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Quenched disorder in tethered membranes

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We study the effect of quenched disorder on the behavior of tethered membranes using both analytic and numerical methods. Disorder is introduced via a random spontaneous curvature and a random preferred metric. The presence of impurity-induced spontaneous curvature is found to stiffen the long-wavelength bending rigidity, stabilizing a new disordered flat phase at temperature $T=0$. This new phase, like that of pure membranes at $T>0$, is characterized by anomalous statistical and elastic properties. Since its origin is purely mechanical, it can, in principle, be constructed from an appropriate array of macroscopic springs.

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Flexible "tethered membranes" are two-dimensional generalizations of one-dimensional polymers [1]. Theory suggests [2] that such membranes can exist in a crumpled state analogous to that of linear polymers. To date, however, such a state has only been observed in simulations on non-self-avoiding phantom membranes [3] and not at all in membranes with self-avoidance [4]. Unlike linear polymers, tethered membranes at temperatures $T\neq 0$ have a flat phase with an infinite orientational persistence length. The nonzero temperature flat phase is characterized by an anomalous elasticity [5–7] with elastic moduli that vanish and a bending rigidity that diverges with decreasing wave number.

Recently Nelson and Radzihovsky [8] considered the effects of quenched impurities leading to random disorder in the preferred metric tensor of D-dimensional manifolds embedded in a d-dimensional space. They found, for $D<4$, that the flat phase remains stable with respect to such randomness at nonzero temperature [9] but becomes unstable at zero temperature because of a disorder-induced softening of the bending rigidity.

In this paper, we extend the work of Nelson and Radzihovsky to include the effects of random spontaneous curvature. This type of randomness could be caused by any inhomogeneity that locally breaks the reflection symmetry relating the two sides of a physical membrane. In the partially polymerized lipid bilayers studied by Mutz, Bensimon, and Brienne [10], for instance, such disorder would naturally be expected to arise from a random distribution of polymerizing crosslinks on the two sides of a bilayer. Using an ϵ expansion below $D=4$, we find that the fixed point controlling the nonzero temperature flat phase is stable with respect to randomness. We find that the presence of a random spontaneous curvature tends to stiffen the bending rigidity and gives rise to a new flat-phase fixed point at $T=0$ not present if there is disorder only in the preferred metric. This disordered phase is characterized by a divergent long-wavelength bending rigidity and anomalous elasticity similar to that found for the thermal flat phase, but with different values for the appropriate scaling exponents. Our numerical simulations of disordered membranes confirm these predictions. We stress that the anomalous properties of the disordered phase are purely mechanical in origin, being the result of impurity-induced rather than thermally induced distortions of the membrane.

To define our problem, we introduce the position vector $\mathbf{R}(\mathbf{x})$ specifying the position in R_d of a mass point in a D-dimensional manifold indexed by the vector $\mathbf{x}$ in R_D. We choose a parametrization of the manifold such that $\mathbf{R}(\mathbf{x})=\mathbf{x}$ in the lowest energy configuration in the absence of disorder. Deviations from this unstretched configuration are described by a displacement variable $\mathbf{u}(\mathbf{x})\in R_D$ and a height variable $\mathbf{h}(\mathbf{x})\in R_{d_c}$, where $d_c=d-D$: $\mathbf{R}(\mathbf{x})=\mathbf{x}+\mathbf{u}(\mathbf{x})+\mathbf{h}(\mathbf{x})$. In the presence of a stress field $\sigma_{ab}(\mathbf{x})$ $(a,b=1,\ldots,D)$ and a field $\mathbf{c}(\mathbf{x})\in R_{d_c}$ favoring mean curvature $\nabla^2\mathbf{h}(\mathbf{x})$, the elastic Hamiltonian is

$$\mathcal{H}=\tfrac{1}{2}\int d^Dx\,[\lambda u_{aa}^2+2\mu u_{ab}^2+\kappa(\nabla^2\mathbf{h})^2 - 2\sigma_{ab}(\mathbf{x})u_{ab}(\mathbf{x})-2\mathbf{c}(\mathbf{x})\cdot\nabla^2\mathbf{h}(\mathbf{x})]\,, \quad (1)$$

