Directed waves in random media

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We investigate an alternative model for the propagation of directed waves in strongly disordered media. The basic ansatz of our approach is that impurity scattering events can be described by the action of random S matrices. This approach has two important advantages over those considered in previous works. First, it yields a numerical discretization in which unitarity is manifestly preserved. Second, the model enables one to compute certain averages over disorder exactly. The beam positions $[\langle x^2\rangle]$ and $[\langle x\rangle^2]$ characterize the transverse fluctuations of a directed wave front, where $\langle\cdots\rangle$ indicates an average over the wave profile for a given realization of randomness, and $[\cdots]$ indicates quenched averaging over all realizations. We confirm the well-known result that the beam width $[\langle \mathbf{x}^2\rangle]$ grows linearly with the propagation distance, t. We also obtain numerically in two and three dimensions (2D and 3D) the behavior of the beam center $[\langle \mathbf{x}\rangle^2]$ as a function of t. The results suggest that $[\langle \mathbf{x}\rangle^2]$ scales as $t^{1/2}$ in 2D and as $\ln t$ in 3D. We show how these scaling laws emerge in a natural way from the problem of two interacting random walkers. Connections to the problem of directed polymers in random media are also explored.

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I. INTRODUCTION AND SUMMARY

The problem of wave propagation in random media is one of longstanding theoretical interest [1,2]. Only recently, however, have we begun to appreciate its connection to other problems in the physics of disordered systems, such as electron localization [3,4], directed polymers in random media [5], and anomalous diffusion [6,7]. Several authors [8–10] have suggested that the diffusion of *directed* wave fronts in disordered media is described, to a good approximation, by the Schrödinger equation for a particle in a random time-dependent potential. In this paper, we propose a model, based on random S matrices, to explore the consequences of this description. An important aim of our study is to contrast the resulting behavior of waves with the types of diffusion known to occur in other disordered systems.

The approximations that reduce the full wave equation [8] to the parabolic Schrödinger equation describing directed waves have been discussed most recently by Feng, Golubovic, and Zhang (FGZ) [10]. Here, we briefly review this reduction starting with the Helmholtz equation for propagation of a *scalar* wave Φ in a random medium. The static solution for Φ satisfies

$$[\nabla^2+k^2n^2(x,y,z)]\Phi(x,y,z)=0\ , \tag{1.1}$$

where $n(x,y,z)$ is a nonuniform index of refraction that describes the landscape of disorder in the host medium. Following FGZ, we decompose $n^2(x,y,z)=n_0^2+\delta n^2(x,y,z)$, where n_0 is the disorder-averaged index of refraction, and $\delta n^2(x,y,z)$ contains local fluctuations due to randomly distributed scattering centers. The problem of directed waves arises in anisotropic media in which the scattering potential set up by these fluctuations varies slowly in the x direction, so as to favor coherent propagation along the z direction. For such a wave parallel to the z axis, we can set $\Phi(x,y,z)=\Psi(x,y,z)e^{ikn_0z}$, thus reducing Eq. (1.1) to

$$-\frac{\partial^2\Psi}{\partial z^2}+2ikn_0\frac{\partial\Psi}{\partial z}=\left[\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}\right]+k^2\delta n^2(x,y,z)\Psi\ . \tag{1.2}$$

Wave propagation according to Eq. (1.2) can be alternatively regarded as the scattering of photons by the fluctuations in n. We are interested in circumstances where the individual scattering events lead to a sequence of small fluctuations in the transverse momentum components of the z-directed paths. We would also like to ignore any backscattering, i.e., large changes in the longitudinal component of the photon momentum. For these conditions to hold, we require $\delta n^2\ll n^2$ and $\partial_z\delta n^2\ll kn_0\delta n^2$. These conditions may be satisfied in anisotropic media [8–10] (e.g., with long fibers along the z axis). The parabolic wave equation is thus obtained by ignoring the second derivative term on the left-hand side of Eq. (1.2). The analogy to the Schrödinger equation now becomes apparent, after the change of variable $z\leftrightarrow t$, which reduces Eq. (1.2) to

$$i\frac{\partial\Psi}{\partial t}=[\gamma\nabla^2-V(x,y,t)]\Psi\ , \tag{1.3}$$

with $\gamma=(2kn_0)^{-1}$ and $V=k\delta n^2$. Equation (1.3) appears in several contexts besides the problem of directed waves in random media. A quantum-mechanical description of motion in *dynamically* disordered media has particular relevance for the problem of diffusion in crystals at finite

temperature [7,11,12]. Random time-dependent potentials have also been used to model the environment of a light test particle in a gas of much heavier particles [13]. Thus although, as we shall discuss later, the applicability of Eq. (1.3) to wave propagation in random media is somewhat limited, the study of its general scaling properties is of much intrinsic interest.

