RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 45, NUMBER 2 15 JANUARY 1992

$1/f^{\alpha}$ noise in dissipative transport

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(Received 8 March 1991)

Motivated by the hypothesis that "self-organized criticality" is a common source of $1/f$ noise, we construct and analyze a class of nonlinear nonequilibrium models describing the dissipative dynamics of interacting particles injected stochastically at the system boundaries. We show that such noisy boundary problems may be analyzed by renormalization-group methods and find that the noise spectrum for the particle number is $1/f$ in all dimensions in the absence of an external driving force or noise. Addition of such a force or of bulk noise changes the spectrum to $1/f^2$, or $1/f^{3/2}$, respectively. These results explain several recent numerical experiments on dissipative transport.

PACS number(s): 05.40.+j, 5.60.+w

Years of research into the sources of $1/f$ noise in many different physical systems have produced little evidence of the simple, universal explanation which the ubiquity of the phenomenon suggests [1]. Recently, though, it has been conjectured [2] that such an explanation resides in the ability of a broad class of nonlinear nonequilibrium, or driven, systems to exhibit scale invariance or "self-organized criticality" (SOC)—infinite correlation lengths and the concomitant algebraic decays of correlations in both space and time for *arbitrary* parameter values. Since temporal scale invariance immediately implies power spectra that behave like $f^{-\alpha}$ for small f, SOC seems on its surface a promising mechanism for generating $1/f$ noise. One must, however, demonstrate that α values close to unity ($0.8 < \alpha < 1.4$ is the rough experimental range), and reasonable noise magnitudes emerge naturally from it.

While several classes of models of dissipative transport capable of generic scale invariance and hence of $1/f^{\alpha}$ fluctuations have been discovered, few of them have produced values of α close to 1 for spatially averaged quantities [1] of interest. For example, the sandpile models often taken as prototypes for SOC have been argued [3] to give $\alpha=2$, under the assumption that the signals from individual avalanches may be linearly superimposed. Models acted upon by external (e.g., thermal) white noise, which are well understood to yield (with few exceptions) generic scale invariance whenever the transport obeys a local conservation law [4], trivially give, as shown below, $\alpha=2$ spectra for the mean value of the entity being transported.

Prompted by the observation of $1/f$ noise in the spectrum of fluctuations of the number of flux lines piercing a region of superconducting material just above the onset of flux flow [5], Jensen [6] has constructed simple lattice-gas models which provide a notable exception to this pessimistic generalization. The models are appealing schematic representations of many transport processes wherein particles are injected stochastically at the boundary of a sample, move interactively and deterministically through it, their total number being locally conserved, and are removed at the other end. They are similar in spirit both to the models (such as sandpiles) studied as paradigms of SOC, and to the nonlinear stochastic theories of dissipative transport ("driven diffusive systems" [7]) used to model superionic conductors. Jensen has studied numerically the fluctuation spectra of the total particle number in a variety of such models, and found $1/f^{\alpha}$ with α very close to unity under a wide range of conditions. Andersen, Jensen, and Mouritsen [8] have studied similar models wherein the particle motion in the bulk of the sample is stochastic, obtaining $1/f^{3/2}$ spectra. Motivated by these findings, we construct and analyze a class of Langevin equations [9] which explain most of the results, and provide simple criteria for the occurrence of $1/f$ noise in such transport processes. We cast the problem in a language suitable for renormalization-group (RG) analysis, which allows a treatment of the nonlinearities neglected in earlier related models. We show that $\alpha=1$ requires that stochastic behavior and driving forces occur only at sample boundaries (not in the interior), for any physical system whose density is bounded. Relaxation of any of these conditions produces crossover to a $1/f^2$ or $1/f^{3/2}$ power spectrum. We also comment on the relationship of these models to the flux flow experiments, to boundary-driven sandpile models, and to other driven diffusive systems.

The idea that noisy diffusive processes might be responsible for $1/f$ noise is hardly new: It has been around for 40 years [10]. We are by no means trying to repudiate the consensus that such theories are probably inadequate to explain $1/f$ noise in condensed-matter systems such as metals [1]. It is unlikely that SOC is the origin of $1/f$ noise in all systems, and particularly in metals, where the extremely small driving currents which produce the phenomenon, and the absence of long-range spatial correlations [1], make activated processes [1] a more plausible explanation. Rather, our intention is to analyze a class of nonlinear dissipative transport processes sufficiently broad to encompass several nonequilibrium systems currently under investigation. The exploration of many such universality classes will ultimately decide the viability of SOC as a realistic explanation for $1/f$ noise in diverse physical situations.

