Effect of boundary conditions on the quantum inverse problem for Alfvén waves and finite-size corrections

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(Received 1 October 1991; revised manuscript received 22 January 1992)

We have extended the previous analysis of the quantum inverse problem for the Alfvén wave propogation by incorporating boundary conditions other than the periodic ones. Our approach is that of Sklyanin [Func. Ana. Appl. **21**, 164 (1986)], which allows one to introduce different boundary conditions at the two ends. A generalized Bethe ansatz is used to deduce the eigenvalues of the Hamiltonian. An immediate outcome of our analysis is the effect of finite-size corrections, which is essential to study scale invariance and conformal properties of the model.

PACS number(s): 52.35.Bj, 03.65.−w

INTRODUCTION

In a recent communication we formulated the quantum inverse problem for the equations describing the propagation of nonlinear Alfvén waves in plasma [1]. Starting from the Lax operator L, the form of the quantum R matrix was deduced, which subsequently was used with the algebraic Bethe ansatz to construct the excited states of the model. The equation governing the propagation of an Alfvén wave was a variant of the nonlinear Schrödinger equation. It can be written as [2]

$$\Psi_{1t}=i\Psi_{1xx}-2C\Psi_1\Psi_2\Psi_{1x}\ ,$$
$$\Psi_{2t}=i\Psi_{2xx}-2C\Psi_1\Psi_2\Psi_{2x}\ .$$

As usual, the standard boundary condition chosen was a periodic one and the volume where the quantization was performed was a box of infinite volume. On the other hand, recently it has been observed that scaling invariance and conformal properties of a quantum integrable model could only be deduced if the space of quantization is of finite volume [3]. In an elegant communication Sklyanin [4] observed that it is possible to quantize an integrable classical model with boundary conditions other than the periodic one. Effectively, he showed that it is possible to impose different boundary conditions on the two ends. These boundary conditions can be introduced via two functions K_+ and K_-. An immediate outcome of this formulation is that the system is now quantized in a finite volume and one can compute all the finite-size corrections together. Previously, such corrections were to be calculated for each model separately as demonstrated by deVega and others [5].

Here we extend our previous formulation to include boundary conditions other than the periodic one and also show how finite-size corrections can be calculated for this particular model. In the following we show how the matrices K_+ and K_- can be constructed with the R matrix deduced previously. Then in the following section we set up the modified algebraic Beth-ansatz equations following Sklyanin [6]. Lastly, the Hamiltonian is diagonalized and eigenstates are constructed, which leads to an integral equation for the eigenvalues of the states, showing explicitly the effect of the finite-size correction. There, integral equations are explicitly solved by the Fourier-transform technique, which finally yields an expression for the $1/L$ correction to the excitation energy and hence the scaling dimension.

FORMULATION

The nonlinear wave equations describing the propagation of Alfven waves in a plasma are written as

$$\begin{aligned}\Psi_{1t}&=i\Psi_{1xx}-2C\Psi_1\Psi_2\Psi_{1x}\ ,\\ \Psi_{2t}&=i\Psi_{2xx}-2C\Psi_1\Psi_2\Psi_{2x}\ .\end{aligned}\tag{1}$$

The system is completely integrable classically and the space part of the Lax pair is written as

$$\Psi_x=L\Psi\ ,\tag{2}$$

$$L=+\begin{bmatrix}\frac{1}{2}(C\Psi_1\Psi_2+\lambda^2) & +\lambda\Psi_1\sqrt{C}\\ \lambda\Psi_2\sqrt{C} & -\frac{1}{2}(\lambda^2+C\Psi_1\Psi_2)\end{bmatrix}\ .\tag{3}$$

It was proved in Ref. [1] that the quantum R matrix associated with Eq. (3) is given as

$$R=\begin{bmatrix}a&0&0&0\\0&b&c&0\\0&c&b&0\\0&0&0&a\end{bmatrix}\tag{4}$$

where a, b, and c are functions of the spectral parameter written as

$$\begin{aligned}a&=\frac{\sinh(u_1-u_2+\eta)}{\sinh(u_1-u_2)}\ ,\\ b&=1\ ,\\ c&=\frac{\sinh\eta}{\sinh(u_1-u_2)}\ .\end{aligned}\tag{5}$$

It may be remarked that these are actually the functions occurring in the R matrix deduced in Ref. [1]; they are connected by the simple substitutions, $\lambda = e^{u_1}$, $\mu = e^{u_2}$, and $1 + C/2 - e^{\eta}$. This later form of the R matrix will be more convenient for the present computation.