For generality, we examine the problem of directed waves in d dimensions. The solution to the appropriate Schrödinger equation is then given by the Feynman path-integral formula [8,14,15]

$$\Psi(\mathbf{x},t)=\int_{(0,0)}^{(\mathbf{x},t)}\mathcal{D}\mathbf{x}(\tau)\exp\left\{i\int_0^t d\tau\left[\frac{1}{2\gamma}\left[\frac{d\mathbf{x}}{d\tau}\right]^2+V(\mathbf{x}(\tau),\tau)\right]\right\}, \tag{1.4}$$

where $\mathbf{x}(\tau)$ now describes a path in $d-1$ dimensions. In writing Eq. (1.4), we have chosen the standard initial condition that at time $t=0$, the wave function is localized at the origin. The beam positions $[\langle\mathbf{x}^2\rangle]$ and $[\langle\mathbf{x}\rangle^2]$ characterize the transverse fluctuations of the wave function Ψ about the forward path of least scattering. Here we use $\langle\cdots\rangle$ to indicate an average with the weight $|\Psi(x,t)|^2$ for a given realization, and $[\cdots]$ to indicate quenched averaging over all realizations of randomness. Roughly speaking, $[\langle\mathbf{x}\rangle^2]$ describes the wandering of the beam center, while $[\langle\mathbf{x}^2\rangle-\langle\mathbf{x}\rangle^2]$ provides a measure of the beam width.

Path integrals similar to Eq. (1.4) also appear in the two closely related problems of directed polymers (DP) [5] and strong localization [16–18]. In the former problem $\Psi(\mathbf{x},t)$ represents the (positive) Boltzmann weight for the ensemble of DP configurations which connect the origin to $(\mathbf{x},t)$: each path contributes an energy cost due to line tension, and a potential energy due to encounters with random impurities [5]. This problem is thus obtained by setting γ and $V(\mathbf{x},\tau)$ imaginary in Eq. (1.4). The quantum tunneling probability of a strongly localized electron is also obtained by summing over all paths connecting the initial and final sites. In this case each path also acquires a random phase due to the effects of magnetic impurity scatterings [16]. This problem can thus be described by an imaginary γ, but a real V in Eq. (1.4). We can thus pose the more general problem of studying the characteristic fluctuations of path integrals of the form Eq. (1.4), when the γ and V can take any values in the complex plane. Numerical and analytical evidence seems to indicate that DP and tunneling problems show similar scaling behavior [16,18]. We shall present some evidence indicating that the point corresponding to real γ and V in the complex plane, i.e., representing directed waves, is the only point in this space that shows new scaling behavior for fluctuations.

A special property of Eq. (1.3) which is valid only for real γ and V is *unitarity*, i.e., the norm $\int d\mathbf{x}|\Psi(\mathbf{x},t)|^2$ is preserved at all times. (In the DP and tunneling problems, the norm clearly decays with the length t.) This additional conservation law thus sets apart the directed wave problem from DP, and in a sense makes its solution more tractable. This unitarity is of course a natural consequence of particle conservation for the Schrödinger equation, but has no counterpart for directed wave propagation. It is likely that a beam of light propagating in a random medium will suffer a loss of intensity, due to either backreflection, inelastic scattering, or localization phenomena [19].

Recent efforts to understand the diffusion of directed waves in random media have focused on the scaling behavior of the beam positions $[\langle\mathbf{x}^2\rangle]$ and $[\langle\mathbf{x}\rangle^2]$ at large t. *Lattice* models have been used here with some success. It has been shown using density-matrix techniques, for instance, that $[\langle\mathbf{x}^2\rangle]$ scales linearly in time as a consequence of unitarity [11]; recent numerical simulations [20,21] also support this view. The scaling behavior of $[\langle\mathbf{x}\rangle^2]$ at large t, however, has yet to be resolved. The first numerical work in this area was done by FGZ [10], who used a discretization procedure in which the norm of the wave function was *not* strictly preserved. In 2D, they found that $[|\langle\mathbf{x}\rangle|]$ grew superdiffusively as t^{ν} with $\nu\approx\frac{3}{4}$, while in 3D, they found a phase transition separating regimes of weak and strong disorder. Recent numerical studies on directed waves in 2D cast doubt on the validity of these results when the time evolution is strictly unitary. Medina, Kardar, and Spohn [20], for instance, find that $[\langle\mathbf{x}\rangle^2]$ scales subdiffusively in 2D as $t^{2\nu}$, with $\nu\approx 0.3$. Likewise, Bouchaud, Touati, and Sornette [21] report behavior compatible with $\nu\approx\frac{1}{4}$. They also conjecture that the wave function becomes "multifractal" in that an infinite number of critical exponents are required to describe its evolution.

Somewhat surprising is the fact that a *continuum* formulation of the wave problem leads to different results. An exact treatment of the continuum Schrödinger equation (1.3) has been given by Jayannavar and Kumar [12]. They show that for a random potential δ correlated in time, $[\langle\mathbf{x}^2\rangle]\sim t^3$ as $t\to\infty$. This behavior is modified when there are short-range correlations in time [13], but the motion remains nondiffusive in that the particle is accelerated indefinitely as $t\to\infty$. Lattice models introduce a momentum cutoff $p_{\max}\sim a^{-1}$, where a is the lattice spacing, and therefore do not exhibit this effect. The momentum cutoff generated by the lattice discretization is in some sense artificial. Nevertheless, in a real fluctuating medium, we do expect on large time scales to recover the lattice result, i.e., normal diffusion. The reason is that dissipative effects do generate an effective momentum cutoff in most physical systems. (Strictly speaking, even in the absence of dissipation, relativistic constraints lead to a velocity cutoff $v=c$.) The presence of such a cutoff for the wave propagation problem, and hence the physical relevance of lattice versus continuum models, is still a matter of debate. While there is no underlying lattice, one suspects on physical grounds that there does exist an effective momentum cutoff for propagating waves, related to the speed of light in the background medium.

