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Directed polymers on fractal substrates

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In this paper we present numerical results for the transverse fluctuations $\langle x^2(t)\rangle$ of a directed polymer in a random potential field on fractal substrates. Because of the randomness of the potential these mean-squared displacements are shown to be enhanced with respect to the regular random-walk behavior on fractals. The enhancement enters in a subordinated way $\langle x^2(t)\rangle \sim t^{\alpha}$, with $\alpha=(2/d_w)\nu$, where d_w is the walk exponent for fractals and ν is the enhancement factor. The minimal energy fluctuations are also calculated and show a similar behavior, $\Delta E \sim t^{(2/d_w)\beta}$. On the Sierpinski gaskets studied up to the numerical accuracy ν and β appear to concur with the values 4/3 and 1/3, correspondingly, independent of the embedding Euclidean dimension.

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The problem of directed polymers in random media [1] has been a subject of extensive research due to its intimate relationship via mapping to fundamental problems such as dynamics of surface growth [2], interfaces in random spin systems [3], and the driven Burger equation [4] and has been shown to have properties previously encountered in the study of spin glasses [5,6]. Of special interest have been the enhanced transverse fluctuations of directed polymers (DP) [1] and the low-temperature (strong-coupling) to high-temperature (weak-coupling) phase transition [7,8]. The DP problem is defined by walks directed in d spacetime dimensions with coordinates $(\mathbf{x},t)$, where $d-1$ is the substrate dimension, which obey the partial differential equation [1]

$$\frac{\partial W}{\partial t}=[D\nabla^2+\eta(\mathbf{x},t)]W\,, \tag{1}$$

subject to the initial condition

$$W(\mathbf{x},0)=\delta(\mathbf{x})\,.$$

Here $W(\mathbf{x},t)$ is the weight of all DP with one end at $(\mathbf{x},t)=(\mathbf{0},0)$ and the other at $(\mathbf{x},t)$. $\eta(\mathbf{x},t)$ is a random potential field which is usually assumed to be Gaussian or white noise with δ-function correlation

$$\langle \eta(\mathbf{x},t)\rangle=0\,,$$
$$\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle=\lambda^2\delta^{d-1}(\mathbf{x}-\mathbf{x}')\delta(t-t')\,. \tag{2}$$

The randomness term given by Eq. (2) causes the DP to be stretched in the transverse direction so that the low-energy sites, even far from the origin, are visited. This randomness-induced stretching contributes to transverse fluctuations which have been shown to grow with time, at low temperatures, faster than expected for a simple Brownian motion. Some analytical results which have been derived for DP in dimensions $d=(1+1)$ and have been confirmed by simulation calculations show that $\langle x^2(t)\rangle \sim t^{\nu}$ with $\nu=4/3$. Claims have been made that the exponent $\nu=4/3$ is independent of the dimension and is therefore superuniversal [1,4,9]. These claims are, however, still under debate since simulation results on equivalent growth models demonstrate dimension-dependent results [2,10,11]. Another issue which has not yet been completely settled is the weak-to-strong–coupling transition in higher dimensions. While low dimensions, $d<3$, are expected to exhibit low-temperature (strong-coupling) behavior which results in enhancement, in higher dimensions, $d>3$, one expects a phase-transition behavior between low temperatures and high temperatures, namely, between enhanced- and regular-transverse fluctuations [2,8].

Several modifications of the original model have been suggested lately, trying to gain more insight into the complexity of this seemingly simple model. Most of the modifications have concentrated on the random potential term $\eta(\mathbf{x},t)$ in Eq. (1) using broad distributions and long-range correlations and have demonstrated possible changes in the values of the characteristic exponent ν [4,12–15].

In this paper we center on substrates of dimension $d-1$ that are characterized by anomalous mean-squared displacements of DP even in the absence of random potentials. For such substrates we choose Sierpinski gaskets which allow us also to follow in more detail the dimensionality dependence between $d=(1+1)$ and $d=(2+1)$. Simple random walks on fractals are known to be dispersive and are described in terms of the walk dimension $d_w=(d_f/2d_s)$ where d_f is the fractal dimension and d_s is the spectral dimension so that [16]

$$\langle x^2(t)\rangle \sim t^{2/d_w}\,. \tag{3}$$

The propagator problem on fractals has been revisited recently and exhibits anomalous behavior as well [17].

