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Dynamics of spiral waves in a spatially inhomogeneous Hopf bifurcation

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We study numerically the behavior of some topological spiral defects, their dynamics being governed by a complex Ginzburg-Landau equation with space-dependent coefficients. We show that the interaction between a spiral and the gradients of the coefficients can counterbalance the repulsion between defects of identical sign and lead to topologically stable patterns. Configurations of several defects with the same topological charge have been observed recently in nonlinear optics experiments.

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The appearance of a Hopf bifurcation in a two-dimensional system can be described by a complex order parameter $A(\mathbf{r},t)$ slowly varying both in space and time, obeying a generalized Ginzburg-Landau equation [1]:

$$\frac{\partial A}{\partial t}=\mu A+\alpha\nabla^2 A-\beta|A|^2A+\boldsymbol{\gamma}\cdot\nabla A\ , \qquad (1)$$

where μ, α, β, and $\boldsymbol{\gamma}$ are complex coefficients and ∇^2 stands for the two-dimensional Laplacian operator. In the limit of real coefficients, topologically stable solutions of (1), called vortices [2], are known to exist. These solutions correspond to phase singularities of the field A, around which the circulation of the phase gradient is equal to $\pm 2\pi$:

$$A(\mathbf{r},t)=R(\mathbf{r},t)e^{i\phi(\mathbf{r},t)},\ \oint_\Gamma \nabla\phi\cdot dl=\pm 2\pi=2\pi\mathcal{T}_c\ ,$$

where Γ stands for any closed path surrounding the singularity and $\mathcal{T}_c$ for the topological charge of this same singularity. The isophase lines are straight lines centered on the singularity and a single vortex is stationary. When the coefficients are allowed to be complex, there exist similar solutions, called spiral waves. These solutions still correspond to phase singularities around which there is a nonvanishing phase circulation, but the isophase lines have a spiral shape, analogous to the Archimedes spiral, and the structure as a whole rotates around its center with constant angular velocity [3]. Topological solutions of the type mentioned above are also called topological defects, and the region close to the singularity is called the defect core.

Because of the translational invariance in space of Eq. (1), the response of a topological defect to a very small perturbation is expected to be an adiabatically slow displacement of the defect solution. This assumption lies at the root of numerous recent numerical and theoretical studies on the movement of a topological defect due to the effect of boundary conditions [4] or the presence of other defects [5,6]. Although the various results do not completely agree, all these studies report, at least for sufficiently small distances, that the interaction between two defects is repulsive when the defects have the same topological charge and attractive otherwise.

The reduced equation (i.e., omitting spatial derivatives)

$$\frac{\partial A}{\partial t}=(\mu_r+i\mu_i)A-(\beta_r+i\beta_i)|A|^2A$$

describes the evolution of a single nonlinear oscillator, with an amplitude $R=(\mu_r/\beta_r)^{1/2}$ and angular frequency $\omega=(\mu_i\beta_r-\mu_r\beta_i)/\beta_r$. Equation (1) can thus be interpreted as describing the large scale spatiotemporal evolution of a set of nonlinear oscillators, coupled together by diffusion and dispersion, and as such, it naturally emerges in various domains of physics. In this Rapid Communication we consider a collection of such nonlinear oscillators, with the addition that their characteristics (R,ω) and their coupling are allowed to be weakly varying functions of space. The interest in studying such a system is not limited to pure theoretical aspects, as it corresponds also to real situations. The one-dimensional Ginzburg-Landau equation with coefficients slowly dependent on space has already been used to describe open flow systems like shear flow or wakes behind a circular cylinder [7]. In two dimensions, this same system has been used to describe the electric field in a transverse section of a laser. Actually it has been shown recently that the Maxwell-Bloch equations [8], which describe the interaction of an electromagnetic field with a two-level nonlinear medium, can be reduced to a complex Ginzburg-Landau equation with space-dependent coefficients [9]. These coefficients depend on the distance r from the center of the laser beam, as a consequence of the presence of finite-size spherical mirrors and space-dependent pumping profiles. Moreover the presence of optical vortices, corresponding to the phase singularities of the electric field, has been confirmed both by precise numerical simulations of the Maxwell-Bloch equations [10] and by preliminary results of experimental researches in laser physics [11].