In this study, we investigate a model for the propagation of directed waves in strongly disordered multiple-scattering media. Our model is formulated on a discrete lattice and reproduces the result that the beam position

$[\langle \mathbf{x}^2 \rangle]$ grows linearly in time. We find also that $[\langle \mathbf{x} \rangle^2]$ scales as $t^{2\nu}$ with $\nu = \frac{1}{4}$ in 2D and as $\ln t$ in 3D. Our approach is noteworthy in several respects. First, our model is formulated in such a way that unitarity is manifestly preserved in numerical simulations, without resorting to complicated checks. Second, we implement scattering events in a manner consistent with the local conservation of probability flux. Third, we perform all averages over disorder *exactly,* whereas previous studies resort to averaging over a necessarily finite number of computer-generated random environments. Finally, we look at scaling behavior in systems that are an order of magnitude larger than those previously considered.

The rest of the paper is divided into two parts. In Sec. II, we develop our model in considerable detail, with emphasis on the simplifying features that permit one to compute averages over disorder exactly. At the end of this section, we present the results of our 2D and 3D numerical simulations. Then, in Sec. III, we interpret our results in light of well-known properties of random walks. We conclude with some final comments on the connection to the DP problem.

II. THE MODEL

Previous numerical investigations of the problem have started by rewriting the Schrödinger equation (1.3) as a difference equation. Such an approach has the advantage of reducing straightforwardly to the continuum description as the unit time increment is shrunk to zero. Unfortunately, the naive discretization of Eq. (1.3) does not preserve the unitarity of time evolution. Since most evidence suggests that it is precisely the constraint of unitarity that gives rise to a new universality class for directed waves, this breakdown is quite serious. Realizing this, previous workers have enhanced the above discretization in ways to mitigate the breakdown of unitarity [10,20,21]. We take a different approach and look for a discretization that manifestly preserves unitarity.

The fundamental motivation for our approach is the path-integral description of quantum mechanics. Rather than discretizing the wave equation (1.3), we seek to implement the sum-over-histories prescription of the path integral (1.4). To this end, let us consider the general problem of a quantum particle on a space-time lattice, initially localized at point A. We propose to assign a complex-valued amplitude to each particle trajectory on the lattice that emanates from A. Additionally, we want to impose the physical requirement that the probability current of the particle satisfies a local conservation law. The normalized wave function of the particle at point B can then be computed by summing the amplitudes of all trajectories that connect A to B. The number of these trajectories is finite due to the discretization of space time. We now show that the sum-over-histories approach, combined with the requirement of probability conservation, gives rise to a model in which the unitarity of time evolution is manifestly preserved.

For concreteness we introduce the model in 2D. A discussion of its generalization to higher dimensions is taken up later. As is customary in the study of directed waves, we identify the time axis with the primary direction of propagation. Our first step, then, is to consider diffusion processes on the 2D lattice shown in Fig. 1. It is amusing to note that this lattice has also been used for discretizing the path integral of a relativistic particle in one dimension [14].

The wave function in our approach takes its values on the links of this lattice. We use $\Psi_{\pm}(x,t)$ to refer to the amplitude for arriving at the site (x,t) from the $\pm x$ direction. At $t=0$, the wave function is localized at the origin, with $\Psi_{\pm}(0,0)=1/\sqrt{2}$. Following the sum-over-histories prescription, our next step is to assign a complex-valued amplitude to each trajectory on the lattice emanating from the origin. Transfer-matrix techniques lend themselves naturally to this purpose. To each site on the lattice, we therefore assign a 2×2 unitary matrix $S(x,t)$. The values of the wave function at time $t+1$ are then computed from the recursion relation

$$\begin{bmatrix} \Psi_+(x-1,t+1) \\ \Psi_-(x+1,t+1) \end{bmatrix} = \begin{bmatrix} S_{11}(x,t) & S_{12}(x,t) \\ S_{21}(x,t) & S_{22}(x,t) \end{bmatrix} \begin{bmatrix} \Psi_+(x,t) \\ \Psi_-(x,t) \end{bmatrix} . \tag{2.1}$$

The S matrices are required to be unitary in order to locally preserve the norm of the wave function.

The S-matrix procedure outlined above weights each trajectory on the lattice with a complex amplitude. Consider, for example, the trajectory in which the particle, incident at the origin from the $-x$ direction, takes two steps in the $+x$ direction and then two steps back. The amplitude $\mathcal{A}$ assigned to this trajectory is given by the product of S-matrix elements:

