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Nonlinear adiabatic reflection of electrons by an electromagnetic wave in a plasma

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A theoretical model is developed for the investigation of the adiabatic reflection of electrons from a spatially localized electromagnetic wave, taking into account the nonlinear trapping in the wave field. A complete analytical description of the adiabatic motion is given via the constancy of the Hamiltonian and the invariance of the adiabatic integral. The set of initial conditions for which reflection occurs is determined in detail as a function of the field amplitude and of the wave frequency for the case of electron-cyclotron waves in perpendicular propagation. The comparison with the results of the Lie transform approach is performed. It is found that trapping effects strongly modify the reflection process, giving rise to a substantial enhancement of the repulsive ponderomotive force. When trapping occurs, reflection is observed for a large set of initial conditions for which the perturbative approach predicts no reflection at any value of the perturbation.

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The adiabatic reflection of charged particles by a spatially localized electromagnetic wave is a peculiar ponderomotive effect with important applications in plasma physics and other fields. Accurate treatments of the ponderomotive force are based on the Lie transform approach [1,2] (LTA). This method, however, cannot properly describe the cases where nonlinear alterations of the particle dynamics, as trapping in the wave field, arise. In the present paper we develop a theoretical model for the investigation of ponderomotive effects in the above-mentioned conditions. The comparison of the present results with those obtained by the LTA shows very significant differences, which are discussed in the following. Moreover, the present results are shown to agree to a great accuracy with those derived from the numerical simulation of the particle motion.

We refer here, for definiteness, to the case of an electron-cyclotron (EC) wave propagating perpendicularly to a uniform magnetic field $\mathbf{B}_0=B_0\mathbf{e}_z$, and characterized by a nonuniform amplitude $E(z)$. This case allows pointing out the essential characteristic of the results. The detailed presentation of the method and the extension to the more general case of oblique propagation will be presented in a future paper.

Let the parameter $\epsilon(z)=eE(z)/mc\omega$, where ω is the wave frequency, be sufficiently small and slowly varying so that the electron motion in the wave field can be considered regular and adiabatic. Moreover, let the flight time of the electrons in the interaction region be larger than the trapping time so that nonlinear effects can take place. Fixing our attention to the initial and final electron states (where $\epsilon=0$), we intend to characterize the ponderomotive reflection by determining the electron momenta of the electrons that are reflected by the wave.

The dynamics of the electrons in the regular regime, for a wave frequency close to the nth cyclotron harmonic, is conveniently described by the following time-independent relativistic Hamiltonian [3]

$$H=\gamma-\nu I/n=H_0(I,P_z)+H_1(z,P_z,\theta,I)\,, \qquad (1)$$

with $H_0(I,P_z)=\Gamma-\nu I/n$, and $H_1=-\alpha(z)P_z^qI^{n/2}\cos n\theta$, where $\Gamma=(1+2I+P_z^2)^{1/2}$, $\nu=\omega/\Omega$ (with $\Omega=eB_0/mc$ the cyclotron frequency for rest electrons), and $q=0,1$, for the extraordinary and ordinary mode, respectively. The given expression for H_1 is valid at lowest order in Larmor radius. The parameter $\alpha(z)$ depends on the harmonic number, the refractive index, and the polarization vector, and is proportional to $\epsilon(z)$. It is characterized by a single maximum $\alpha_M<1$, which can be considered as the nonlinearity parameter. All variables are dimensionless. In Eq. (1) the canonical momentum $\mathbf{P}$ is related to the kinetic momentum $\mathbf{p}$ by the relation $\mathbf{p}=\mathbf{P}+\mathbf{A}$, where $\mathbf{A}$ is the vector potential; θ,I are angle action variables; and, in absence of perturbation, $I=p_\perp^2/2$ represents the perpendicular energy. In adiabatic conditions, θ,I and z,P_z are the fast and slow variables, respectively, and the action integral $J=(n/2\pi)\int d\theta I$ is an adiabatic invariant, which is constant during the motion except at the separatrix crossing, where it may change abruptly.