In the Sklyanin approach, which is actually based on the factorized scattering matrix formalism with the reflection coefficient of Cherednik [7], the primary requirement is to search for the two functions K_+ and K_-, which will lead to the different boundary conditions at the two ends of the space axis. In the following we give some salient features of the Sklyanin approach, which we then apply in sections, and the modified algebraic Bethe ansatz (MABA) is then used to construct the excited states along with the integral equation for the eigenvalues. Our computation explicitly displays the effect of finite-size correction that is essential for understanding the scale invariance and conformal properties.

SKLYANIN APPROACH

Let us consider the integrable equation (1) on a finite segment of the space axis. Let T denote the matrix associated with Eq. (2) satisfying the relation

$$\frac{\partial T(\lambda,x,x_-)}{\partial x} = L(\lambda,x)T(\lambda,x,x_-) , \tag{6}$$

with the condition that $T(\lambda,x_-,x_-) = \mathbb{1}$ where $\mathbb{1}$ stands for the unit matrix. It is also pertinent that L satisfies

$$\{L^{(1)}(\lambda),L^{(2)}(\mu)\} = [r_-,L^{(1)}(\lambda)+L^{(2)}(\mu)] \times \delta(x_1-x_2) \tag{7}$$

where

$$L^{(1)}(\lambda) = L(\lambda,x_1)\otimes \mathbb{1} , \quad L^{(2)}(\mu) = \mathbb{1}\otimes L(\mu,x_2) , \tag{8}$$

where $\{a,b\}$ denotes the Poisson bracket between the elements of the matrices a and b. The matrix function r_- depends only on $\lambda-\mu$ and satisfies the classical Yang-Baxter equation. Here we consider the integrable model on a finite portion of the real axis $[x_+,x_-]$. Let us now consider three more matrices $r_+ = [r_+(\lambda+\mu)]$, K_+, and K_- that satisfy the relations

$$[r_-,K^{(1)}K^{(2)}] = K^{(2)}r_+K^{(1)} - K^{(1)}r_+K^{(2)} , \tag{9}$$

where $K^{(1)} \equiv K\otimes \mathbb{1}$ and $K^{(2)} \equiv \mathbb{1}\otimes K$.

The R matrix given in Eq. (4) satisfies

$$R(\lambda-\mu)L^{(1)}(\lambda)L^{(2)}(\mu) - L^{(2)}(\mu)L^{(1)}(\mu)R(\lambda-\mu) , \tag{10}$$

hence the generator of mutually commuting conserved quantities in the case of periodic boundary conditions is

$$T(\lambda) - \mathrm{Tr}T(\lambda) - A(\lambda)+D(\lambda) . \tag{11}$$

To generalize the situation when K_+ and K_- give the boundary condition at x_+ and x_-, Sklyanin showed that K_+ and K_- are to be determined from the following equations:

$$\begin{aligned} R(\lambda_{12})K_-^1(\lambda_1)R(\bar{\lambda}_{12}-\eta)K_-^2(\lambda_2) &= K_-^2(\lambda_2)R(\bar{\lambda}_{12}-\eta)K_-^1(\lambda_1)R(\lambda_{12}) \\ R(-\lambda_{12})K_+^{1t_1}(\lambda_1)R(-\bar{\lambda}_{12}-\eta)K_+^{2t_2}(\lambda_2) &= K_+^{2t_2}(\lambda_2)R(-\bar{\lambda}_{12}-\eta)K_+^{1t_2}(\lambda_1)R(-\lambda_2) , \end{aligned} \tag{12}$$

where we have used λ_1 and λ_2 as the two distinct values of the spectral parameter and $\lambda_{12} = \lambda_1-\lambda_2$, $\bar{\lambda}_{12} = \lambda_1+\lambda_2$, and the symbol t_i stands for the transposition in the space v_i. Introducing the matrix $v(\lambda)$ as

$$U(\lambda) = T(\lambda)K_-(\lambda)\sigma_2 T^2(-\lambda)\sigma_2 = \begin{vmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{vmatrix} , \tag{13}$$

Sklyanin proved that $U(\lambda)$ also satisfies

$$R(\lambda_{12})U^1(\lambda_1)R(\bar{\lambda}_{12}-\eta)U^2(\lambda_2) = U^2(\lambda_2)R(\bar{\lambda}_{12}-\eta)U^1(\lambda_1)R(\lambda_{12}) . \tag{14}$$

Then the generator of the commuting conserved densities is given as

$$t(\lambda) = \mathrm{Tr}[K_+(\lambda)U(\lambda)] , \tag{15}$$

which can be used as the generator of the Hamiltonians. In all the above equations we have always used the notation that for any matrix $C(\lambda)$,

$$C^1(\lambda) = C(\lambda)\otimes \mathbb{1} \quad \text{and} \quad C^2(\lambda) = \mathbb{1}\otimes C(\lambda) .$$

So for the specific model under consideration we start from the R matrix given in Eq. (4) and seek solutions of Eqs. (12).