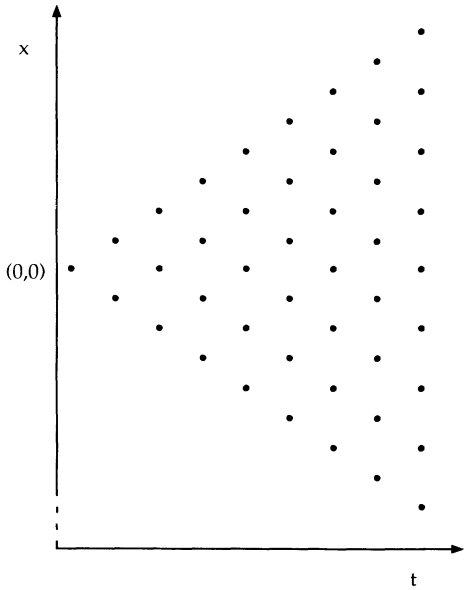

FIG. 1. Lattice discretization for directed waves in $d=2$. The wave function $\Psi_{\pm}(x,t)$ is defined on the links of the lattice, while random scattering events occur at the sites.

$$\mathcal{A}=S_{21}(0,0)S_{21}(1,1)S_{22}(2,2)S_{22}(1,3)\ . \tag{2.2}$$

In general, a trajectory of L links on the lattice is weighted with an amplitude derived from the product of L S-matrix elements. The value of the wave function $\Psi_{\pm}(x,t)$ is obtained by summing the individual amplitudes of all directed paths which start at the origin and arrive at the point (x,t) from the $\pm x$ direction. To simulate the effect of a random potential, we choose the S matrices randomly from the group of 2×2 unitary matrices. We thus achieve a unitary discretization of the path integral in Eq. (1.4), in which the phase change from the random potential $V(x,t)$ is replaced by an element of the matrix $S(x,t)$. The recursion relation in Eq. (2.1) is the coarse-grained analog of the Schrödinger equation (1.3); unlike a simple difference equation, however, Eq. (2.1) enforces the local conservation of probability flux and leads to a sum-over-histories solution for the wave function. Unitarity is manifestly preserved.

Besides these advantages, the S-matrix approach also has a natural physical interpretation of the problem of directed waves in random media. The basic idea is simple: at time t, we imagine that a random scattering event occurs at each site in the lattice at which either $\Psi_{+}(x,t)$ or $\Psi_{-}(x,t)$ is nonzero. The matrices $S(x,t)$, which relate the ingoing and outgoing amplitudes at each lattice site, can then be regarded as scattering matrices in the usual sense. A picture of a typical scattering event is shown in Fig. 2. A lattice S-matrix approach for the study of electron localization and the quantum Hall effect has been used by Chalker and Coddington [22]. A related model has also been recently proposed [23] to investigate the localization of wave packets in random media. These models also include backscattering and hence involve a larger matrix at each site.

We are interested in the beam positions

$$[\langle x^2(t)\rangle]=\sum_x [P(x,t)]x^2 \tag{2.3}$$

and

$$[\langle x(t)\rangle^2]=\sum_{x_1,x_2} [P(x_1,t)P(x_2,t)]x_1x_2\ . \tag{2.4}$$

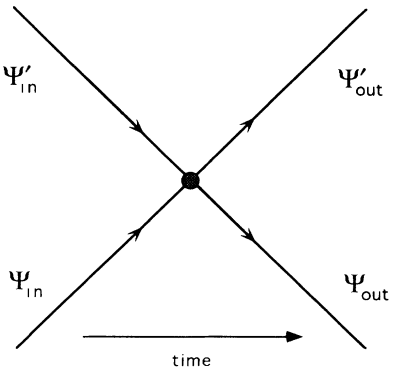

FIG. 2. Scattering event at a lattice site. Time flows in the horizontal direction. A 2×2 S matrix relates ingoing and outgoing amplitudes.

Here, $P(x,t)$ is the probability distribution function (PDF) on the lattice at time t, defined by

$$P(x,t)=|\Psi_{+}(x,t)|^2+|\Psi_{-}(x,t)|^2\ . \tag{2.5}$$

(Defining the weights directly on the bonds does not substantially change the results.) Note that unlike the DP problem, $P(x,t)$ is properly normalized, i.e.,

$$\sum_x P(x,t)=1$$

and Eqs. (2.3) and (2.4) are not divided by normalization such as $\sum_x P(x,t)$. This simplification is a consequence of unitarity, and makes the directed wave problem tractable.

The average [ ] in Eqs. (2.3) and (2.4) is to be performed over a distribution of S matrices that closely resembles the corresponding distribution for V in the continuum problem. However, by analogy to the DP problem [5], we expect any disorder to be relevant. Hence, to obtain the asymptotic scaling behavior, we consider the extreme limit of strong scattering in which each matrix $S(x,t)$ is an *independently* chosen, random element of the group U(2). With such a distribution we lose any preasymptotic behavior associated with weak scattering [13]. The results are expected to be valid over a range of length scales $a\ll x\ll\xi$, where a is a length over which the change of phase due to randomness is around 2π, and ξ is the length scale for the decay of intensity and breakdown of unitarity. Since the parabolic wave equation was obtained from the full wave equation (1.1) by assuming that the scattering potential varied slowly along the propagation direction ($\partial_z\delta n^2\ll kn_0\delta n^2$), it is fair to inquire if the conditions for the validity of such path integrals are ever satisfied in transmission of light. By way of a partial answer, we provide an idealized *macroscopic* realization in which a beam of light is incident upon a lattice of beam splitters arranged as in Fig. 3. Each splitter partially reflects and partially transmits the beam, both in the forward direction. (Note that as long as the beam width is smaller than the size of each slab, the beam does not encounter variations of n along the t direction, and will not be backscattered.) In this strong-scattering limit, the effect of an impurity at (x,t) is therefore to redistribute the incident probability flux $P(x,t)$ at random in the $+x$ and $-x$ directions. On average, the flux is scattered symmetrically so that the disorder-averaged PDF describes the event space of a classical random walk:

