading fluid is more effective in wetting the medium. Analyses of interface profiles $h(x)$ appear to indicate a power-law increase of the rms fluctuation $C(r)=\{\langle[h(r)-h(0)]^2\rangle\}^{1/2}$ with the distance r, $C(r)\sim r^\zeta$, where the roughness exponent $\zeta \sim 0.73$–0.88 [2–4].

Due to the random geometry of the pores, the dynamics of fluid flow on the scale of pore size is very complicated even in the quasistatic limit, where the fluid motion is governed by capillary forces at the interface [1–4]. Nevertheless, attempts have been made to understand the roughening behavior of the interface on a phenomenological level by analyzing models which incorporate the essential roughening and smoothening mechanisms [5,6]. Koplik and Levine, and later Kessler, Levine, and Tu (KLT) considered the following equation of motion for a coarse-grained interface height $h(\mathbf{x},t)$,

$$\frac{\partial h}{\partial t} = D\nabla^2 h + F + f(\mathbf{x},h) , \qquad (1)$$

where f is the driving force [5]. The main difference between (1) and the usual Edwards-Wilkinson [7] (EW) or Kardar-Parisi-Zhang [8] (KPZ) equation is that here the noise term $f(\mathbf{x},h)$, which represents the random geometry of the pores, is *quenched* rather than fluctuating in time. This type of disorder may lead to interface pinning at sufficiently small F, a well-known phenomenon in the study of domain walls in impure magnets [5,9]. Simulations of (1) by KLT in 1+1 dimensions produced a scaling regime with $\zeta \simeq 0.75$, as compared to $\zeta = \frac{1}{2}$ for both the EW and KPZ equations [7,8].

While the numerical result of KLT is very encouraging, a number of open problems remain. It is not clear whether the exponent 0.75 has any particular significance, nor is it clear why there is a discrepancy among the reported experimental results. Motivated by these questions, we consider in this paper a simple (1+1)-dimensional solid-on-solid model for which pinning occurs as a result of directed percolation of strong pins *along* the interface. This mechanism yields a critical depinning transition at the directed percolation threshold. The interface just below the transition is self-affine, though the roughness exponent $\zeta_c \simeq 0.63$ is smaller than the value 0.81 found in Ref. [4]. At the onset of flow, there exists a regime where the interface is *rougher* than at the transition. We find that growth in this regime is heterogeneous, the interface consists of a mixture of pinned and moving parts which exhibit different scaling properties.

Consider a square lattice where each cell $\mathbf{R}$ is assigned a random pinning force $f(\mathbf{R})$ uniformly distributed in the interval [0,1]. For a given applied pressure $p>0$, one can divide the cells into two groups, those with $f(\mathbf{R}) \leq p$ (free or F cells), and those with $f(\mathbf{R})>p$ (pinning or P cells). Denoting by q the density of P cells on the lattice, we have $q=1-p$ for $0<p<1$ and $q=0$ for $p \geq 1$. Growth proceeds in a strip geometry with periodic boundary conditions in the direction parallel to the interface. Under the solid-on-solid condition, the interface is completely specified by a set of integer column heights h_i, $i=1,\ldots,L$. At $t=0$, all columns are assumed to have the same height $h_i=0$. During growth, we randomly select a column, say i, and compare its height with those of neighboring columns. If h_i is greater than either h_{i-1} or h_{i+1} by *two* or more units, the height of the lower of the two columns $i-1$ and $i+1$ is incremented by one unit. (In the case of a tie, we choose one of the two with equal probability.) In the opposite case, $h_i<\min\{h_{i-1},h_{i+1}\}+2$, column i advances by one unit provided the to-be-occupied cell is an F cell, i.e., $f(\{i,h_i+1\}) \leq p$. Otherwise no action takes place. This completes a growth event. Time is measured in units of such events per column.

In the absence of P cells, our model is a variant of the EW model [7] which has been used to model capillary phenomena without quenched disorder [10]. The presence of P cells decreases the growth velocity of the interface. In particular, there exists a critical density q_c (defined in the limit $L \to \infty$) above which growth eventually comes to a halt, i.e., the whole interface is pinned.

To find q_c, we note that growth stops if and only if (i) $h_{i+1}-h_i=0, \pm 1$ for all i, and (ii) all cells above the interface are P cells. Thus the condition for pinning is that there exists a string of P cells across the system, connected horizontally or diagonally one after another. In the limit $L \to \infty$, such directed strings appear when the density of P cells exceeds a critical value q_c known as the directed percolation threshold, which is about 0.539 in the present case [11].