In linear conditions, i.e., when trapping in the wave field does not occur, a ponderomotive Hamiltonian $H_P(P_z,\alpha,J)$ can be found by means of the LTA [4]. The average of H over the θ variation is expressed as

$$\bar{H}=H_0(J,P_z)+H_P(\alpha,J,P_z)\,, \qquad (2)$$

where J is constant on the whole motion and is equal to the unperturbed value of the action I_0, and the ponderomotive Hamiltonian reads

$$H_P=-\frac{\alpha(z)^2}{4}P_z^{2q}\frac{\partial}{\partial I}\left[\frac{I^n}{\partial H_0/\partial I}\right]\Bigg|_{I=J}. \qquad (3)$$

The reflection implies the inversion of P_z sign in the unperturbed initial and final states. From the Hamiltonian

(2), the evolution of P_z can be obtained as a function of α. The condition for reflection is given determining the value α_R, corresponding to P_z equal to zero, and requiring $\alpha_R \leq \alpha_M$.

For the case under investigation, we can at first observe that for the ordinary mode, reflection can never occur due to wave polarization, i.e., to the H_P dependence on P_z^2 in Eq. (3). Therefore we analyze here the case of the extraordinary mode, and refer explicitly to the second harmonic ($n=2$); in this case $\alpha = \epsilon v N e_-$, N being the wave refractive index, and e_- the right component of the polarization vector. Note that the following analysis is valid for wave frequencies not close to the upper hybrid frequency, where the given expression for H_1 does not hold.

The analytical determination of the reflection regions is performed in the weakly relativistic approximation, putting $\Gamma = 1 + P_z^2/2 + I(1 - P_z^2/2) - I^2/2$ in the expression for H_0. The integration of the equation for $dP_z^2/d\alpha^2$ leads to the following expression for α_R:

$$\alpha_R^2 = \frac{1}{2(1-I_0)} \left\{ P_{z0}^2 \left[3 - \frac{1}{2I_0} \left(2\rho + \frac{1}{2} P_{z0}^2 \right) \right] + I_0 \ln \left| 1 - \frac{P_{z0}^2}{2\rho - I_0} \right| \right\} \quad (4)$$

valid for $I_0 > 2\rho$, where $\rho = 1 - v/2$. When $I_0 < 2\rho$, no reflection can occur for any α_M; only electrons with finite values of the perpendicular energy are reflected.

Substantial modifications of the above conditions occur when trapping is taken into account. A complete description of the average motion relevant to the Hamiltonian (1), valid in both the linear and nonlinear regime, is given by the two equations

$$H(z, P_z, \theta, I) = H_0(I_0, P_{z0}) ,$$
$$J(\alpha(z), P_z, H_0) = \bar{J} , \quad (5)$$

which represent the constancy of the Hamiltonian and the adiabatic invariance of J. In the case of nonlinear motion, the constant $\bar{J}$ has different values in each stage of the motion: before trapping, $\bar{J} = I_0$; and in the trapping regime, $\bar{J} = J_s$, J_s being the area enclosed by the separatrix when it is crossed [note that J_s coincides with I_0 when the separatrix splits the (θ, I) plane in two regions only].

The condition for reflection can then be written as

$$J(\alpha_R, P_z = 0, H_0) = \bar{J} . \quad (6)$$

From Eq. (6), the reflection region in momentum space can be then obtained by means of the condition $\alpha_R(I_0, P_{z0}) \leq \alpha_M$. Thus the above model solves the problem of the electron reflection by the wave. It is found again that, owing to the P_z factor in H_1, in the case of the ordinary mode (at any harmonic) Eq. (6) does not admit solutions for α_R.