$$[P(x,t)]=\frac{t!}{[(t-x)/2]![(t+x)/2]!}\ . \tag{2.6}$$

Substituting this into Eq. (2.3), we find $[\langle x^2(t)\rangle]=t$, in agreement with previous studies [11]. Consider now the position of the beam center $[\langle x(t)\rangle^2]$, given by Eq. (2.4). Unlike $[P(x,t)]$, the correlation function $[P(x_1,t)P(x_2,t)]$ does not have a simple form. An exact calculation of $[\langle x(t)\rangle^2]$ thus proves rather difficult.

One way to proceed is to perform numerical simulations, based on Eq. (2.1), in which averages over disorder are computed by sampling a finite number of computer-

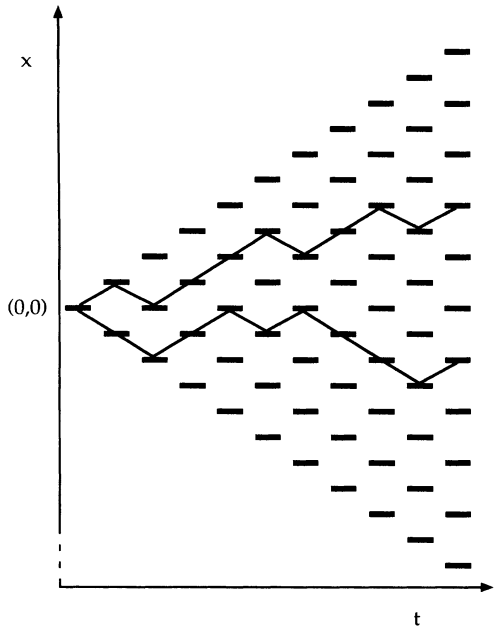

FIG. 3. Lattice of beam splitters in $d=2$. In black: a pair of paths contributing to $W(r,t)$, the disorder-averaged probability that two paths are separated by $2r$ at time t.

generated random environments. For the purpose of computing $[\langle x(t)\rangle^2]$, however, this S-matrix algorithm has a large amount of unnecessary overhead. All the information required to compute beam positions is contained in the function $P(x,t)$. Moreover, we are not interested in those quantities, such as transverse probability currents, for which a complete knowledge of $\Psi_{\pm}(x,t)$ is required. A better algorithm, for our purposes, would be one that directly evolves $P(x,t)$ rather than the wave functions $\Psi_{\pm}(x,t)$.

One may wonder if such an algorithm exists, since in general it is not possible to simulate the dynamics of the Schrödinger equation without reference to the wave function. Consider, however, the scattering event shown in Fig. 2. Probability flux is locally conserved; hence,

$$|\Psi_i|^2+|\Psi_i'|^2=|\Psi_o|^2+|\Psi_o'|^2 . \tag{2.7}$$

As the S matrix that connects these waves is uniformly distributed over the group U(2), its action distributes the outgoing waves uniformly over the set of spinors whose components satisfy Eq. (2.7). A straightforward calculation shows in turn that the ratio

$$q=\frac{|\Psi_o|^2}{|\Psi_i|^2+|\Psi_i'|^2} \tag{2.8}$$

is uniformly distributed over the interval [0,1]. This result, which holds for all scattering events on the lattice, can be used to evolve $P(x,t)$ directly, without reference to the wave functions $\Psi_{\pm}(x,t)$.

Let us examine in detail how this is done. At $t=0$, $P(x,t)$ is localized at the origin:

$$P(x,t=0)=\delta_{x,0} . \tag{2.9}$$

As before, we imagine that at times $t>0$, disorder-induced scattering events occur at all sites on the lattice where $P(x,t)$ is nonzero. Now, however, we implement these events by assigning to each lattice site a random number $0\le q(x,t)\le 1$. The probability distribution function $P(x,t)$ is then directly evolved according to the recursion relation

$$P(x,t+1)=q(x-1,t)P(x-1,t)+\{1-q(x+1,t)\}P(x+1,t) . \tag{2.10}$$

When the numbers q in Eq. (2.10) are distributed uniformly between 0 and 1, this set of rules for evolving $P(x,t)$ is equivalent to the previous one for evolving $\Psi_{\pm}(x,t)$. In fact, we will see later that except in very special circumstances, Eq. (2.10) leads to the same scaling behavior as long as $[q]=\frac{1}{2}$.