Our aim in this paper is to make a qualitative study of the dynamics of topological spiral defects, in the presence of spatial inhomogeneities. We restrict our study to the case where the coefficients of the complex Ginzburg-Landau model are function of the radius r only, so that the rotational symmetry of the equation is preserved. We show numerically that spatial inhomogeneities can confine defects of the same topological charge in a small limited region of space, and we report about the various stable patterns associated with a given total topological charge. These solutions (which are unstable for spatially independent coefficients) broaden the range of application of the Ginzburg-Landau picture, extending direct comparisons between these models and experimental results from

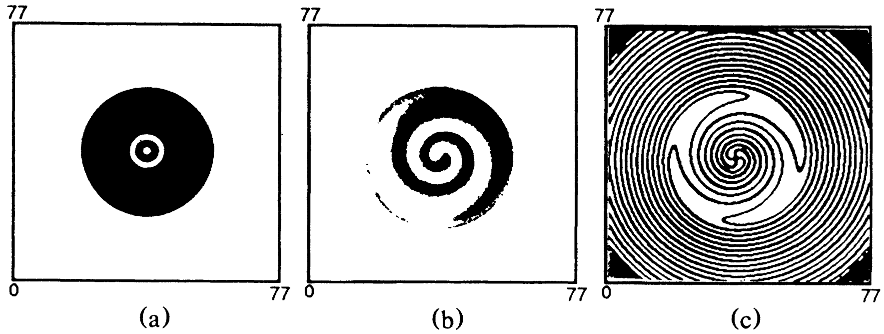

FIG. 1. $\mu_r(r)=-2\tanh[4(r-10)]-1$, $\beta_i=0.5$, $\mu_i(r)=-\tanh(0.1r)$, $\beta_r(r)=-\tanh(0.1r)+1.3$, and $\alpha_r=\alpha_i=0.2$. The initial topological charge is equal to +1, and the contrast of the various pictures has been reinforced by choosing white and black colors for lower and higher intensities, respectively. (a) Spatial distribution of the modulus of A, (b) spatial distribution of the real part of A, and (c) lines where the phase of A is equal to 0, $\pi/2$, $-\pi/2$, and π, respectively.

different branches of nonequilibrium physics ranging from optics to chemical oscillations and hydrodynamics [12]. For example, Arecchi *et al.* have recently reported about the long-term stability of patterns containing six vortices with the same topological charge in an optical oscillator with photorefractive gain medium [13].

The amplitude equations describing the Hopf bifurcation are obtained by a perturbative calculation in the neighborhood of the bifurcation, the scale of variation of the order parameter A being given by the square root of the distance to the bifurcation threshold. Thus, it can be seen that if the parameters of the system vary spatially on a much larger scale than the typical variation of A, the resulting Ginzburg-Landau can be written as

$$\frac{\partial A}{\partial t}=\mu(r)A+\alpha(r)\nabla^2 A-\beta(r)|A|^2A+\gamma(r)\cdot\nabla A\,, \qquad (2)$$

where μ, β, α, and γ are complex coefficients slowly varying in space. Numerical simulations of Eq. (2) were performed using second-order space and time finite difference codes on a massively parallel computer (Connection Machine) in the case of $\boldsymbol{\gamma}=\mathbf{0}$. The number of collocation points varied from 128×128 to 512×512, the time step being equal to 0.1 and the space step varying from 0.1 to 0.3 without any qualitative difference. Although we used a square numerical grid with rigid boundary conditions for our simulations, the results are in perfect agreement with the axisymmetry of (2). For completeness, we have checked that a cylindrical grid yields the same qualitative results.

When all the coefficients are constant (and $\boldsymbol{\gamma}=\mathbf{0}$), the homogeneous trivial solution $A(\mathbf{r},t)=0$ is stable for negative μ, but unstable otherwise. As we want to simulate the radial profile of the pumping process of a laser, we have chosen an axisymmetric hyperbolic tangent shape for $\mu(r)$ so to enhance high light intensities around the beam center and vanishing electric field away from it. Other coefficients were chosen with numerous and various shapes.

