PHYSICAL REVIEW A VOLUME 45, NUMBER 12 15 JUNE 1992

General construction of force-free current filaments

N. A. Salingaros
Department of Mathematics, Southern Methodist University, Dallas, Texas 75275
and Department of Physics, University of Texas at Arlington, Arlington, Texas 76019
(Received 10 February 1992)

A complete construction of cables with vanishing Lorentz self-force is given (solution of the cylindrically symmetric force-free problem for nonconstant α). A key component of the result is that boundary conditions are intrinsic and cannot be imposed, which implies that many previous solutions of the problem are in fact spurious. Two force-free current ropes are constructed as an illustration. These results have applications in magnetic fusion; space physics, solar physics, and astrophysics; force-reduced superconducting magnet windings; and superconducting cables for power transmission.

PACS number(s): 52.25.Fi, 41.20.Bt

I. INTRODUCTION

The topic of magnetically force-free current filaments is of central importance in both astrophysics and laboratory plasma physics. Currents in plasmas are often observed to evolve into distinct force-free filaments. It has been suggested by Bostick and co-workers [1–3] that laboratory plasma currents are naturally filamented. This is further supported by the work of Lerner [4,5] on astrophysical plasmas. A general plasma description in terms of current filaments would be very different in application from the classical magnetohydrodynamic theory. Recently, steps towards such a theoretical description have been taken by Peratt [6] and the present author [7–9]. The definition and properties of force-free current filaments together constitute one of the main components of a fiber-theory description of plasmas.

The cylindrically symmetric force-free problem is equally relevant for superconducting cables [10–12]. Helically wound cables with vanishing Lorentz self-force would appear to be ideal for transmitting high currents without the forces that usually lead to quenching or tearing of the conductor [13–15]. Currents in both plasmas and superconductors can attain their highest value only when the current flow is optimized in this way.

This paper presents a scheme for constructing force-free states with cylindrical symmetry. The method sheds light on a subtle point that can lead to serious errors. It has been standard practice in the past to define force-free states as solutions of the equation $\mathbf{B}\times(\nabla\times\mathbf{B})=\mathbf{0}$ without worrying about their physical realization. Nevertheless, the conventional approach not only unnecessarily complicates the mathematical problem, but also introduces the possibility of an interpretative error. We recast the force-free problem in a simple global form, using the fact that the magnetic field depends explicitly on the spatial extent of the generating current distribution [14,16]. In this case, equilibrium is most easily established in terms of the currents and not via the magnetic fields.

A general formulation of axisymmetric plasma equilibrium in a cylinder has been given previously in Ref. [16]. Those results apply equally to helically wound superconducting cables. The approach of Ref. [16] is used here to derive the following result: a helically wound force-free cable is uniquely defined by its axial current distribution $j_z(r)$. This leads to a general prescription for constructing force-free configurations with cylindrical symmetry. Note that this paper solves the general problem $\mu_0\mathbf{j}=\alpha(r)\mathbf{B}$ with nonconstant proportionality factor α. Two examples are provided as an illustration of the method.

II. GENERAL CONSTRUCTION OF FORCE-FREE CURRENTS IN A CYLINDER

It is shown in Ref. [16] how the kinetic pressure profile of an axisymmetric equilibrium plasma is computed directly from the components of the current-density-distribution vector. The force-free case with $\nabla p=\mathbf{0}$ is the simplest example of magnetohydrostatic equilibrium and a concise formulation and complete solution of the problem is possible.

Assume a current-density distribution in a cylinder of radius a with both translational and rotational symmetry, and no radial part, $\mathbf{j}(r)=(0,j_\theta(r),j_z(r))$ in cylindrical polar coordinates (r,θ,z). With this symmetry, the magnetic self-field is computed via an Ampère integral as

$$\mathbf{B}(r)=\mu_0\left[0,\frac{1}{r}\int_0^r s j_z(s)ds,\ \int_r^a j_\theta(s)ds\right]_{\text{cyl}} . \tag{1}$$

We examine a situation with zero pressure throughout: according to the convention $p(a)=0$ at the conductor boundary, we take $p=0$ inside the current volume. The equilibrium equation for $p(r)=0$ and no externally generated field ($B_0=0$) follows from radial force balance [see Eq. (4) of Ref. [16]]. Defining the double integral $f(r)$, the equilibrium condition becomes, using the integral identity in the Appendix,

$$f(r)\equiv\int_r^a\frac{j_z(t)}{t}\int_0^t s j_z(s)ds\,dt , \tag{2a}$$

$$\mathbf{j}(r)\times\mathbf{B}(r)=\mathbf{0}\Longleftrightarrow f(r)=\tfrac{1}{2}\left[\int_r^a j_\theta(s)ds\right]^2 . \tag{2b}$$