So far, then, we have sketched two algorithms that can be used to investigate the scaling behavior of $[\langle x(t)\rangle^2]$. Method A evolves the wave functions $\Psi_{\pm}(x,t)$ through a field of random S matrices. Method B evolves the PDF $P(x,t)$ directly, with much less overhead. The *exact* equivalence of these two methods depends crucially on our choice of a uniform distribution for the S matrices that appear in Eq. (2.1). If the S matrices are not chosen from a uniform distribution over the group U(2), then the ratio q defined by Eq. (2.8) will not be distributed over the interval [0,1] in the same way at all lattice sites. It is easily seen, moreover, that a nonuniversal distribution for q invalidates the logic behind Eq. (2.10). We emphasize, however, that the scaling behavior of $[\langle \mathbf{x}\rangle^2]$ should not depend sensitively on the details of the distribution used to generate the S matrices in Eq. (2.1); a broad range of distributions should belong to the same universality class of diffusive motion. Consequently, the simplifying assumption of a uniform distribution should not destroy the generality of the results for directed waves in random media and/or quantum mechanics in a random time-dependent potential. Method B thus retains the essential elements of the problem, while from a computational point of view it is greatly preferred.

In fact, the greatest virtue of method B is that it permits an even further simplification. Indeed, though faster, more efficient, and conceptually simpler, method B still shares an obvious shortcoming with method A. In both, averages over disorder are performed in an approximate way by sampling a finite number of computer-generated realizations of randomness. We now show how method B can be extended to compute these averages in an exact way.

Define the new correlation function

$$W(r,t)=\sum_x [P(x,t)P(x+2r,t)] . \tag{2.11}$$

From Eq. (2.9), we have the initial condition

$$W(r,t=0)=\delta_{r,0} . \tag{2.12}$$

The value of $W(r,t)$ is the disorder-averaged probability that two paths, evolved *in the same realization of randomness,* are separated by a distance $2r$ at time t. We can compute this probability as a sum over all pairs of paths

that meet this criteria. A typical configuration of paired paths is shown in Fig. 3. Consider now the evolution of two such paths from time t to time $t+1$. Clearly, at times when $r \neq 0$, the two paths behave as independent random walks. On the other hand, when $r=0$, there is an increased probability that the paths move together as a result of participating in the same scattering event. These observations lead to a recursion relation for the evolution of $W(r,t)$:

$$W(r,t+1)=\left[\frac{1+\epsilon\delta_{r,0}}{2}\right]W(r,t) + \left[\frac{1-\epsilon\delta_{r,1}}{4}\right]W(r-1,t) + \left[\frac{1-\epsilon\delta_{r,-1}}{4}\right]W(r+1,t)\ , \tag{2.13}$$

with $\epsilon=4([q^2]-[q]^2)\geq 0$. The value of ϵ measures the tendency of the paths to stick together on contact. As mentioned before, a uniform distribution of S matrices over U(2) gives rise to a uniform distribution of q over the interval [0,1]. In this case, $\epsilon=\frac{1}{3}$.

Starting from Eq (2.13), we have found $W(r,t)$ numerically for various values of $0<\epsilon<1$. The position of the beam center was then calculated from

$$[\langle x(t)\rangle^2]=t-2\sum_r W(r,t)r^2\ . \tag{2.14}$$

The results for $t<15\,000$, shown in Fig. 4, suggest unambiguously that $[\langle x(t)\rangle^2]$ scales as $t^{2\nu}$, with $\nu=\frac{1}{4}$. We emphasize here the utility of the S-matrix model for directed waves in random media. Not only does our final algorithm compute averages over disorder in an exact way, but it requires substantially less time to do so than simulations which perform averages by statistical sampling. We have in fact confirmed our 2D results with these slower methods on smaller lattices ($t<2000$).

We now consider the S-matrix model in higher dimensions. Most of the features of the 2D model have simple analogs. The wave function takes its values on the links of a lattice in d dimensions. Random $N\times N$ S matrices are then used to simulate scattering events at the sites of the lattice. The value of N is equal to one-half the coordination number of the lattice. When the matrices $S(\mathbf{x},t)$ are distributed uniformly over the group U(N), the same considerations as before permit one to perform averages over disorder in an exact way. In addition, one obtains the general result for $d\geq 2$ that $[\langle \mathbf{x}^2\rangle]$ scales linearly in time.

The computation of $[\langle \mathbf{x}\rangle^2]$ in $d>2$, of course, requires significantly more computer resources. In 3D, methods which rely on sampling a large number of realizations of randomness begin to lose their practical value. We have computed $[\langle \mathbf{x}\rangle^2]$ on a 3D body-centered-cubic lattice, starting from the appropriate generalization of Eq. (2.13). The results for $t<3000$, shown in Fig. 5, indicate that $[\langle \mathbf{x}\rangle^2]$ scales logarithmically in time.

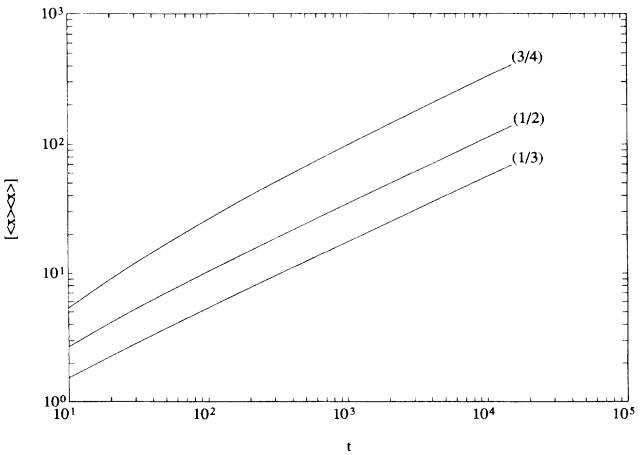

FIG. 4. Log-log plate of the wandering of the beam center $[\langle \mathbf{x}\rangle^2]$ versus the propagation distance t in $d=2$. The value of ϵ [see Eq. (2.13)] is indicated in parentheses.