CONSTRUCTION OF K_+ AND K_-

We now solve Eq. (12) for K_+ and K_- when the R matrix is given by Eq. (4). For this purpose we have seen that it is convenient to write out these matrices in terms of basis matrices e_{ij} with zero everywhere except at the (ij)th position. The general R matrix can be written as

$$R(u) = a(u)\sum_K e_{KK}\otimes e_{KK} + b(u)\sum_{\substack{p,q \\ p\neq q}} e_{pp}\otimes e_{qq} + C(u)\sum_{\substack{p,q \\ p\neq q}} e_{pq}\otimes e_{qp} . \tag{16}$$

Furthermore, we assume $K_\pm$ to have the diagonal form

$$K_\pm(u) = \sum_{j=1}^{2} K_\pm^{jj}(u)e_{jj} , \tag{17}$$

so that Eqs. (12) result in the following algebraic form:

$$\frac{K_\mp^{22}(u_2)K_\mp^{22}(u_1) - K_\mp^{11}(u_1)K_\mp^{11}(u_2)}{K_\mp^{11}(u_2)K_\mp^{22}(u_1) - K_\mp^{22}(u_2)K_\mp^{11}(u_1)} = \frac{b(-u_1-u_2-\eta)C(-u_1+u_2)}{C(-u_1-u_2-\eta)b(-u_1+u_2)} . \tag{18}$$

In the above analysis we have tacitly assumed that K_+ is a diagonal matrix. This assumption stems from the fact that in the original model of Cherednik where the factorized S matrix with reflection was considered, these K_+ functions were nothing other than the matrices T^i representing the reflection of either the first, second, etc. particle from the boundary. It has nothing to do between two different particles, therefore, it is quite natural that it will have a diagonal structure in the product space. Equation (18) is a simple algebraic equation which in the present case, where a, b, c, etc. are given by Eq. (5), can be solved immediately to yield

$$K_+=\begin{bmatrix}\sinh(u-\eta/2+\xi_+) & 0\\ 0 & \sinh(-u-\eta/2+\xi_+)\end{bmatrix} \tag{19}$$

with a similar expression for K_-.

COMMUTATION RULES FOR THE SCATTERING DATA

The commutation rules for the scattering data, given by Eq. (14), can now be written out in full, and we get

$$\begin{aligned}A(u_1)B(u_2)=&\frac{b(u_1+u_2-\eta)a(u_2-u_1)}{b(u_2-u_1)a(u_1+u_2-\eta)}B(u_2)A(u_1)\\&-\frac{b(u_1+u_2-\eta)C(u_2-u_1)}{a(u_1+u_2-\eta)b(u_2-u_1)}B(U_1)A(u_2)\\&-\frac{C(u_1+u_2-\eta)}{a(u_1+u_2-\eta)}B(u_1)D(u_2)\end{aligned} \tag{20}$$

along with

$$\begin{aligned}D(u_1)B(u_2)\equiv&\frac{C(u_1-u_2)}{a(u_1+u_2-\eta)b(u_1+u_2-\eta)b(u_1-u_2)}[C^2(u_1+u_2-\eta)-a^2(u_1+u_2-\eta)B(u_1)]D(u_2)\\&+\frac{a(u_1-u_2)}{a(u_1-u_2-\eta)b(u_1+u_2-\eta)}[a^2(u_1-u_2-\eta)-c^2(u_1+u_2-\eta)]B(u_2)D(u_1)\\&-B(u_2)A(u_1)\frac{C(u_1-u_2)C(u_1+u_2-n)a(u_1-u_2)}{a(u_1+u_2-\eta)b(u_1-u_2)}\left[\frac{a(u_2-u_1)}{b(u_2-u_1)}+\frac{a(u_1-u_2)}{b(u_1-u_2)}\right]\\&+B(u_1)A(u_2)\frac{C(u_1+u_2-\eta)}{a(u_1+u_2-\eta)}\frac{1}{b(u_1-u_2)}\left[\frac{C(u_1-u_2)C(u_2-u_1)}{b(u_2-u_1)}+\frac{a^2(u_1-u_2)}{b(u_1-u_2)}\right].\end{aligned} \tag{21}$$