We have investigated the motion of the interface at a fixed q both above and below the directed percolation threshold by means of simulation. Before presenting our numerical results, let us briefly recall some properties of two-dimensional directed percolation. For $q<q_c$, a typical connected cluster extends over a distance of the order of $\xi_\parallel$ in the parallel direction and a distance of the order of $\xi_\perp$ in the perpendicular direction. For $q>q_c$, there appears a directed percolating cluster which extends over the whole system. This cluster is known to possess a network structure of nodes and compartments [12]. Each compartment has an anisotropic shape similar to the connected clusters below q_c, characterized by $\xi_\parallel$ in the parallel direction and $\xi_\perp$ in the perpendicular direction. On both sides of the percolation transition, the two lengths have the power-law behavior

$$\xi_\parallel \sim |q-q_c|^{-\nu_\parallel}, \ \xi_\perp \sim |q-q_c|^{-\nu_\perp}. \tag{2}$$

The latest series calculation yields $\nu_\parallel=1.733 \pm 0.001$ and $\nu_\perp=1.097 \pm 0.001$ [13].

Our simulation data indicate that, near the depinning transition, $\xi_\perp$ sets a characteristic scale for the height while $\xi_\parallel$ sets characteristic scales for both the distance parallel to the interface and the time. Figure 1 shows a scaling plot of the mean interface height $H(t)=\langle\overline{h_i(t)}\rangle$ against the time t for $0.3 \leq q \leq 0.59$ and $L=16384$. Here and elsewhere the overbar denotes an average over all columns and $\langle\,\rangle$ denotes an average over different realizations of the random pinning forces. To bring data at small t onto the scaling curve, a constant 0.25 is added to $H(t)$. The best data collapse is achieved by choosing $q_c=0.5385$, using the exponents $\nu_\parallel$ and $\nu_\perp$ quoted above [14]. Denoting by Φ_+ (Φ_-) the upper (lower) branch of the collapsing curve, we have

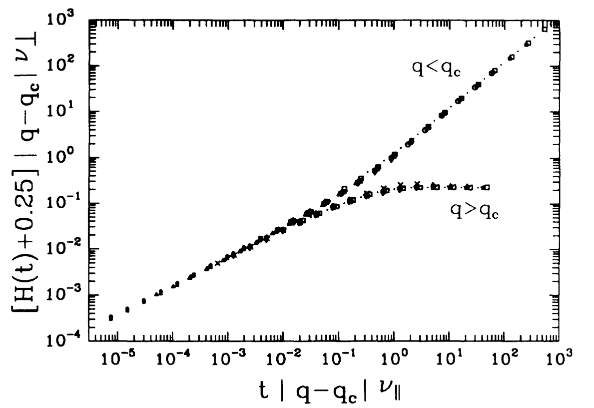

FIG. 1. A scaling plot of the mean surface height vs time for $0.30 \leq q \leq 0.59$.

$$H(t) \simeq \xi_\perp \Phi_\pm(t/\xi_\parallel) \tag{3}$$

for $q<q_c$ ($q>q_c$). The pinned phase is represented by the lower branch, where $H(t)$ grows to a finite value of the order of $\xi_\perp$ which is the average perpendicular dimension of a compartment of the infinite directed percolating cluster. In the moving phase, there is a crossover from a power-law growth $H(t) \sim t^{\nu_\perp/\nu_\parallel}$ at $t \ll \xi_\parallel$ to a linear behavior $H(t)=vt$ at $t \gg \xi_\parallel$. The steady-state velocity (in the limit $L \to \infty$) can be expressed as

$$v(p) \sim \xi_\perp/\xi_\parallel \sim (p-p_c)^{\nu_\parallel-\nu_\perp}, \tag{4}$$

where $\nu_\parallel-\nu_\perp \simeq 0.636$.