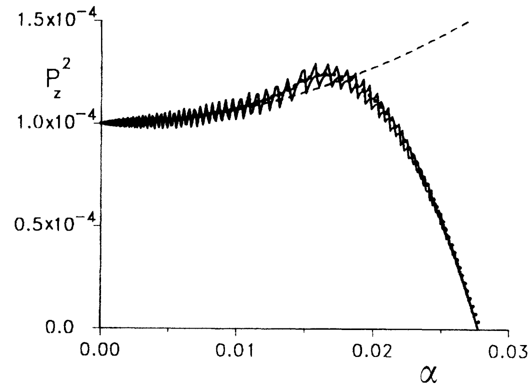

FIG. 1. Behavior of P_z^2 vs α for initial conditions $P_{z0} = 0.01$, $I_0 = 1.25 \times 10^{-4}$, and $\rho = 0.02$. The solid line is the solution of the equation of the motion, the dotted line is the adiabatic result obtained from Eq. (9), and the dashed line is the LTA result.

To illustrate how the nonlinear trajectories affect the electron reflection, in Fig. 1 we show as an example for the extraordinary mode at the second harmonic the behavior of P_z vs α as computed from Eqs. (5), in comparison with the behavior deduced by the perturbation approach (2), for a case with $I_0 < 2\rho$. The evolution of P_z as obtained by solving the equations of motion is also shown. Note the very good agreement between the adiabatic result [from Eqs. (5)] and the numerical result. Note also the inversion of the ponderomotive force, and the corresponding reflection, when the LTA foresees no reflection for any α_M. In general, due to trapping, when the LTA predicts reflection, an increase of the repulsive ponderomotive force and consequent decrease of α_R is observed.

To analyze the reflection process, we consider again the weakly relativistic approximation, where the adiabatic invariant J can be computed analytically, and the determination of α_R is then reduced to the determination of zeros of a transcendental function. Using the procedure presented in Ref. [3], we introduce the new action $\xi = I/|I_r|$, normalized over $I_r(P_z) = \rho - P_z^2/2$ (which represents the unperturbed resonant value for $P_z^2 \leq 2\rho$), and the function $h = (H - 1 - P_z^2/2)/I_r^2$, which, for $n=2$, reads

$$h(\theta, \xi) = s\xi - \xi^2/2 - \beta\xi \cos 2\theta , \quad (7)$$

where $s = I_r/|I_r|$ and $\beta = \alpha(z)/|I_r|$. The function h, which depends on two parameters only, s and β, describes the electron motion in the fast variables θ, ξ. The adiabatic integral J can be expressed as $J = |I_r| j(s, \beta, h)$, with $j(s, \beta, h) = \int \xi \, d\theta/\pi$. The computation of j is easily performed in terms of elliptic functions for any value of s, β, and h:

$$j(s, \beta, h) = \frac{1}{\pi} \int d\xi \frac{2h - \xi^2}{\{[\xi^2 - 2\xi(s - \beta) + 2h][-\xi^2 + 2\xi(s + \beta) - 2h]\}^{1/2}} , \quad (8)$$

where the extrema are the roots of denominator. The Eqs. (5) reduce to the following equation, which gives the function $P_z(\alpha)$ in both the linear and nonlinear regime:

$$|I_r(P_z)| j(s, \beta, h) = \bar{J} . \quad (9)$$

Note that β is a function of P_z and α, and h depends on P_z and on the initial momenta through H_0. The analysis of the reflection process corresponds to the case $P_z=0$ in Eq. (9):

$$|\rho| j(s, \beta_R, h_R)=\bar{J}, \tag{10}$$

with $\beta_R \equiv \beta(\alpha, P_z=0)=\alpha/|\rho|$, and $h_R \equiv h(P_z=0)=(H_0-1)/\rho^2$.