where

$$u_{ab}(\mathbf{x})=\tfrac{1}{2}(\partial_a u_a+\partial_b u_a+\partial_a\mathbf{h}\cdot\partial_b\mathbf{h}) \quad (a,b=1,\ldots,D)$$

is a truncated strain containing all relevant nonlinear terms. The fields $\sigma_{ab}(\mathbf{x})$ and $\mathbf{c}(\mathbf{x})$ are random variables with Gaussian probability distributions and variances

$$[\sigma_{ab}(\mathbf{x})\sigma_{cd}(\mathbf{x}')]=[\Delta_\lambda\delta_{ab}\delta_{cd}+2\Delta_\mu I_{abcd}]\delta^D(\mathbf{x}-\mathbf{x}')\,, \quad (2)$$

$$[\mathbf{c}_i(\mathbf{x})\mathbf{c}_j(\mathbf{x}')]=\Delta_\kappa\delta_{ij}\delta^D(\mathbf{x}-\mathbf{x}')\,,$$

where $I_{abcd} \equiv \frac{1}{2}(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})$. Here and in what follows, square brackets [ ] denote ensemble averages over quenched random variables. The stress tensor can also be interpreted as a variation $\delta g^0_{ab}(\mathbf{x})$ in the local preferred metric tensor $g^0_{ab}(\mathbf{x})$ via $\sigma_{ab}(\mathbf{x}) = \frac{1}{2}[\lambda\delta_{ab}\delta g^0_{cc}(\mathbf{x}) + \mu\delta g^0_{ab}(\mathbf{x})]$. Similarly, $\mathbf{c}(\mathbf{x}) = \kappa R_0^{-1}(\mathbf{x})$ where $R_0^{-1}(\mathbf{x})$ is the local spontaneous mean curvature.

We can characterize random tethered membranes by linear response functions

$$\chi_{u_a u_b}(\mathbf{q}) = \left[\frac{\partial\langle u_a(\mathbf{q})\rangle}{\partial f^u_b(-\mathbf{q})}\right], \quad \chi_{h_i h_j}(\mathbf{q}) = \left[\frac{\partial\langle h_i(\mathbf{q})\rangle}{\partial f^h_j(-\mathbf{q})}\right], \tag{3}$$

and by Edwards-Anderson [11] correlation functions

$$\begin{aligned} C_{u_a u_b}(\mathbf{q}) &= [\langle u_a(\mathbf{q})\rangle\langle u_b(-\mathbf{q})\rangle], \\ C_{h_i h_j}(\mathbf{q}) &= [\langle h_i(\mathbf{q})\rangle\langle h_j(-\mathbf{q})\rangle], \end{aligned} \tag{4}$$

where $\mathbf{q}$ is the wave vector and f^u_a and f^h_i ($i = D+1,\ldots,d$) are, respectively, external forces conjugate to u_a and h_i.

The divergences of these quantities at zero wave vector are controlled by critical exponents which we define via

$$\chi_{u_a u_b}(\mathbf{q}) \sim q^{-(2+\eta_u)}, \quad \chi_{h_i h_j}(\mathbf{q}) \sim q^{-(4-\eta_h)}, \tag{5}$$

and

$$C_{u_a u_b}(\mathbf{q}) \sim q^{-(2+\eta'_u)}, \quad C_{h_i h_j}(\mathbf{q}) \sim q^{-(4-\eta'_h)}. \tag{6}$$

In general, we can expect the primed and unprimed exponents to differ. Ward identities associated with rotational invariance imply [6] $\eta_u + 2\eta_h = 4 - D$ in the $T \neq 0$ flat phase, and $\eta'_u + 2\eta'_h = 4 - D$ in the new $T = 0$ disordered phase. The primed and unprimed exponents in the disordered phase are related by an exponent $\phi_T = \eta'_h - \eta_h = \eta_u - \eta'_u$ associated with the renormalization of temperature near $T = 0$ [12].

The elastic properties of random tethered membranes, like those of nonrandom membranes [6,7], are characterized by a non-Hookean stress-strain relation $[\langle u_{ab}\rangle] \sim \sigma_{\text{ext}}^{1-\eta_\sigma}$, where σ_{ext} is an external stress associated with forces applied at the boundary of the membrane. In the $T \neq 0$ flat phase $\eta_\sigma = 2\eta_u/(D+\eta_u)$ [6,7], and in the $T = 0$ disordered flat phase we find $\eta_\sigma = 2\eta_u/(D'+\eta_u)$, where $D' = D + \phi_T$.