The fact that $\xi_\parallel$ appears as a characteristic time has to do with the way a segment of the interface gets pinned and depinned in our model. In the following we shall focus on the case $q<q_c$, where each pinned segment has a finite extent of order $\xi_\parallel$. Figure 2(a) shows a typical set of interface configurations $h_i(t)$ at uniform time intervals $n\tau$ just below the critical value q_c, where $\tau=5$. (Time increases from bottom to top.) The difference between successive curves $\Delta h_i(t)=h_i(t+\tau)-h_i(t)$ is plotted in Fig. 2(b). In both cases each curve is shifted vertically upwards by an amount proportional to t. The flat segments in Fig. 2(b), which correspond to the dark areas in Fig. 2(a), are parts of the interface which remain stationary over each time interval. A bump in Fig. 2(b) is a part of the interface which has moved during this time period. Rather than a more or less uniform growth, the pictures clearly indicate a separation of pinned and moving parts. Each train of bumps in Fig. 2(b) is associated with a pair of pinning and depinning processes at the upper and lower strings of P cells. Their constant slope shows that there is

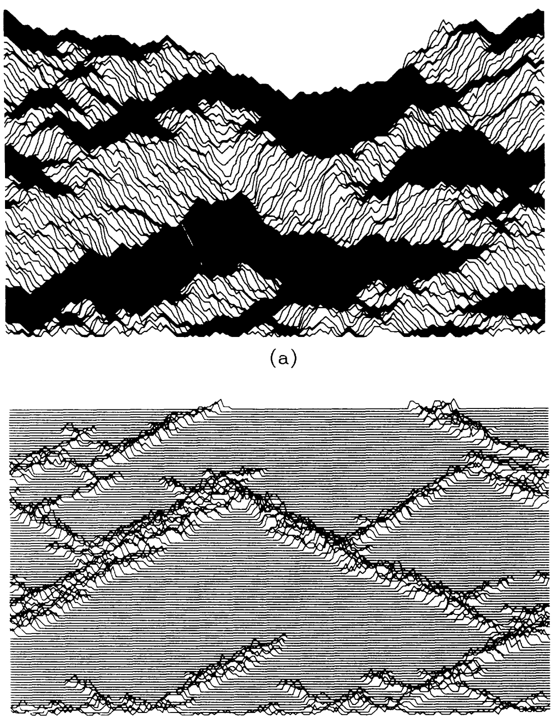

FIG. 2. (a) Snapshots of the upward moving interface just above the pinning threshold at uniform time intervals. (b) Difference in height between successive curves in (a). In both cases each curve is shifted vertically by an amount proportional to t.

a typical pinning and depinning velocity along the interface. The characteristic time in (3) is simply the time it takes for the interface to get over a long string of P cells, which has a typical size $\xi_\parallel$. To be consistent with (3), the typical size of moving segments in the perpendicular direction should be of the order $\xi_\perp$.

It should also be evident from Fig. 2(a) that the pinned and moving parts of the interface behave differently upon a rescaling of horizontal and vertical distances. Each pinned segment traces out a typical path on a directed percolation cluster, which has a power-law roughness characterized by an exponent $\zeta_c = \nu_\perp/\nu_\parallel \simeq 0.63$ as confirmed by direct measurements. On the other hand, the moving segments tend to have a slope of order 1 or larger. Consequently, self-affine scaling is not expected to hold for an interface which contains both types of segments when the coarse-graining length is less than $\xi_\parallel$.

What happens if one attempts to describe the roughness of such a heterogeneous interface with a single exponent ζ? A common approach to determining ζ is to measure the first or second moment of the distribution of $|h_{i+r} - h_i|$ and fit the data to a power law. In our case, this procedure generates a reasonably good power law over a certain range of distances, but the exponent depends on the moment considered. For example, let us consider the case $q = 0.530$ which is only slightly less than the critical value 0.5385. Denoting by $C_m(r) = \sqrt[m]{\langle |h_{i+r} - h_i|^m \rangle}$ the mth root of the mth moment of the distribution at a given r, we define an effective exponent

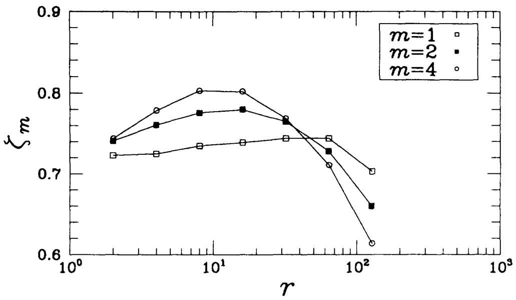

FIG. 3. The effective exponent $\zeta_m(r)$ from the first, second, and fourth moments of the height difference distribution against r. Here $q = 0.530$.

$$\zeta_m(r) = \log[C_m(2r)/C_m(r)]/\log 2 . \tag{5}$$

Figure 3 is a plot of this quantity against r for the first, second, and the fourth moment for an $L = 32\,768$ system after 2^{15} time steps, and averaged over several different runs. Data from the first and second moments yield an effective exponent 0.73 and 0.77 over a decade, respectively, while those from the fourth moment reach as high as 0.8. The increase in ζ_m for higher moments comes from the larger weights assigned to the steep (moving) parts of the interface which, in our case, have different scaling properties than the less steep (pinned) parts.