We model particles which are injected stochastically at the left boundary of a d-dimensional system, move interactively and dissipatively through the medium, and are removed at the other end. The motion in the bulk may be either deterministic or probabilistic, and the particles may also be propelled from left to right by a driving force. This process can be described by the Langevin equation [9]

$$\frac{\partial n(\mathbf{x},t)}{\partial t}=-\nabla\cdot \mathbf{J}\{n\}+\delta(x_\parallel)J_0(\mathbf{x},t)\,. \qquad (1)$$

Here $n(\mathbf{x},t)$ is a coarse-grained density field, $\mathbf{J}\{n\}$ is the current associated with the local conservation of $n(\mathbf{x},t)$, and J_0 is the rate at which particles are injected at the system's left boundary, the $(d-1)$-dimensional hyperplane $x_\parallel=0$. [Here $\parallel$ and $\perp$ denote, respectively, the direction transverse to, and the $(d-1)$ directions in, that plane.] A boundary condition such as $n(x_\parallel=L,t)=0$ is typically imposed at the right edge of the system, though one can also consider a semi-infinite volume by sending $L\to\infty$. Periodic boundary conditions are assumed in the $\perp$ directions.

To have any chance of obtaining from (1) $1/f$ spectra in the total particle number or $\mathbf{k}=\mathbf{0}$ Fourier mode, $N(t)\equiv n(\mathbf{k}=\mathbf{0},t)$, one must prevent the density from increasing without bound. To see this, imagine taking $L=\infty$ and treating J_0 as purely stochastic, $J_0(\mathbf{x}_\perp,t)=\eta(\mathbf{x}_\perp,t)$, where η is a white-noise variable with correlations $\langle\eta(\mathbf{x}_\perp,t)\eta(\mathbf{x}'_\perp,t')\rangle=2\Gamma\delta(\mathbf{x}_\perp-\mathbf{x}'_\perp)\delta(t-t')$ for some constant Γ. Then $N(t)$ executes a random walk, $\partial N/\partial t=\eta(\mathbf{k}_\perp=\mathbf{0},t)$, whereupon the power spectrum $\langle|N(f)|^2\rangle$ diverges like $1/f^2$, the trivial Brownian result. This conclusion is very general, depending only on the conservation of $n(\mathbf{x},t)$ embodied in the $\nabla\cdot\mathbf{J}\{n\}$ term of (1), and not on the form of $\mathbf{J}\{n\}$ or the spatial dependence of the noise [though obviously $\eta(\mathbf{k}=\mathbf{0},t)$ must not vanish [11]]. The boundary condition $n(x_\parallel=L)=0$, which allows particles to escape through the boundary at $x_\parallel=L$, solves the problem for finite L, but a solution that works even when $L=\infty$ is to include in J_0 a term that "pins" the density in the first layer, making it saturate before it random walks to arbitrarily large values, i.e.,

$$J_0(\mathbf{x},t)=-rn(\mathbf{x},t)+\tilde{\eta}(\mathbf{x},t)\,, \qquad (2)$$

where $\tilde{\eta}$ is a noise variable, and r a positive constant. This choice for J_0 incorporates the energetic constraint in real systems against the density (of, e.g., flux lines in the superconductors of Ref. [5]) growing unboundedly; in lattice-gas models [6-8] this constraint is manifest in the hard-core condition. This saturation effect is a key to obtaining $1/f$ noise, rather than $1/f^2$, from Eq. (1) [11]. [Note that Eq. (2) is equivalent to imposing a *noisy* boundary condition, i.e., $n(x_\parallel=0,\mathbf{x}_\perp,t)=\eta(\mathbf{x}_\perp,t)$. Casting general, nonequilibrium problems that involve noisy boundaries in the form of the Langevin equation (1), however, enables one to apply the RG to analyze the nonlinearities.]