Our first result deals with the stability of the topological defect. We have observed that although the shape of a spiral wave is usually altered by the gradients of the coefficients, the phase singularity at the defect core is still present and moves as in the presence of an external force. With our choice of $\mu(r)$, $\alpha(r)$, . . . , this pseudoforce is sufficient to maintain a defect in the region where $|A|$ is nonvanishing. Because of the axisymmetry of (2), the center of the grid plays a special role becoming an attractive stable point to which all single topological defect are driven by the action of the coefficient gradients. Figure 1 shows a typical stable situation with a single topological defect localized at the center of the grid and rotating around its core with constant angular velocity. When the total topological charge is higher than one, each spiral de-

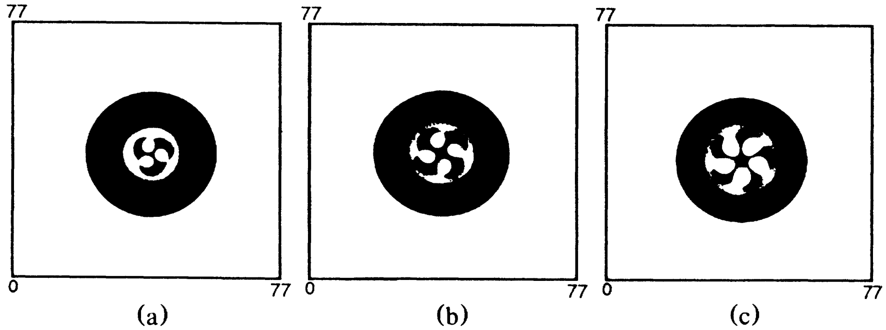

FIG. 2. Spatial distribution of the modulus of A in the presence of (a) three, (b) four, and (c) five spiral defects. The parameters are the same as in Fig. 1, but with different initial topological charges. Each phase singularity rotates around its core, while the whole pattern rotates around the center of the grid.

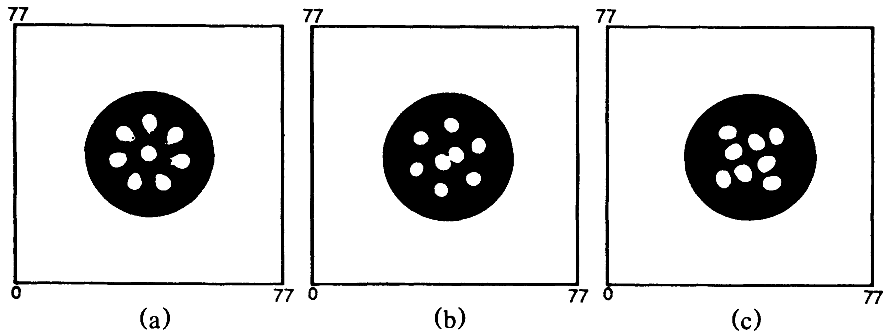

FIG. 3. Spatial distribution of the modulus A for the same parameters of Fig. 1 and total topological charge equal to 8. The differences in the asymptotic states are due to the initial configurations only. (a) Seven defects rotate around one defect localized in the center of the grid. (b) All the defects rotate around the center of the box and in the same direction. There are, however, two circular trajectories, the inner one containing two defects, and the outer one containing six other defects. (c) The two trajectories contain the same number of defects.