In contrast to the roughness exponent ζ, different moments of the height fluctuation yield nearly the same power law for the interface width,

$$w_m(t) = \{\langle \overline{[h_i(t) - \overline{h_i(t)}]^m} \rangle\}^{1/m} \sim t^{\beta_c} \tag{6}$$

for $t \ll \xi_\parallel$ with $\beta_c \simeq 0.63$. Figure 4 shows a scaling plot for the mean-square surface width $w^2(t)$ using similar scaling factors as in Fig. 1. From the data collapse we conclude

$$w(t) \simeq \xi_\perp \Psi_\pm(t/\xi_\parallel) , \tag{7}$$

where $\Psi_\pm(x) \sim x^{\nu_\perp/\nu_\parallel}$ for $x \ll 1$. For $x \gg 1$, $\Psi_-(x)$ (for the pinned phase) tends to a constant while $\Psi_+(x)$ (for

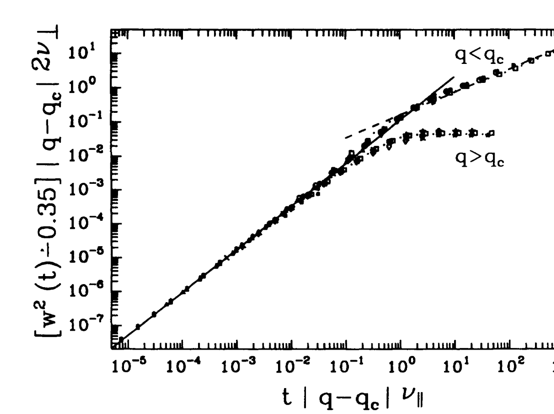

FIG. 4. A scaling plot of the mean-square surface width vs time for $0.30 \le q \le 0.59$. The upper data collapsing curve crosses over from a power law with an exponent $2\beta_c = 1.266$ (solid line) to another power law with the KPZ exponent $2\beta = \frac{2}{3}$ (dashed line) at $t \sim \xi_\parallel$.

the moving phase) crosses over to $x^{1/3}$. The latter behavior is typical for a moving interface which is rough and whose velocity depends on its orientation. Although in the $q=0$ limit the interface velocity in our model has no slope dependence, such a dependence is expected at a finite concentration of the P cells whose effectiveness in retarding the interface motion varies with the overall slope.

To summarize, we discussed here a particular pinning mechanism—directed percolation of strong pins—for a driven interface in a two-dimensional disordered medium. This mechanism gives rise to a continuous depinning transition at the percolation threshold. The critical behavior around the transition was shown to be related to the geometrical properties of percolating clusters. Approaching the transition from the moving phase, the interface breaks up into pinned and moving segments with different scaling properties. Measurement of moments of interface height fluctuations nevertheless gave effective roughness exponents comparable to experimental findings.

While there are obvious similarities between the model we studied here and the fluid displacement experiments and simulations, the solid-on-solid condition we adopted does introduce an anisotropy which is not expected to be present in experiments. Direct comparison of effective exponents may be further complicated by finite-size effects, different boundary conditions, etc. Another important difference is the use of constant pressure in our simulations, whereas displacement experiments are typically carried out at a constant flow rate. While the two ensembles should give identical results for an infinite system in the moving phase, their relationship in a finite system needs to be further explored. However, we believe that the pinning mechanism proposed here and the critical behavior we observed around the depinning transition could bear a broader significance. In particular, it would be interesting to examine the possibility that a moving interface close to the pinning threshold is not self-affine on length scales of experimental interest.

Note added. After this work was completed, Havlin kindly sent us a copy of recent work [15] which independently proposes the possible relevance of directed percolation as a mechanism for interface pinning in a medium with quenched disorder, along with a presentation of experiments which seem to match well with the proposed theoretical picture.

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