To see this, consider first the simple diffusive or mean-field theory obtained by taking the current $\mathbf{J}$ to be $-\nu\nabla n$, where ν is a diffusion constant [12]:

$$\frac{\partial n(\mathbf{x},t)}{\partial t}=\nu\nabla^2 n(\mathbf{x},t)-r\delta(x_\parallel)n(\mathbf{x},t)+\delta(x_\parallel)\tilde{\eta}(\mathbf{x}_\perp,t)\,. \qquad (3)$$

Since particles are added stochastically at $x_\parallel=0$, with some positive average rate, h_0 say, we take $\tilde{\eta}(\mathbf{x}_\perp,t)=h_0+\eta(\mathbf{x}_\perp,t)$, where η represents white noise of strength 2Γ, as above. The field $n(\mathbf{x},t)$ is conveniently separated into steady state and fluctuating parts: $n(\mathbf{x},t)=n_0(x_\parallel)+h(\mathbf{x},t)$, where $n_0(x_\parallel)=\langle n(\mathbf{x},t)\rangle$, and $\langle h(\mathbf{x},t)\rangle=0$; $n_0(x_\parallel)$ satisfies $\partial^2 n_0/\partial x_\parallel^2=0$, and the boundary conditions $n_0(0)=h_0/r$ and $n_0(L)=0$, i.e., $n_0=h_0(1-x_\parallel/L)/r$. The fluctuating piece $h(\mathbf{x},t)$ then obeys an equation identical to (3), but with $\tilde{\eta}$ replaced by η. For an infinite system, this equation is readily solved by Fourier transformation:

$$h(\mathbf{k},f)=g_0(\mathbf{k},f)\eta(\mathbf{k}_\perp,f)/[1+rI(\mathbf{k}_\perp,f)]\,, \qquad (4)$$

with $g_0^{-1}(\mathbf{k},f)\equiv(-if+\nu k^2)$ and $I(\mathbf{k}_\perp,f)\equiv\int dk_\parallel \times g_0(\mathbf{k},f)/2\pi$.

For small f, $I(\mathbf{k}_\perp=\mathbf{0},f)\sim f^{-1/2}$, whence $h(\mathbf{k}=\mathbf{0},f)\sim\eta(\mathbf{k}_\perp=\mathbf{0},f)/f^{1/2}$. Hence $\langle|N(f)|^2\rangle$ diverges like $1/f$ as $f\to 0$. More precisely, the $1/f$ behavior is obtained for $f<f_r\sim r^2/\nu$; for $f>f_r$ one gets $1/f^2$. The only significant effect of incorporating the correct boundary conditions [13] in the finite region $0\le x_\parallel\le L$ is to make the spectrum saturate at $f_L\sim\nu/L^2$, approaching a constant value $\sim\Gamma L^2/r^2$ as $f\to 0$, rather than diverging. This reflects the boundedness of $N(t)$ in the finite region $0\le x_\parallel\le L$.

Thus a pinned, noisy boundary in the mean-field (linear) diffusion problem produces $1/f^\alpha$ noise with $\alpha=1$ in any dimension [14]. One can verify that this result holds in situations wherein the system is "pinned" and subjected to white noise in a boundary region more realistic than the first layer of sites. Consider, e.g., the model

$$\frac{\partial n(\mathbf{x},t)}{\partial t}=\nu\nabla^2 n(\mathbf{x},t)-r_1(x_\parallel)n(\mathbf{x},t)+r_2(x_\parallel)\tilde{\eta}(\mathbf{x},t)\,. \qquad (5)$$

Provided $r_1(x_\parallel)$ and $r_2(x_\parallel)$ decay from $x_\parallel=0$ with some characteristic lengths (exponentially, say), the $1/f$ result derived for the δ function case continues to hold.

Before arguing about the effect of nonlinearities on this result, we study two different linear (mean-field) theories. First consider model (3) in the presence of a uniform driving force that gives the particles a drift velocity across the sample from left to right. Such forcing (present, e.g., in the overdamped motion of charged particles in an electric field), breaks the $x_\parallel\to -x_\parallel$ symmetry of Eq. (3), and so permits the extra linear term $-\kappa\partial n/\partial x_\parallel$ on the right-hand side, for positive constant κ. The solution of this problem is still given, for $f>f_L\sim\nu/L^2$, by Eq. (4), with g_0 now given by $g_0^{-1}\equiv(-if+i\kappa k_\parallel+\nu k^2)$; correspondingly, $I(\mathbf{k}_\perp=\mathbf{0},f)\to\text{const}$ as $f\to 0$, whereupon

$\langle |N(f)|^2\rangle$ grows like $1/f^2$ for small f. Thus the uniform driving produces a noise spectrum which is Brownian in any dimension, rather than $1/f$. More precisely, the spectrum crosses over from $1/f$ to $1/f^2$ for f's less than $f_\kappa \sim \kappa^2/v$, before ultimately saturating at $\sim \Gamma L^2/r^2$ for f's below f_L.