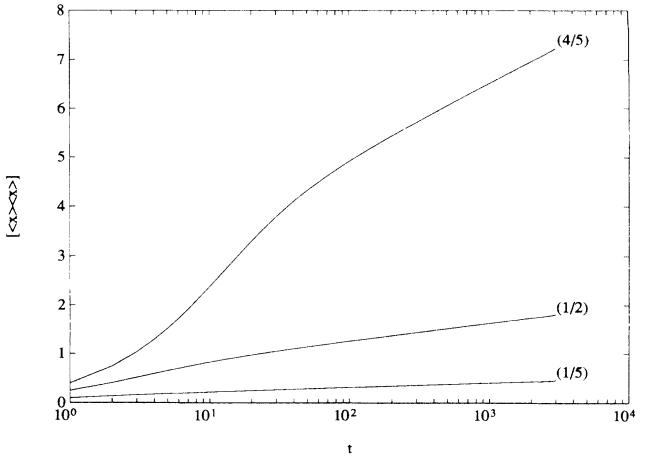

FIG. 5. Semilogarithmic plot of the wandering of the beam center $[\langle \mathbf{x}\rangle^2]$ versus the propagation distance t in $d=3$. The value of ϵ [see Eq. (2.13)] is indicated in parentheses. The results were obtained on a truncated lattice in which the norm of the wave function was preserved to 10^{-6}.

III. ANALYSIS

In this section, we examine our numerical results in light of well-known properties of random walks. Consider a random walker on a $D=(d-1)$-dimensional hypercubic lattice. We suppose, as usual, that the walker starts out at the origin, and that at times $t=0,1,2,\ldots$ the walker has probability $0<p\leq 1/2D$ to move one step in any lattice direction and probability $1-2Dp$ to pause for a rest. The mean time m_0 spent by the walker at the origin grows as [6]

$$m_0\sim\begin{cases} t^{1/2} & (D=1)\\ \ln t & (D=2)\\ \text{const} & (D=3)\ . \end{cases} \tag{3.1}$$

From the numerical results of Sec. II, it is clear that the

same scaling laws describe the wandering of the beam center, $[\langle \mathbf{x} \rangle^2]$, in $d = D + 1$ dimensions, for $d = 2$ and 3. We now show that this equivalence is not coincidental; moreover, it strongly suggests that $d_u = 3$ is a critical upper dimension for directed waves in random media.

To this end, let us return to our model for directed waves in $d = 2$. The recursion relation for $W(r,t)$ can be used to transform the right-hand side of Eq. (2.14), with the result

$$[\langle x(t) \rangle^2] = \epsilon \sum_{\tau=0}^{t-1} W(0,\tau) . \tag{3.2}$$

In the preceding section, we saw that the disorder-averaged correlation function $W(r,t)$ describes the time evolution of two paths in the same realization of randomness. We can also regard $W(r,t)$ as a probability distribution function for the relative coordinate between two interacting random walkers. In this interpretation, the value of ϵ in Eq. (2.13) parametrizes the strength of a contact interaction between the walkers. If $\epsilon = 0$, the walkers do not interact at all; if $\epsilon = 1$, the walkers bind on contact.

According to Eq. (3.2), the wandering of the beam center $[\langle x(t) \rangle^2]$ is proportional to the mean number of times that the paths of these walkers intersect during time t. If $\epsilon = 0$, the number of intersections during time t obeys the scaling law in Eq. (3.1), since in this case, the relative coordinate between the walkers performs a simple random walk. The numerical results of Sec. II indicate that the same scaling law applies when $0 < \epsilon < 1$: the contact attraction does *not* affect the asymptotic properties of the random walk. To elaborate this point, we expand $W(r,t)$ as a power series in ϵ:

$$W(r,t) = \sum_{n=0}^{\infty} \epsilon^n W_n(r,t) .$$

The zeroth-order term in this series, $W_0(r,t)$, describes a simple random walk, while higher-order terms represent corrections due to the contract attraction ϵ. Substituting into Eq. (3.2) and using the $D = 1$ result of Eq. (3.1) give

$$[\langle x(t) \rangle^2] \sim \epsilon t^{1/2} + O\left(\frac{\epsilon^2}{t^{1/2}} \right) . \tag{3.3}$$

The scaling properties of higher-order corrections follow from simple dimensional arguments, with the result that the series converges rapidly for large t. We conclude that $\nu = \frac{1}{4}$ exactly in $d = 2$.

The above argument is readily generalized to $d > 2$, in which ϵ has the units of $[t]^{d-1}$. The result is that the wandering of the beam center, $[\langle \mathbf{x} \rangle^2]$, in $d = D + 1$ dimensions obeys the scaling laws in Eq. (3.1), with next-order corrections smaller by relative factors of $O(\epsilon / t^{d-1})$. Moreover, the argument leads to an upper critical dimension $d_u = 3$ above which the typical wandering of the beam center remains finite even as the propagation distance $t \to \infty$. In summary, three classes of behavior are thus encountered in this model. For $\epsilon = 0$, i.e., no randomness, the incoming beam stays centered at the origin, while its width grows diffusively. For $0 < \epsilon < 1$, the beam center $[\langle \mathbf{x} \rangle^2]$ also fluctuates, but with a dimension-dependent behavior as in Eq. (3.1). In the limit of $\epsilon = 1$, interference phenomena disappear completely. (This limit can be obtained by replacing the beam splitters of Fig. 3 with randomly placed mirrors.) In this case, the beam width is zero, and the beam center performs a simple random walk.