In order to study DP on Sierpinski gaskets we have solved numerically the path integral

$$W(\mathbf{x},t)=\int_{(0,0)}^{(\mathbf{x},t)} D\mathbf{x}'(t')\exp\left\{-\int_0^t dt'\left[\left(\frac{1}{4D}\right)\left(\frac{d\mathbf{x}'}{dt'}\right)^2+\eta(\mathbf{x}',t')\right]\right\}, \tag{4}$$

using the transfer-matrix approach as discussed by Kardar [18]. We used a discretized version of Eq. (4) where the sites lie on a grid. Figure 1 describes the (1+1) example of our approach with two possible rules for the transfer matrix. The DP in Fig. 1(a) corresponds to the Kardar and Zhang solution of Eq. (1) [1] and follows the recursion relation

$$W(x,t)=e^{\eta(x,t)}\{W(x,t-1)+e^{-1/4}[W(x-1,t-1)+W(x+1,t-1)]\}, \tag{5}$$

where D was set to unity. Figure 1(b) corresponds to a different case which is basically a random-walk process that allows for up or down random steps, as indicated, in the presence of the same random noise. For this case the recursion relation (5) is modified by omitting the first term in the curly brackets. The random term η was randomly chosen from the range $\eta\in[-\lambda\sqrt{3},\lambda\sqrt{3}]$ for case 1(a) and from a Gaussian distribution of width λ in case 1(b).

These two models were extended by us to Sierpinski gaskets. Figure 2 provides an example of DP extension to fractals: a (d_f+1) structure in spacetime with a gasket as the substrate of $d_f=1.58$. This structure over which the DP walks is essentially a triangular prism (called a "Toblerone" in Ref. [19]). For a randomness-free problem the transverse fluctuation of the DP follows Eq. (3), namely, the Brownian results for fractals. Once we introduce $\eta(\mathbf{x},t)$, as defined in Eq. (2), we observe an enhancement. Our numerical calculations were done on systems of dimensions (1+1) and (2+1) and on (d_f+1) dimensions for Sierpinski gaskets embedded in Euclidean dimensions 2, 3, and 4, for which $d_f=1.585$, 2, and 2.322, correspondingly (to be referred to in the figures as $d_{SG}=2$, $d_{SG}=3$, $d_{SG}=4$). For the (2+1)-dimensional system the size of the transverse square lattice was 512×512. The Sierpinski gaskets were taken at the 8th, 7th, and 7th iteration stages for the embedding dimensions 2, 3, and 4, correspondingly. The number of realizations was typically of the order of 10^3–10^4. In the calculation of the transverse mean-squared displacement we used the fractal extension of Eq. (5) and Fig. 1(a) for two values of the randomness parameter $\lambda=1$ and $\lambda=4$.

The results for the transverse mean-squared displacements for both $\lambda=1$ and 4 show enhancement relative to the regular $\langle x^2(t)\rangle$ and the amount of the enhancement depends on λ. This enhancement, however, need not be superdiffusive since $2/d_w$ may belong to strongly dispersive cases depending on the underlying fractal. We therefore introduce the concept of an enhancement factor ν. Figures 3(a) and 3(b) display the transverse mean-squared displacement for the different dimensions and fits well the behavior

$$\langle x^2(t)\rangle\sim t^{(2/d_w)\nu}. \tag{6}$$

The figures include also the Euclidean limits $d=(1+1)$ and $d=(2+1)$ for which $2/d_w=1$. In Fig. 3(a) we present the transverse fluctuations for $\lambda=1$. Only the (1+1)-dimensional case has approached the asymptotic behavior with $\langle x^2(t)\rangle\sim t^{4/3}$. For the Euclidean case of (2+1) dimensions and for the Sierpinski gaskets (d_f+1) we observe that the mean-squared displacements follow Eq. (6) but with values of ν lying in the range $1<\nu<4/3$. This may support the idea of a possible crossover to the asymptotically, fully enhanced limit which we found for $\lambda=4$ [Fig. 3(b)]. In Table I we summarize our fitted exponents which correspond to $\lambda=1$. As stated, the exponents are larger than expected for regular random-walk motion on fractals $(2/d_w)$ but still smaller than the values obtained for the stronger randomness parameter. In the case of $\lambda=4$ all the substrates demonstrated the same enhancement, as shown in Fig. 3(b), namely, up to the ac-

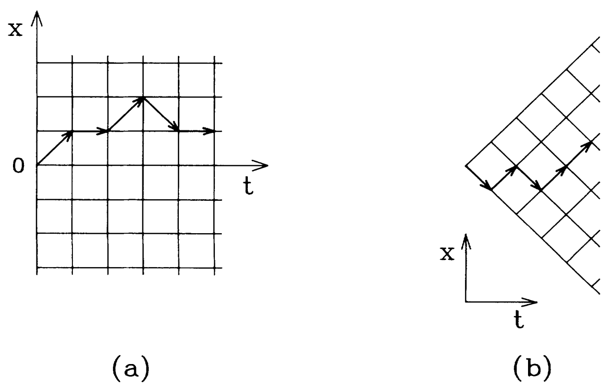

FIG. 1. Directed paths on different substrates. (a) Three possible steps with weight factors corresponding to Eq. (5). (b) The random-walk process.