Next consider model (3) without a driving force, but with white noise that conserves the number of particles in the bulk. Such noise is represented by the random variable $\bar\eta(\mathbf{x},t)$, with $\langle \bar\eta(\mathbf{x},t)\bar\eta(\mathbf{x}',t')\rangle = -2\bar\Gamma \nabla^2\delta(\mathbf{x}-\mathbf{x}')\times\delta(t-t')$, for some $\bar\Gamma$. Fourier transformation now yields the solution [15] $\langle |N(f)|^2\rangle \sim 1/f^{3/2}$ for f small, but still larger than the finite-size cutoff f_L proportional to L^{-2}. It is easy to verify that the combination of a driving force and conserving bulk noise produces a $1/f^2$ spectrum.

We must now determine how the hitherto neglected nonlinear parts of the current $\mathbf{J}(n)$ affect the spectrum. As usual, only terms consistent with the symmetry of the problem can occur, those of lowest order in both $n(\mathbf{x},t)$ and gradient operators being most *relevant*, or having the strongest effect on small-f properties [16]. Consider first nonlinear diffusion without external driving, i.e., with $x_\parallel \to -x_\parallel$ invariance. The lowest-order nonlinearity which respects this symmetry (and the conservation of particle number in the interior) is $\mathbf{J}_{\rm NL} \sim (\lambda_\parallel \nabla_\parallel + \lambda_\perp \nabla_\perp) n^2$. Again separating the field n into steady state $(n_0(x_\parallel))$ and fluctuating $(h(x,t))$ parts, and setting $\lambda_\parallel = \lambda_\perp \equiv \lambda$ (which simplifies the notation without changing the results), one obtains

$$\frac{\partial h}{\partial t} = (v - 2\lambda n_0)\nabla^2 h - r\delta(x_\parallel) h + u(n_0,h) - \lambda \nabla^2 h^2 + \delta(x_\parallel)\eta(x_\perp,t)\,, \qquad (6)$$

where

$$u(n_0,h) \equiv -2\lambda \left[\frac{\partial^2 n_0}{\partial x_\parallel^2}\right] h - 4\lambda \left[\frac{\partial n_0}{\partial x_\parallel}\right]\left[\frac{\partial h}{\partial x_\parallel}\right].$$

Before considering the nonlinear $\nabla^2 h^2$ term of (6), we discuss the linear terms which involve the static density profile, $n_0(x_\parallel)$; n_0 is determined by the equation $\partial J_\parallel/\partial x_\parallel = 0$ (where $J_\parallel \equiv \hat{\mathbf{x}}_\parallel \cdot \mathbf{J}$), or $\partial^2(v n_0 - \lambda n_0^2)/\partial x_\parallel^2 = 0$, and the boundary conditions $n_0(0) = h_0/r$, and $n_0(L) = 0$. The solution takes the form $n_0(x_\parallel) = A - (B - C x_\parallel/L)^{1/2}$ for constants A, B, and C, which are independent of L. Thus $\partial n_0/\partial x_\parallel$ is proportional to $1/L$, so the $u(n_0,h)$ pieces of (6) can be neglected in the large-L limit. [They produce crossover effects for $f \lesssim O(L^{-2})$, as discussed above [17].] Next consider the $\lambda n_0 \nabla^2 h$ term. For small λ it is easy to show that $n_0(x_\parallel)$ is linear in $x_\parallel$: $n_0 \sim h_0(1 - x_\parallel/L)/r$, with nonlinear corrections of order $\lambda h_0/vr$. In this limit, it can be verified by RG power counting [16] that the $n_0\nabla^2 h$ term has no effect on the small k and f behavior of model (6). This is not surprising, since the $n_0\nabla^2 h$ term is very similar to the ordinary $\nabla^2 h$ one, but is reduced by a factor of the small parameter $\lambda h_0/vr$.

Thus for small λ and large L, model (6) differs from (3) only by the presence of $\nabla^2 h^2$. Again, power counting shows that this nonlinearity is *irrelevant* for any d, so the mean-field $1/f$ spectrum survives the inclusion of nonlinear fluctuations, i.e., the upper critical dimension is unusually low. (The coupling constant λ decreases like $b^{-(d+1)/2}$ when lengths are rescaled by the factor b.) Details of this calculation, which, owing to the lack of translational invariance in (6), is considerably more complicated than in typical problems, will be given elsewhere.