To conclude, we compare the situation here to the one of directed polymers in random media [5]. In the replica approach to the DP problem, the nth moment of the weight $\Psi(x,t)$ is obtained from the statistics of n directed paths. Disorder averaging again produces an attractive interaction between these paths, with the result that the paths can be regarded as the world lines of n quantum particles interacting through a short-range potential. The large t behavior of nth-order moments is then related to the ground-state wave function of the corresponding n-body problem in $d - 1$ dimensions. In $d = 2$, the Bethe ansatz can be used to find an exact solution for particles interacting through δ-function potentials: Any amount of randomness (and hence attraction) leads to the formation of a bound state. The behavior of the bound-state energy can then be used to extract an exponent of $\nu = \frac{2}{3}$ for the superdiffusive wandering of the single DP in the quenched random potential.

By contrast, the replicated paths encountered in the directed wave problem [such as the two paths considered for Eq. (2.11)], although interacting, cannot form a bound state. This point was first emphasized by Medina, Kardar, and Spohn [20], who showed that the formation of a bound state was inconsistent with the constraints imposed by unitarity on the lattice. This result also emerges in a natural way from our model of directed waves. In $d = 2$, for instance, it is easy to check that $W(\mathbf{r},t) \sim (1 - \epsilon \delta_{r,0})^{-1}$ is the eigenstate of largest eigenvalue for the evolution of the relative coordinate. Hence, as $t \to \infty$, for randomness δ correlated in space and time, there is no bound state. This result holds in $d \geq 2$ and is not modified by short-range correlations in the randomness. The probability-conserving nature of Eq. (2.13) is crucial in this regard [24]. Small perturbations that violate the conservation of probability lead to the formation of a bound state. In the language of the renormalization group, this suggests that the scaling behavior of directed waves in random media is governed by a fixed point that is unstable with respect to changes that do not preserve a strictly unitary time evolution. Numerical and analytic results support the idea that this fixed point belongs to a new universality class of diffusive behavior.

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[1] A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, New York, 1978).
[2] *Scattering and Localization of Classical Waves in Random Media,* edited by P. Sheng (World Scientific, Singapore, 1990).
[3] B. A. Altshuler and P. A. Lee, Phys. Today **41**(12), 36 (1988).
[4] *Chance and Matter,* Proceedings of the Les Houches Summer School of Theoretical Physics, Haute Savoie, 1986, Session XLVI, edited by J. Souletie, J. Vannimenus, and R. Stora (Elsevier, Amsterdam, 1987).
[5] M. Kardar and Y.-C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987); G. Parisi, J. Phys. (France) **51**, 1595 (1990).
[6] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 128 (1990), and references therein.
[7] J. P. Bouchaud, Europhys. Lett. **11**, 505 (1990).
[8] R. Dashen, J. Math Phys. **20**, 894 (1979).
[9] H. de Raedt, A. Lagendijk, and P. de Vries, Phys. Rev. Lett. **62**, 47 (1988).
[10] S. Feng, L. Golubovic, and Y.-C. Zhang, Phys. Rev. Lett. **65**, 1028 (1990).
[11] A. A. Ovchinnikov and N. S. Erikhman, Zh. Eksp. Teor. Fiz. **67**, 1474 (1974) [Sov. Phys. JETP **40**, 733 (1975)].
[12] A. M. Jayannavar and N. Kumar, Phys. Rev. Lett. **48**, 553 (1982).
[13] L. Golubovic, S. Feng, and F. Zeng, Phys. Rev. Lett. **67**, 2115 (1991).
[14] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
[15] R. Tao, Phys. Rev. A **43**, 5284 (1991).
[16] E. Medina, M. Kardar, Y. Shapir, and X.-R. Wang, Phys. Rev. Lett. **62**, 941 (1989); **64**, 15 (1990).
[17] Y.-C. Zhang, Phys. Rev. Lett. **62**, 979 (1989); Europhys. Lett. **9**, 113 (1989).
[18] E. Medina and M. Kardar, Phys. Rev. Lett. **66**, 3187 (1991), and unpublished.
[19] For a review of localization of light, see S. John, Phys. Today **44**(5), 32 (1991).
[20] E. Medina, M. Kardar, and H. Spohn, Phys. Rev. Lett. **66**, 2176 (1991).
[21] J. P. Bouchaud, D. Touati, and D. Sornette, Phys. Rev. Lett. **68**, 1787 (1992).
[22] J. T. Chalker and P. D. Coddington, J. Phys. C **21**, 2665 (1988).
[23] C. Vanneste, P. Sebbah, and D. Sornette (unpublished); M. Bernhardt and J. T. Chalker (unpublished).
[24] F. Igloi, Europhys. Lett. **16**, 171 (1991).