The problem posed here is the following: for a given j_z compute the unique j_θ that will produce a force-free current (in the absence of an externally generated field). Inverting and differentiating Eq. (2b) shows that

$$j_\theta(r) = -\frac{f'(r)}{\sqrt{2f(r)}} \; . \tag{3}$$

This j_θ [Eq. (3)], together with the j_z from which it is derived, gives an axisymmetric force-free current distribution inside a cylinder. It is easy to check that the magnetic self-field inside a cylinder [Eq. (1)] assumes the form

$$\mathbf{B}(r) = \mu_0 \left[0, -\frac{f'(r)}{j_z(r)}, \sqrt{2f(r)} \right]_{\text{cyl}} , \quad r \le a \; . \tag{4}$$

The proportionality factor for a general force-free cable with $\mu_0 \mathbf{j} = \alpha(r)\mathbf{B}$ is a function of the radius, $\alpha(r) = j_z(r)/\sqrt{2f(r)}$.

We now derive conditions on the current components in order for this problem to be well posed. From Eq. (2a), the condition $f(a)=0$ shows that a limit must be taken to define $j_\theta(a)$ at the edge in Eq. (3), and exists only when $f'(a)=0$. Using the definition of f [Eq. (2a)], $f'(a)=0$ implies that j_z vanishes at the outer radius a (otherwise the total axial current I is zero). This condition $j_z(a)=0$ is a crucial part of the general solution and implies the geometrical quantization of force-free currents [14]. Now call the limit of the azimuthal current at the plasma edge K for convenience [Eq. (3) as r goes to a] and use l'Hôpital's rule to obtain

$$K \equiv \lim_{r \to a} j_\theta(r) = \frac{\lim_{r\to a} f''(r)}{K} \Longrightarrow f''(a) = K^2 \; . \tag{5}$$

Substituting $j_z(a)=0$ into the second derivative of f from Eq. (2a) leads to the condition $I j_z'(a) = -2\pi a K^2$, where I is the total axial current. This is useful because it gives the value of $K = j_\theta(a)$ directly from the axial current, so it is not necessary to evaluate the limit in Eq. (3) every time. We also have $j_\theta(0)=0$ on the axis, which follows from computing $f'(0)=0$ and $f(0)\neq 0$ in Eq. (3). This condition is required in general (see discussion in the Appendix of Ref. [16]). The end values of j_θ are therefore

$$j_\theta(0)=0, \quad j_\theta(a) = \left[\frac{-I}{2\pi a} j_z'(a) \right]^{1/2} \; . \tag{6}$$

We now summarize the general prescription for constructing force-free states with cylindrical symmetry: (i) choose any differentiable axial current-density profile $j_z(r)$ such that $j_z(a)=0$ (not necessarily the first zero of j_z); (ii) compute $f(r)$ from $j_z(r)$ in Eq. (2a); (iii) $j_\theta(r)$ follows from Eq. (3) and its end values from Eq. (6). The total vector $\mathbf{j} = (0, j_\theta, j_z)$ defines a force-free current in a cylinder of radius a with an internal magnetic field given by Eq. (4). The novelty of the derivation resides in the boundary conditions, which are discussed next.

III. BOUNDARY CONDITIONS

Standard treatments of the force-free problem contain a fundamental misunderstanding arising from the boundary conditions. Solutions of the differential equation $\mathbf{B}\times(\nabla\times\mathbf{B})=0$ actually give the magnetic field of a force-free current distribution that extends continuously out to infinity in all directions [17]. (That corresponds to what is usually called a "force-free magnetic field.") By cutting off the solution at some convenience boundary, the current distribution is also confined to within that boundary. By altering the generating current distribution the magnetic field is thereby altered [14,16,18] and the resulting situation is in most cases no longer force free. Only very specific boundary conditions can preserve the force-free character of an initially infinite current distribution.

This result is reflected in the well-known virial theorem [19–21], which states that there can be no force-free state in a finite volume. The precise link with the boundary conditions has not been clear, however, and this has led to an ambiguity as to what force-free configurations are possible. In particular, we can indeed have a force-free cable with finite radius, as long as its length is (theoretically) infinite.