In the presence of a driving force which breaks the $x_\parallel \to -x_\parallel$ symmetry, the leading allowed nonlinearity in $J_\parallel$ is n^2 (this is clearly the most relevant nonlinear operator that can occur for any symmetry). This produces a nonlinear fluctuating term of the form $\partial h^2/\partial x_\parallel$. Again, one can check that this term is irrelevant, and fails to alter the $1/f^2$ spectrum of the linear theory with driving. One can, however, consider situations with slightly different symmetries, e.g., problems with a "particle-hole" invariance [4] under the transformation $h \to -h$, $x_\parallel \to -x_\parallel$. In such systems the κn term of $J_\parallel$ is forbidden (so that the linear theory yields a $1/f$ spectrum), but the n^2 piece is still allowed. Power counting shows that this term is irrelevant, so the $1/f$ result is valid, for all dimensions down to the upper critical dimension $d_c = 1$. With bulk noise, the $1/f^{3/2}$ result holds down to $d_c = 2$.

The results described here are, with one exception, in agreement with the numerics of Jensen [6] who finds, for several different one- and two-dimensional lattice-gas algorithms (which are varied by the introduction of next-near-neighbor interactions, of pinning centers that trap particles temporarily, or of stochastic particle sources at more than one boundary), a spectrum consistent with $\alpha = 1$ or $\alpha = 2$ in the respective absence and presence of a driving force. The single discrepancy is the one-dimensional model without driving, where the $\alpha = 1$ expected theoretically has so far been observed only when identical stochastic boundary conditions are applied at the two sample boundaries, thus producing a constant density profile. For boundary conditions yielding a nonuniform profile, the exponent $\alpha = 1.5$ seems to occur instead. It is not yet clear whether this signals a crossover from $\alpha = 1$ to $\alpha = 2$.

Our results are also in agreement with the numerics of Andersen *et al.* [8], who study models similar to that of Ref. [6], but with conserving noise in the bulk, finding $\alpha = \frac{3}{2}$ in the absence of driving. Jensen [14(d)] has studied sandpile models [2] in which particles (or, more precisely, units of slope), are added stochastically at certain (closed) boundaries of the system, move through the system according to the deterministic algorithms of Ref. [2], and can leave at other (open) boundaries. If, as seems reasonable, the macroscopic behavior of such systems can be described by our model without a driving force or bulk noise, then one would expect a $1/f$ spectrum for the total slope. This is exactly what is observed numerically.

It is tempting to attribute to model (1) the $1/f$ fluctuations observed in the flux flow experiments of Ref. [5], at certain current levels just above the threshold value where depinning and vortex motion first occur. One might argue that at threshold the driving and impurity pinning forces exactly balance, so the vortices experience no net driving force. Moreover, vortices presumably get introduced stochastically at the edge of the sample, there are obvious energetic constraints against their density diverging, and thermal noise in the bulk is negligibly small, so the problem seems well described by our model without driving, which gives $1/f$. The true situation is more complicated, however. The experiments actually record fluctuations in the number of vortices piercing a small region of the sample, rather than the whole. Our model does predict $1/f$ behavior for this quantity, but only for f's in the range $\Delta L^{-2} \ll f \ll L_{\parallel}^{-2}$, where ΔL and $L_{\parallel}$ are respectively the width of the small region and the shortest distance separating this region from the edge of the sample where the vortices are introduced. It is also likely that impurities play a crucial role right at the depinning transition. Significantly above threshold (i.e., in the "quasilinear" regime of Ref. [5]) the impurities can be ignored more plausibly, and the situation should reasonably be described by our model with a driving force. In this case the fluctuations in the number of vortices piercing a small region is straightforwardly found to behave like $1/f^2$ for $f > f_c$ and is constant for $f < f_c$, just as is observed experimentally. However, the characteristic frequency f_c decreases in our calculation like ΔL^{-2}, whereas it seems to be independent of width in the experiments.

We thank M. P. A. Fisher, J. Krug, and R. F. Voss for helpful discussions. T.H. is grateful to IBM for financial support. T.H.'s research is supported in part by the NSF through Grant No. DMR-90-96267.

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[12] One should allow for different diffusion constants, $\nu_{\parallel}$ and $\nu_{\perp}$ in the $\parallel$ and $\perp$ directions, but this would not alter our results significantly.

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