As in previous treatments of membranes without randomness, it is useful to derive renormalization-group (RG) recursion relations in terms of reduced variables $\hat{\mu} = S_D\mu T/\kappa^2$, $\hat{\lambda} = S_D\lambda T/\kappa^2$, $\hat{\Delta}_\mu = S_D\Delta_\mu T/\kappa^2$, $\hat{\Delta}_\lambda = S_D\Delta_\lambda T/\kappa^2$, and $\hat{\Delta}_\kappa = \Delta_\kappa/\kappa T$, where $(2\pi)^D S_D$ is the solid angle subtended by a D-dimensional sphere. To study behavior in the vicinity of $T = 0$, it is useful to introduce a measure of random spontaneous curvature which remains finite as $T \to 0$, and we define $\hat{g}_\mu = \hat{\mu}\hat{\Delta}_\kappa$. Using a replica formalism, we derive [12] recursion relations for the variables $\{\hat{v}\} = \{\hat{\mu},\hat{\lambda},\hat{\Delta}_\mu,\hat{\Delta}_\lambda,\hat{g}_\mu\}$ by momentum-shell integration, removing components $\mathbf{h}(\mathbf{q})$ and $\mathbf{u}(\mathbf{q})$ with $e^{-l} < q < 1$.

To describe the long-wavelength behavior of membranes with nonzero spontaneous curvature, it is sufficient to consider an attractive invariant subspace of $\{\hat{v}\}$ defined by $\hat{\lambda}/\hat{\mu} = \hat{\Delta}_\lambda/\hat{\Delta}_\mu = -1/3$. This three-dimensional space (shown in Fig. 1) is parametrized by $\hat{\mu} \propto T$ as a measure of renormalized temperature, $\hat{g}_\mu$ as a measure of random spontaneous curvature, and $\hat{\Delta}_\mu$ as a measure of random stress. The recursion relations are

$$\begin{aligned} \frac{d\hat{\mu}}{dl} &= \hat{\mu}\left(\epsilon + 2\hat{\Delta}_\mu - \frac{d_c+12}{6}\hat{g}_\mu - \frac{d_c+24}{12}\hat{\mu}\right), \\ \frac{d\hat{g}_\mu}{dl} &= \hat{g}_\mu\left(\epsilon + 4\hat{\Delta}_\mu - \frac{d_c+18}{6}\hat{g}_\mu - \frac{d_c+36}{12}\hat{\mu}\right), \\ \frac{d\hat{\Delta}_\mu}{dl} &= \hat{\Delta}_\mu\left(\epsilon + 2\hat{\Delta}_\mu - \frac{d_c+6}{3}\hat{g}_\mu - \frac{d_c+12}{6}\hat{\mu}\right) + \frac{d_c}{12}\hat{g}_\mu^2, \end{aligned} \tag{7}$$

where $\epsilon = 4 - D$. We note in the above that a membrane constructed with random spontaneous curvature ($\hat{g}_\mu$) but without random stress ($\hat{\Delta}_\mu$) will always generate random stress upon renormalization.

The only completely stable fixed point (for $D < 4$) is the point P4, associated with the $T \neq 0$ flat phase, with $\hat{\mu} \neq 0$, and $\hat{\Delta}_\mu = \hat{\Delta}_\kappa = 0$. Another fixed point, labeled P5, lies in the $\hat{\mu} = 0$ plane, with $\hat{\Delta}^*_\mu, \hat{\Delta}^*_\kappa \neq 0$. At exactly $T = 0$, P5 controls the long-wavelength properties of any membrane with nonzero random spontaneous curvature ($\Delta_\kappa > 0$) and nonzero shear and compression moduli ($\mu > 0$, $\lambda + 2\mu/D > 0$). The associated $T = 0$ disordered phase is rough but macroscopically flat, with frozen distortions described by the correlation functions $C_{h_i h_j}(\mathbf{q})$ and $C_{u_a u_b}(\mathbf{q})$. Various exponents and fixed-point values for P4 and P5 are given to $O(\epsilon)$ in Table I.