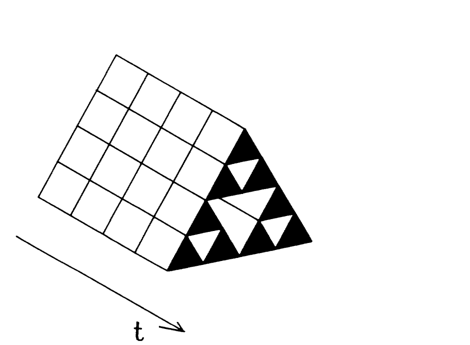

FIG. 2. Triangular prism structure. The vertical space is given by a Sierpinski gasket embedded in a two-dimensional Euclidean space and the longitudinal direction is indicated by time t ($d_{SG}=2$).

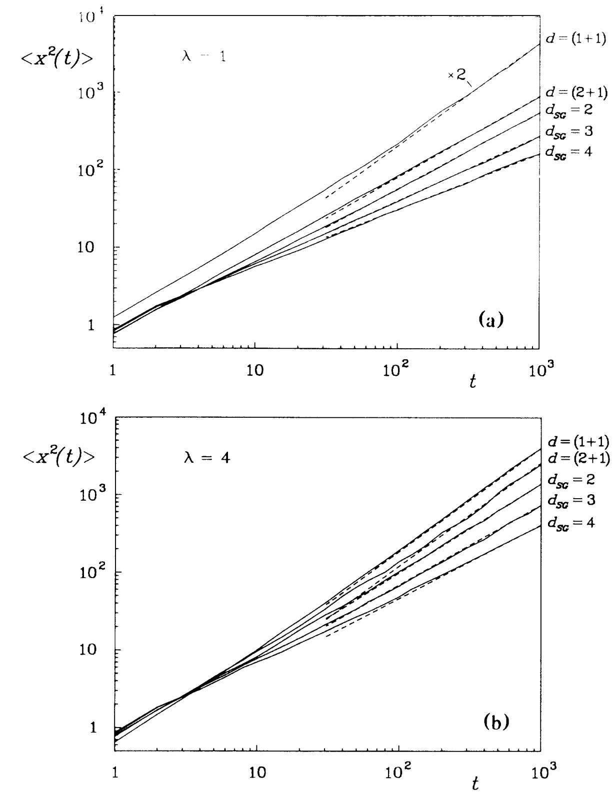

FIG. 3. The mean-squared transverse fluctuations $\langle x^2(t)\rangle$ as a function of time t for transverse spaces given by Euclidean lattices $(1+1)$ and $(2+1)$ and by Sierpinski gaskets of embedding dimension d_{SG}, as indicated. Solid lines denote simulation results obtained from paths of the type shown in Fig. 1(a). Dashed lines give the slopes for the long-time behavior. (a) The randomness parameter $\lambda=1$. The slopes are fitted to the data and are presented in Table I. The data for $d=(1+1)$ are shifted vertically by a factor of 2. (b) The randomness parameter $\lambda=4$. The slopes give the predicted enhanced exponents $8/3d_w$.

curacy of our calculations we found that $\nu=4/3$ fitted, independent of the dimension. Since the fractals fall into the category of low-dimensional systems $(d_s<2)$ they should belong to the low-temperature, strong-coupling limit which is energy, rather than entropy dominated. Although, for the d_f values we studied, we obtained λ-dependent values of ν such that $\nu>1$, but not necessarily

TABLE I. Exponents for the mean-squared transverse fluctuations of directed polymers in Euclidean lattices of dimension d and for Sierpinski gaskets embedded in Euclidean spaces of dimension d_{SG}. From left to right are listed the exponents for the regular behavior, for the enhanced behavior, and the slopes fitted to the data obtained for $\lambda=1$ as presented in Fig. 3(a).