An example of reflection region for a given value of α_M and $\rho>0$ is shown in Fig. 2, which puts into evidence the substantial difference with respect to the LTA. Note that the reflection region extends to lower I_0 values and to larger P_{z0}, with a maximum P_{z0} on the resonant curve. The main modifications occur in the trapping region characterizing the electrons that cross the separatrix during the interaction. To determine this region, we observe that a separatrix exists in phase space for any value of β when $s=1$, and for $\beta \geq 1$ when $s=-1$. The value of the function h at the separatrix is $h_s=(1-\beta)^2/2$ for $\beta<1$, and $h_s=0$ for $\beta \geq 1$, so that $h_s \geq 0$ for any β. This implies that electrons with initial conditions corresponding to $H_0(I_0, P_{z0})<1$ can never be trapped in the wave field. In the region of momentum space defined by the condition $H_0<1$, the LTA will then give results very close to those obtained from Eq. (10). Inside the trapping region a particular role is played by the electrons, which can experience a net energy change due to the nonlinear interaction with the wave. They define the nonlinear-interaction region, introduced in Ref. [3], which extends around the resonance curve. This region, which saturates for $\alpha_M=\rho$, is defined by the inequalities $P_{z0}^2 \leq 2\rho$, $I_- \leq I_0 \leq I_+$, where $I_\pm = I_r(P_{zs})(1 \pm j_s/2)$ for $\beta(\alpha_M, P_{zs}) \leq 1$, and $I_+=2I_r$, $I_-=0$ for $\beta(\alpha_M, P_{zs})>1$, the latter case corresponding to the saturated region. In the above expressions the index s refers to the separatrix crossing, and j_s is given by

$$j_s=4[\arcsin\sqrt{\beta}+\sqrt{\beta(1-\beta)}]/\pi .$$

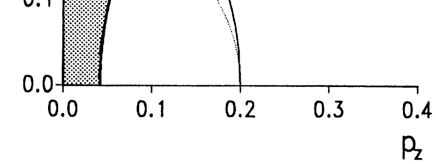

FIG. 2. Reflection region in momentum space for $\rho=0.02$ and $\alpha_M=6\times10^{-2}$. The shaded region corresponds to reflected electrons, the thick dotted curve to the linear result, and the dashed curve to $H_0(P_{z0}, I_0)=1$, which delimits the trapping region. The thin dotted curve is the linear resonance curve $I_0=\rho-P_{z0}^2/2$, while the solid thin line delimits the nonlinear-interaction region, where a net energy variation can occur.

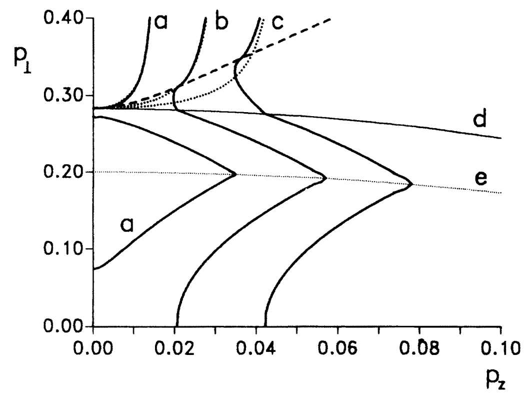

FIG. 3. Reflection region in momentum space for $\rho=0.02$ and different α_M values. Curves a, b, c refer to $\alpha_M=2$, 4, 6×10^{-2}, respectively. The solid lines are the results obtained from Eq. (9) and the dotted lines are the linear results obtained from Eq. (4). The dashed curve is $H_0(P_{z0}, I_0)=1$, the thin curve (e) is the linear resonance curve, and the thin solid line (d) delimits the nonlinear-interaction region.

Electrons with initial momentum belonging to this region experience an energy transition $\Delta\gamma \approx \nu(\rho-P_{z0}^2/2-I_0)$, with probability $\frac{1}{2}$ [3]. In this condition, electrons with initial momentum P_{z0} and energy γ_0 are split by the reflection process in two populations with two different final energies γ_0 and $\gamma_0+\Delta\gamma$, and the same final $P_z=-P_{z0}$.