At small but nonzero T, we expect, *a priori*, that $\hat{\mu}(l) \sim e^{l\phi_T}$ near P5, with $\phi_T > 0$ if the fixed point is unstable. Explicit calculation to $O(\epsilon)$ gives $\phi_T = 0$, making P5 marginally stable within the linearized flow equations, but nonlinear contributions make P5 weakly unstable. Disordered membranes at very low T will, therefore, be described by the $T = 0$ fixed point only at length scales L less

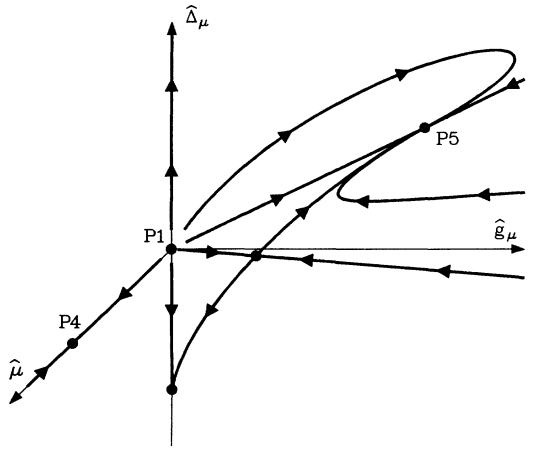

FIG. 1. Flow diagram in the $\hat{\lambda}/\hat{\mu} = \hat{\Delta}_\lambda/\hat{\Delta}_\mu = -1/3$ subspace for $d_c = 1$, $D < 4$. All fixed points shown, except the $T \neq 0$ fixed point P4, lie in the $\hat{\mu} = 0$ (i.e., $T = 0$) plane. P1 is the unstable Gaussian fixed point and P5 is the physical $T = 0$ fixed point. The two unlabeled fixed points in the $\hat{\Delta}_\mu < 0$, $\hat{\mu} = 0$ half-plane are unphysical.

TABLE I. Fixed-point values and exponents η to $O(\epsilon)$ at the $T\neq 0$ and $T=0$ physical fixed points. Both fixed points lie in the $\lambda/\mu=\Delta_\lambda/\Delta_\mu=-1/3$ subspace.

Fixed point	$\hat{\mu}$	$\hat{\Delta}_\mu$	$\hat{g}_\mu$	η_h	η_h'	η_u	η_u'
P4	$12\epsilon/(24+d_c)$	0	0	$12\epsilon/(24+d_c)$	$24\epsilon/(24+d_c)$	$d_c\epsilon/(24+d_c)$	0
P5	0	$3\epsilon/(6+d_c)$	$6\epsilon/(6+d_c)$	$3\epsilon/(6+d_c)$	$3\epsilon/(6+d_c)$	$d_c\epsilon/(6+d_c)$	$d_c\epsilon/(6+d_c)$

than a crossover length $L_c \propto e^{-1/aT}$, where a depends on the unrenormalized parameters of the membrane.

Membranes with a random stress field $\sigma_{ab}(\mathbf{x})\neq 0$ but with $\mathbf{c}(\mathbf{x})=0$ are described by an invariant subspace of $\{\hat{v}\}$ in which $\hat{\Delta}_\kappa=0$. The Hamiltonians $\mathcal{H}[\sigma_{ab}]$ describing such membranes have a global $\mathbf{h}(\mathbf{x})\rightarrow -\mathbf{h}(\mathbf{x})$ reflection symmetry which is absent for $\mathbf{c}(\mathbf{x})\neq\mathbf{0}$. At nonzero T, all RG trajectories in this subspace eventually flow to P4, but at exactly $T=0$, the $\hat{\Delta}_\mu(l)$ and $\hat{\Delta}_\lambda(l)$ diverge [8] with l. The divergence at $T=0$ is caused by an impurity-induced destruction of the bending rigidity $\kappa(l)$. It is believed that this behavior is associated with the spontaneous breaking of reflection symmetry in such membranes, via the formation of a "buckled" state with frozen curvature $[\langle\nabla^2\mathbf{h}\rangle\cdot\langle\nabla^2\mathbf{h}\rangle]\neq 0$. The tendency of random stress to soften the bending rigidity, and of either random spontaneous curvature or nonzero temperature to stiffen it, can be seen most clearly by considering the recursion relation for $\bar{\kappa}\equiv\kappa/T$:

$$\frac{1}{\bar{\kappa}}\frac{d\bar{\kappa}}{dl}+\hat{\eta}_h=\hat{\mu}+\hat{g}_\mu-\hat{\Delta}_\mu\,, \tag{8}$$

where we have set $\hat{\lambda}/\hat{\mu}=\hat{\Delta}_\lambda/\hat{\Delta}_\mu=-1/3$, and we renormalize $\mathbf{h}$ according to $\mathbf{h}(e^l\mathbf{q})=e^{l(4+D-\hat{\eta}_h)}\mathbf{h}'(\mathbf{q})$. The softening of κ for $\hat{g}_\mu=0$ is important in the low-T, disorder dominated regime $\hat{\mu}\ll\hat{\Delta}_\mu$. We note that the predicted destruction of κ at $T=0$ can occur only for exactly reflection symmetric membranes, since the symmetry-breaking variable $\hat{g}_\mu(l)$ grows rapidly under renormalization when $\hat{\mu}\ll\hat{\Delta}_\mu$.

To confirm the existence of a flat, zero-temperature disordered phase, we have carried out numerical simulations of disordered membranes. We treat triangular networks with a harmonic tethering potential $V_0(r)=\frac{1}{2}\omega_0^2(r-a)^2$ as a function of separation r between nearest-neighbor particles, and a repulsive potential $V_1(r)=ae^{-r^2/2b^2}$ between next-nearest neighbors. To introduce random spontaneous curvature, we define the triple product

$$\tau=\mathbf{r}_{CA}\cdot(\mathbf{r}_{BA}\times\mathbf{r}_{DA})(4/3r_0^3) \tag{9}$$

associated with neighboring plaquets Δ_{ABC} and Δ_{CDA}, where $\mathbf{r}_{BA}=\mathbf{r}_B-\mathbf{r}_A$, etc., and r_0 is the nearest-neighbor length at $T=0$ in the absence of disorder. In the limit $r\simeq r_0$, corresponding to a very stiff tethering potential, $\tau\simeq\sin(\theta)$, where θ is the angle between normals to the plaquets. The bending energy for neighboring plaquets is

$$V_b=\tfrac{1}{2}\kappa'(\tau-\sin\theta_0)^2\,, \tag{10}$$

where the preferred angle θ_0 for each pair of plaquets is chosen randomly from a Gaussian probability distribution. The repulsive potential V_1 has been introduced to break the $\theta\rightarrow\pi-\theta$ degeneracy of V_b. All simulations have been carried out using the parameters $\kappa'=0.5\omega_0^2$, $a=2.0\omega_0^2$, $b=0.5a$, giving $r_0\simeq 1.04a$, and a variance $[\theta_0^2]=(\pi/4)^2$. For small deviations of the membrane from the flat reference configuration, we can relate the discrete and continuum parameters via $\kappa'\simeq\kappa$, $\omega_0^2\simeq\mu a^2$, and $[\theta_0^2]\simeq\Delta_\kappa/\kappa^2$, giving $\hat{g}_\mu a^2\simeq 1$ for the values chosen.

We simulate networks of hexagonal shape and of sizes $9\leq L\leq 59$ between opposite corners, using free boundaries. For each L, we construct $32\leq M\leq 128$ membranes (with M smaller for larger L) using different sets of random angles θ_0. To simulate the membranes, we use the molecular-dynamics algorithm of Ref. [5], which is basically a discretization of the Langevin equation. We first equilibrate each membrane at a temperature $T\simeq 0.25\omega_0^2$. To find the energy minima, we then turn the temperature of the heat bath to zero, and let the kinetic energy dissipate through a small viscous force $\mathbf{f}=-\Gamma\dot{\mathbf{r}}$ on each particle. The simulation time required to find an equilibrium configuration for a given membrane is largely determined by the period $2\pi/\omega_b$ of the lowest frequency bending mode, and we choose $\Gamma\simeq\omega_b\simeq 0.005\omega_0-0.05\omega_0$ to achieve nearly critical damping of that mode. Ensemble-averaged quantities for each L are obtained by