Up until now, the identification of allowable boundary conditions has not been recognized as an essential component of the force-free problem. The solution given above contains its own boundary conditions because it is formulated in terms of the currents and not the magnetic field. The restriction of allowed solutions by the conditions $j_z(a)=0$ and $j_\theta(0)=0$ is intrinsic to the cylindrical geometry. These conditions are derived and not imposed, and correspond to built-in boundary conditions.

The magnetic field inside a force-free cable [Eq. (4)] satisfies definite (derived) boundary conditions. From Eq. (1) it follows that for any current distribution, the value of the axial field component at the edge $r=a$ vanishes, $B_z(a)=0$. This is a strict condition. Many solutions of the force-free problem with cylindrical symmetry proposed in the literature do not satisfy $B_z(a)=0$; they are consequently incorrect. Other solutions are incomplete insofar as they do not include the solution boundaries, which, as shown above, are an essential part of the solution.

It is instructive to apply the above method to the Bessel-function model, which is the standard textbook example in cylindrical coordinates [21,22]. Starting from $j_z(r) = cJ_0(kr)$ and proceeding as outlined in the preceding section, we obtain $j_\theta(r) = cJ_1(kr)$, as expected (see Ref. [14]). The essential condition $j_z(a)=0$ implies that the radius of the current must coincide with a zero of J_0, which in turn guarantees that $B_z(a)=0$. This is the geometrical quantization condition derived in Refs. [14,18] and missing from conventional treatments.

Note also the following point. The remarkable ease in obtaining the general solution [Eq. (4)] ought to be contrasted with having to solve a nonlinear second-order partial differential equation in the conventional treatments (see Refs. [12,13,23] and Sec. 3.5.4 of Ref. [22]). This facility is a direct consequence of the integral formulation of pinch equilibrium given in Ref. [16]. The intrinsic boundary conditions, as well as the ease of derivation, distinguish the present solution from previous treatments of the problem.

IV. SOME EXAMPLES

As an application of the above method, we list two polynomial solutions for force-free currents with cylindrical symmetry in a cylinder of radius a. In general, one may take any appropriate j_z profile, such as, for example, those given in Sec. 5 of Ref. [16], and immediately write down a model for a force-free cable.

(a) Axial current distribution of the form $j_z = c(1-\zeta)$, $\zeta = r/a \le 1$. This is the simplest example of an axial current-density distribution that goes to zero at the edge. The appropriate azimuthal current distribution for a force-free state is computed from Eqs. (2a) and (3) and the internal magnetic field is computed from Eq. (4). It is easy to cancel the polynomial factors that are responsible for the 0/0 behavior in the expressions for both the azimuthal current and magnetic-field components. We then have

$$\mathbf{j}(\zeta) = c\left[0, \frac{1}{\sqrt{2}}\frac{\zeta(3-2\zeta)}{\sqrt{2+4\zeta-3\zeta^2}}, 1-\zeta\right]_{\text{cyl}},$$
$$\mathbf{B}(\zeta) = \frac{\mu_0 ca}{6}\left(0, \zeta(3-2\zeta), \sqrt{2}(1-\zeta)\sqrt{2+4\zeta-3\zeta^2}\right)_{\text{cyl}}. \tag{7}$$

The current-distribution profiles for this particular force-free cable are unusual because the graph of j_z is a straight line. The magnetic-field profiles, however, are qualitatively similar to the Bessel-function profiles up to the first zero of J_0 (see below).

(b) Axial current distribution of the form $j_z = c(1-\zeta^2)$, $\zeta = r/a \le 1$. Starting from an inverse-parabolic axial current distribution, the azimuthal current distribution giving a force-free state is computed from Eq. (3), and its internal magnetic field from Eq. (4)

$$\mathbf{j}(\zeta) = c\left[0, \frac{\sqrt{6}}{2}\frac{\zeta(2-\zeta^2)}{\sqrt{5-2\zeta^2}}, 1-\zeta^2\right]_{\text{cyl}},$$
$$\mathbf{B}(\zeta) = \frac{\mu_0 ca}{4}\left[0, \zeta(2-\zeta^2), \frac{2}{\sqrt{6}}(1-\zeta^2)\sqrt{5-2\zeta^2}\right]_{\text{cyl}}. \tag{8}$$

The magnetic-field profiles are displayed in Fig. 1 with the Bessel-function profiles included for comparison. Note that both examples (a) and (b) have magnetic fields that resemble each other, as well as the Bessel-function model (see Fig. 1). A key component of the general result is that force-free cables will be qualitatively similar.