	$2/d_w$	$8/3d_w$	Fit
$d=(1+1)$	1.0	1.333	1.33
$d=(2+1)$	1.0	1.333	1.05
$d_{SG}=2$	0.861	1.148	0.99
$d_{SG}=3$	0.774	1.032	0.84
$d_{SG}=4$	0.712	0.950	0.72

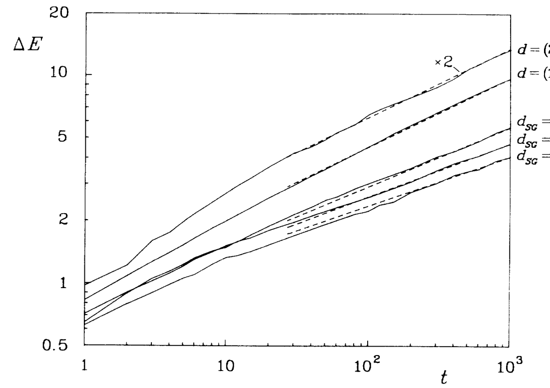

FIG. 4. The minimum energy fluctuations ΔE as a function of time for transverse spaces given by Euclidean lattices $(1+1)$ and $(2+1)$ and by Sierpinski gaskets of embedding dimension d_{SG}, as indicated. Solid lines denote the simulation results obtained from paths following a random-walk picture as demonstrated in Fig. 1(b). Dashed lines denote the predicted exponents $2/3d_w$.

4/3, we conclude that these may originate from a different approach to the asymptotic limit depending on the random potential parameter λ.

The enhancement of the spatial fluctuations has been related to the minimal energy distribution. A relationship has been proposed between the transverse fluctuations and the scaling of the energy fluctuations, ΔE, with time [1,3]. In the $d=(1+1)$ system it has been confirmed that ΔE scales as $t^{1/3}$. In order to check a possible extension of the relationship between ΔE and $\langle x^2(t)\rangle$ to fractal substrates we calculated ΔE for the different d_f. For each of the gaskets, in a given configuration of the random potential field, we determined the minimum energy using the recursion relationship which is the discretized version of the integral in the exponent of Eq. (4) taken along the minimum energy path with zero-diffusional contribution

$$H(x,t)=\eta(x,t)+\min[H(x-1,t-1),\ H(x+1,t-1)]\,, \tag{7}$$

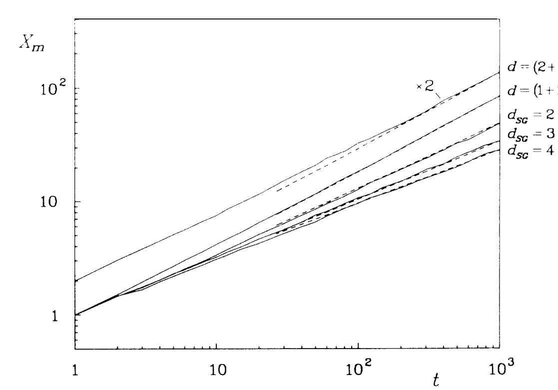

FIG. 5. The transverse displacement $X_m(t)$ of the paths of minimum energy for the parameters as in Fig. 4. Dashed lines denote the predicted exponents $4/3d_w$.

and defined

$$E_{\min}(t) = \min[H(x,t)]_x . \quad (8)$$

Here we used the random-walk approach defined in Fig. 1(b) and η was chosen from a Gaussian distribution. Within this framework we were confined to the strong-coupling limit. The root-mean squared of the $E_{\min}$ was calculated as $\Delta E(t) = \langle E_{\min}^2 - \langle E_{\min}\rangle^2\rangle^{1/2}$ where the average was taken over the random potential configurations. We observed that ΔE follows the power law $\Delta E \sim t^{\beta/d_w}$. In Fig. 4 $\Delta E(t)$ is plotted as a function of time on log-log scales. Solid lines give the simulation results, and for longer times the predicted slopes are given by dashed lines. The energy fluctuations which correspond to the (2+1)-dimensional case were calculated by the energy difference between the minimal energy path and the path terminating at $\mathbf{x}=\mathbf{0}$. Again reasonable agreement is obtained in all cases with the value of $\beta = 1/3$.

Following Huse and Henley [3] we measured the length X_m which is defined as the transverse location on the substrate at $E_{\min}(t)$ averaged over all configurations, $X_m(t) = \langle x_{\min}{}^2\rangle^{1/2}$. As we see in Fig. 5 X_m scales with time with the same exponent as in Eq. (6), namely, $X_m \sim t^{\nu/d_w}$. Each X_m was calculated under the same conditions as ΔE. It follows therefore that $(\Delta E)^2 \sim X_m$, which generalizes the result obtained in the case of $d = (1+1)$.

Concluding, we point out that our results agree with values $\nu = 4/3$ and $\beta = 1/3$ independent of the embedding Euclidean dimension. However, we cannot rule out the possibility of slight variations of ν and β as functions of d_s (or d_f) along the results of Wolf and Kertész [10] or Kim and Kosterlitz [11]. Such variations would be difficult to observe over the range of dimensionalities covered here.

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