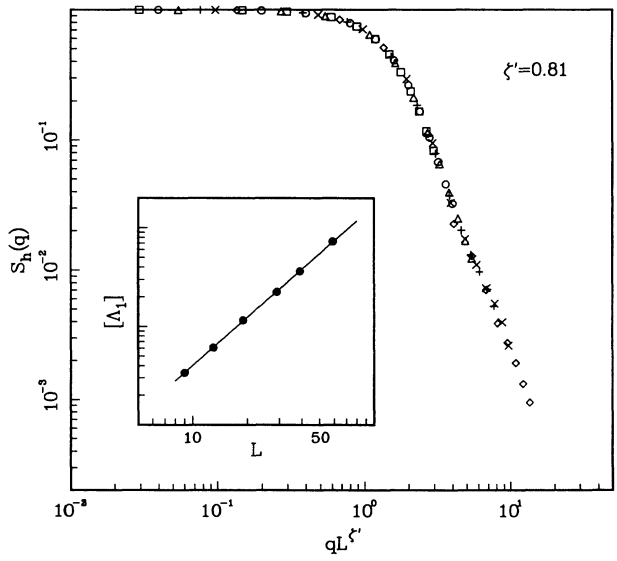

FIG. 2. Configuration-averaged scattering function S_h vs $qL^{\zeta'}$, with $\zeta'=0.81$. Symbols for different sizes L are □=9, ○=13, △=19, +=29, ×=39, ◇=59. Inset: Averaged smallest eigenvalue $[\Lambda_1]$ of I_{ij} vs L. Slope of line corresponds to the same ζ' as for $S_h(q)$.

averaging over the M resulting configurations.

Following the analysis of pure membranes in Ref. [4], we establish a coordinate system for each configuration by diagonalizing the normalized moment-of-inertia tensor $I_{ij}=(1/N)\sum_a r_{ia}r_{ja}$, where $\alpha=1,\ldots,N$ is a particle index, and $\mathbf{r}_a$ is measured from the membrane's center of mass. The unit eigenvector $\hat{\mathbf{e}}_1$ associated with the smallest eigenvalue Λ_1 defines the out-of-plane direction, with $h_a=\hat{\mathbf{e}}_1\cdot\mathbf{r}_a$, and the two remaining eigenvectors define the plane R_D. The configuration averages of the two larger eigenvalues scale with L as $[\Lambda_2],[\Lambda_3]\sim L^2$, and the smallest eigenvalue like $[\Lambda_1]\sim L^{2\zeta'}$, with $\zeta'\simeq 0.81\pm 0.02$ (see Fig. 2, inset), corresponding to $\eta_h'=2(1-\zeta')\simeq 0.4$.

This anisotropic scaling behavior confirms the existence of a flat but rough disordered phase, as predicted above. The measured value of the wandering exponent ζ' is definitely less than the value $\zeta'=\zeta=1$ predicted by harmonic elastic theory and appropriate for sufficiently small disorder and temperature, and is greater than the values $\zeta=0.62-70$ obtained by most authors [4,13,14] for the characteristic exponent of the $T\neq 0$ phase.

We have also determined ζ' by calculating the anisotropic scattering function $S_h(q)=[|(1/N)\sum_a e^{iqh_a}|^2]$. For $q>1/a$, $S_h(q;L)$ for different sizes L can be plotted (see Fig. 2) as a universal function $S_h=S_h(qL^{\zeta'})$. This method gives the same value $\zeta'=0.81\pm 0.03$ as obtained above from the moment-of-inertia tensor.

We have measured the elastic response in the disordered phase by applying small forces to the boundaries of our membranes, reequilibrating each configuration at $T=0$, and then measuring the induced strain at several different values of the applied stress $\sigma_{\rm ext}$. Details of this part of the simulation will be given elsewhere [15], but we note here that our results are consistent with $\eta_h\simeq 0.4$ and $\eta_u\simeq 1.2$, implying that $\phi_T\simeq 0$. We have also obtained $\phi_T=0$ to $O(1/d)$ for arbitrary $D<4$ in a large-d expansion [12], and above we obtained $\phi_T=0$ to $O(\epsilon)$ for arbitrary d in the ϵ expansion. Taken together, these results strongly suggest that $\phi_T=0$ for all d and D, but we have thus far been unable to prove rigorously this conjecture.

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