The above polynomial examples have only one geometrical force-free state. The existence of higher geometrical states (allowable current radii), as in the Bessel-function model, is a consequence of the oscillatory properties of the solution in that case. Because of their widespread use in the literature, special "layered" solutions may give the misleading impression that force-free states necessarily possess an infinite number of reversals.

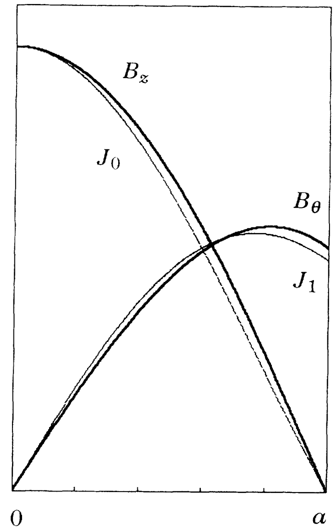

FIG. 1. Magnetic-field profiles for a force-free current distribution with $j_z \propto a^2 - r^2$, compared to the Bessel functions J_0 and J_1.

That is not true in general. As demonstrated in Refs. [16,18], field reversal at the edge is due to an antiparallel externally generated axial field, and is not directly related to the self-field.

V. CONCLUSION

Force-free currents with cylindrical symmetry were constructed following a method for describing pinch equilibrium introduced previously. The result corrects a misunderstanding that invalidates many of the published solutions to the force-free problem. It was shown that a force-free cable is uniquely defined by its axial current distribution j_z. Explicit expressions for the current-density distribution, the magnetic field, and the proportionality factor $\alpha(r)$ were provided. Two polynomial examples were derived as an illustration of the method. These results revise and extend the standard treatments of this topic and show that improved insight is possible by approaching the problem from the current instead of the magnetic field.

APPENDIX: AN INTEGRAL IDENTITY

Let $\Lambda(r)$ denote the indefinite integral of the azimuthal current-density distribution [16], so that $j_\theta(r) = \Lambda'(r)$. Then

$$\int_r^a j_\theta(t)\int_t^a j_\theta(s)ds\,dt = \int_r^a \Lambda'(t)\int_t^a \Lambda'(s)ds\,dt = \tfrac{1}{2}[\Lambda(a)-\Lambda(r)]^2 = \tfrac{1}{2}\left[\int_r^a j_\theta(s)ds\right]^2. \tag{A1}$$

[1] W. H. Bostick, V. Nardi, and W. Prior, J. Plasma Phys. **8**, 7 (1972).
[2] W. H. Bostick, V. Nardi, and W. Prior, Ann. N.Y. Acad. Sci. **251**, 2 (1975).
[3] V. Nardi, W. H. Bostick, J. Feugeas, and W. Prior, Phys. Rev. A **22**, 2211 (1980).
[4] E. J. Lerner, Laser Part. Beams **4**, 193 (1986).
[5] E. J. Lerner, Laser Part. Beams **6**, 457 (1988).
[6] A. L. Peratt, *Physics of the Plasma Universe* (Springer-Verlag, New York, 1992).
[7] N. A. Salingaros, IEEE Trans. Plasma Sci. **17**, 854 (1989).
[8] N. A. Salingaros, Phys. Scr. **43**, 416 (1991).
[9] N. A. Salingaros and R. Carrera, Fusion Technol. **19**, 1302 (1991).
[10] C. J. Bergeron, Appl. Phys. Lett. **3**, 63 (1963).
[11] A. M. Campbell and J. E. Evetts, *Critical Currents in Superconductors* (Taylor & Francis, London, 1972).
[12] D. G. Walmsely, J. Phys. F **2**, 510 (1972).
[13] G. E. Marsh, J. Appl. Phys. **68**, 3818 (1990).
[14] N. A. Salingaros, Appl. Phys. Lett. **56**, 617 (1990).
[15] N. A. Salingaros, J. Appl. Phys. **69**, 531 (1991).
[16] N. A. Salingaros, Plasma Phys. Controlled Fusion **34**, 191 (1992).
[17] N. A. Salingaros, Phys. Essays **1**, 92 (1988).
[18] N. A. Salingaros, Phys. Scr. **43**, 316 (1991).
[19] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Dover, New York, 1961).
[20] E. N. Parker, Phys. Rev. **109**, 1440 (1958).
[21] H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University, Cambridge, England, 1978).
[22] E. R. Priest, *Solar Magnetohydrodynamics* (Reidel, Dordrecht, 1982).
[23] H. P. Furth, M. A. Levine, and R. W. Waniek, Rev. Sci. Instrum. **28**, 949 (1957).