fect interacts both with the gradients of the coefficients and with all the other defects. For the same choice of parameters, the system then usually admits several stable states with various total topological charges. However, when the initial topological charge is higher than a critical value, the repulsion between defects prevail over the action of the gradients and some of the spiral defects are ejected from the active region (where $|A|$ is nonvanishing). Figures 1, 2, and 3 illustrate such situations, with total topological charges equal to 1, 3, 4, 5, and 8, respectively. It should be noticed that the distance between a defect and the center of the grid increases with the total topological charge. The symmetry of these regular patterns, arising from the balancing of the repulsion between defects and the drift force induced by the gradient of the coefficients, affects not only the positions of the defect cores but also the phase ϕ of the order parameter A. The entire structure rotates around the center of the plane while each spiral rotates around its own core, keeping, however, the phase difference between two adjacent defects constant. Another interesting feature of (2) is the coexistence of different configurations of defects with the same total topological charge. Figure 3 shows some stable patterns in the case of eight singularities; a single defect in the center of the grid is found at equilibrium with a rotating circle of seven defects [Fig. 3(a)]; groups of defects rotating on circles of different radii find their balance by moving with different angular velocities [Figs. 3(b) and 3(c)]. Moreover, for certain initial conditions, a spontaneous creation of a pair of topological defects with opposite charge occurs during transients. This yields asymptotic patterns where the value of the total topological charge is different from the total number of defects. Similar stable configurations of defects, which, however, do not rotate, neither around their core nor around the center of the grid, have been obtained in the case of purely real coefficients of Eq. (2).

The spontaneous creation, random displacement and mutual annihilation of defects in a regime of phase instability [14] has already been identified as the primary mechanism of the onset of turbulence in the complex Ginzburg-Landau equation (1) [15]. This is still the case when the coefficients depend on space. But now, due to the existence of regular patterns with high topological charge, the first steps of the transition to a turbulent state are associated with a complexification of the motion of the phase singularities and with a progressive destruction of the spatial order. This is illustrated by the four instantaneous plots of Fig. 4 which show the time evolution of an initial configuration with total topological charge equal to 4, after transients have been discarded. Interactions and drift forces are now responsible for the erratic motion of the defects, generating partial loss of spatial correlation. The possibility of detecting and studying these intermediate states makes equation (2) an excellent prototype for the analysis of the transition to turbulent states. Our present efforts are leading in this direction.

Stable patterns with total topological charges different

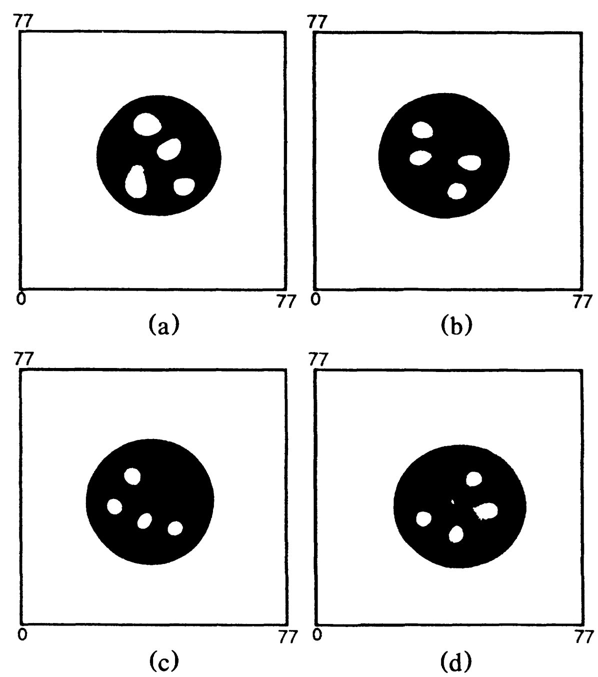

FIG. 4. Same parameters as in Fig. 1, but $\beta_i(r) = \tanh(0.4r) - 1.4$. The four panels show the time evolution in the case of irregular motion of defects.

from zero have been shown to be coexisting and commonplace for partial differential equations with spatially inhomogeneous coefficients. A practical use of this phenomenon in nonlinear optics can be immediately envisaged. Multistable patterns can be utilized to increase the alphabet of a coded signal, i.e., the amount of information transmitted in an optical channel. The experimental results of Ref. [13] confirms that our predictions may have effective applications in this field. In addition to nonlinear optics where the spatial inhomogeneities are intrinsic to the system [9], we may also imagine modifying experimental systems known to contain spiral waves solutions. For example in the case of the Belousov-Zhabotinsky chemical reactions [16], the space variation of the coefficients may be obtained by an inhomogeneous distribution of the species concentrations. Such an experimental apparatus (which already exists [17,18]) should be able to originate some regular beautiful arrangement of spiral defects.

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