The behavior of the reflection region with α_M is represented in Figs. 3 and 4, where the reflection regions obtained from Eq. (10) for different α_M values are compared with the linear regions for positive and negative ρ. In both cases strong modifications with respect to the linear cases are observed. From the analysis of Eqs. (8) and (10), the following considerations can be made. For $\rho>0$, the reflection region starts enlarging from the resonant point $P_{z0}=0$, $I_0=\rho$, when $\alpha_M=0$, and for increasing α_M, it extends up to a maximum P_{z0} given by

$$P_{z0}^2=2\{[\nu^2+\alpha_M(\alpha_M+2\rho)]^{1/2}-\nu\}$$

$$\approx \alpha_M(\alpha_M+2\rho)$$

for $I_0=I_r(P_{z0})$, i.e., on the resonant curve, and finally

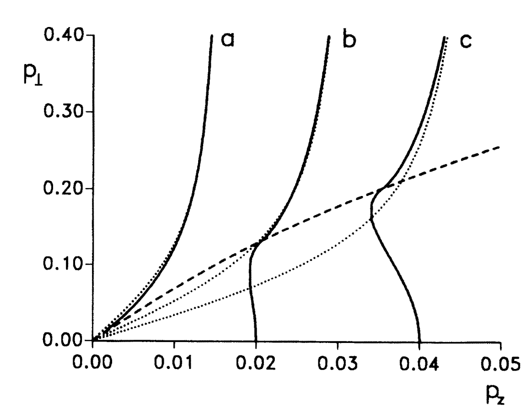

FIG. 4. Same as in Fig. 3 for $\rho=-0.02$.

overlaps the saturated-nonlinear-interaction region when $a_M = \sqrt{2\rho} - \rho$ (Fig. 3). For $\rho < 0$, the reflection region extends up to the axis $I_0 = 0$ ($p_\perp = 0$) for $P_{z0} \leq \max(a_M + \rho, 0)$ (Fig. 4).

In conclusion, we have developed a theoretical model that allows the computation of the adiabatic reflection of the electrons due to the ponderomotive force in the presence of trapping. The investigation is based on the relativistic Hamiltonian formulation of the problem and allows the determination of the reflection region in momentum space. A strong enhancement of the repulsive ponderomotive force and a substantial modification of the reflection region in momentum space with respect to the LTA have been pointed out. The case of the extraordinary mode at the second harmonic has been analyzed in detail. The obtained results are quite general, since it is found that for a trapped particle the ponderomotive force is repulsive. No reflection is found in the case of the ordinary mode because of polarization effects.

Finally, we note that the enhancement of the ponderomotive force due to trapping effects, discussed here, may play a significant role in the self-focusing process of the localized electromagnetic wave. Modifications with respect to the estimates found in the literature [5] should mainly occur for extraordinary waves with frequency around the cyclotron harmonics. On the other hand, a large interval of parameters exists where trapping strongly modifies the reflection mechanism, while self-focusing is negligible. In fact, referring to a Maxwellian plasma of temperature T_e, reflection occurs mainly in the strongly nonlinear regime [3], where $a \gtrsim T_e/mc^2$, while the condition for an effective self-focusing requires $a \gtrsim (T_e/mc^2)^{1/2}$.

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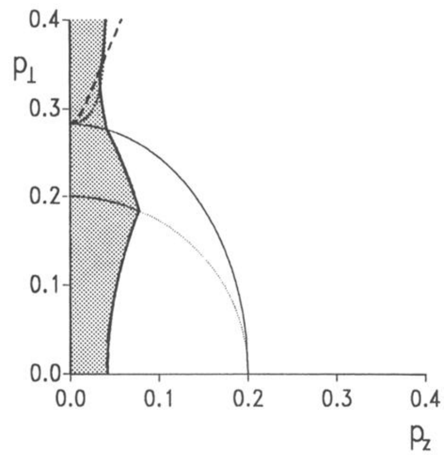
[1] C. Grebogi, A. N. Kaufman, and R. G. Littlejohn, Phys. Rev. Lett. **43**, 1668 (1979).

[2] J. R. Cary and A. N. Kaufman, Phys. Fluids **24**, 1238 (1981).

[3] D. Farina and R. Pozzoli, Phys. Fluids B **3**, 1570 (1991).

[4] A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, Berlin, 1983).

[5] A. G. Litvak, in *Voprosy Teorii Plazmy,* edited by M. A. Leontovich (Atomizdat, Moscow, 1980), Vol. 10, p. 164 [*Reviews of Plasma Physics* (Consultant Bureau, New York, 1986), Vol. 10, p. 293].



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