For our specific problem, the specific values of a, b, and c given in Eq. (5) are to be used. The complicated nature of the commutation rules Eq. (21) required a different set of scattering data to be defined. This set of data makes the diagonalization of the Hamiltonian easier and is of great convenience in setting up the Algebraic Bethe-ansatz equation. Actually, the original scattering data follow from a four-term commutation rule and it is highly inconvenient to construct the Bethe eigenstates with the help of these. The latter set of data follows a set of commutation rules [Eq. (24)] similar in form to the usual ones and can be manipulated easily. This set of data (A,B,C,D) is defined through the algebraic adjunct of the matrix u. Mathematically, we write

$$\begin{aligned}\bar U(u)=&\begin{bmatrix}\bar D(u) & -\bar B(u)\\ -\bar C(u) & \bar A(u)\end{bmatrix}\\=&2tr_2P_{12}^{-1}U(u)R_{12}(2U) &(22)\\=&\begin{bmatrix}-C(2u)A(u)+b(2u)D(u) & -a(2u)B(u)\\ -a(2u)C(u) & b(2u)A(u)-C(2u)D(u)\end{bmatrix}. &(23)\end{aligned}$$

If we now recalculate the commutation relation between D and B, we get for our present problem

$$\begin{aligned}\bar D(V_1)B(V_2)=&B(v_1)A(v_2)\frac{\sinh\eta\sinh(2v_1+\eta)\sinh(2v_2-\eta)}{\sinh(v_1+v_2)\sinh2v_2\sinh2v_1}-B(v_1)\bar D(v_2)\frac{\sinh\eta\sinh(2v_1+\eta)}{\sinh(v_1-v_2)\sinh2v_1}\\&+B(v_2)\bar D(v_1)\frac{\sinh(v_{12}+\eta)\sinh(v_{12}+\eta)}{\sinh v_{12}\sinh v_{12}}.\end{aligned} \tag{24}$$

With these commutation rules at hand we can now proceed to set up the modified algebraic Bethe ansatz to construct the excited states. To proceed we observe that from Eq. (15)

$$\begin{aligned}t(u)=&\mathrm{Tr}[K_+(u)U(u)]\\=&\sinh(u+\eta/2+\xi_+)A(u)+\sinh(\xi_+-u-\eta/2)D(u)\\=&\frac{\sinh(2u+\eta)\sinh(\xi_++u-\eta/2)}{\sinh2u}A(u)+\sinh(\xi_+-u-\eta/2)D(u).\end{aligned} \tag{25}$$

Next, we define $|0\rangle$ to be the vacuum state with the property that

$$A(\lambda)|0\rangle = e^{-i(L\lambda^2/2)|0\rangle} , \quad D(\lambda)|0\rangle = e^{i(L\lambda^2)|0\rangle} , \tag{26}$$

and

$$C(\lambda)|0\rangle = 0 ,$$

hence excited states are created by repeated application of $B(\lambda_1)$ on $|0\rangle$. Let us consider the one-particle state

$$B(\lambda_1)|0\rangle = 0$$

and consider the effect of operating with $\tau(v)$ on $B(\lambda_1)|0\rangle$. This will be an eigenstate of $\tau(v)$ if we zero the unwanted terms arising out of the operation of commuting A and b with $B(\lambda_1)$ [8]. Finally, it leads to

$$e^{iL\lambda_1^2} = \frac{\sinh(2\lambda_1-\eta)\sinh(\alpha+\lambda_1)}{\sinh(2\lambda_1)\sinh(\alpha-\lambda_1)} . \tag{27}$$

For the two-particle state, we consider $\tau(v)B(\lambda_1)B(\lambda_2)|0\rangle$. The computation is more complicated, but quite straightforward, leading to the following equations for the eigenvalues:

$$e^{iL\lambda_1^2} = \frac{\sinh(2\lambda_1-\eta)\sinh(\alpha+\lambda_1)}{\sinh(\alpha-\lambda_1)} \times \frac{\sinh(\lambda_{12}-\eta)\sinh(\lambda_{12}-\eta)}{\sinh(\lambda_{12}+\eta)\sinh(\lambda_{12}+\eta)} , \tag{28}$$

with a similar expression for λ_2. In general, for the n-particle state $B(\lambda_1)B(\lambda_2)\cdots B(\lambda_n)|0\rangle$ we get

$$e^{i\lambda_n^2 L} = \frac{\sinh(2\lambda_n-\eta)\sinh(\alpha+\lambda_n)}{\sinh(\alpha-\lambda_n)} \prod_{j=1}^{\eta} \frac{N(\lambda_{nj},\bar{\lambda}_{nj})}{D(\lambda_{nj},\bar{\lambda}_{nj})} , \tag{29}$$

where

$$N(\lambda_{nj},\bar{\lambda}_{nj}) = \sinh(\lambda_{nj}-\eta)\sinh(\bar{\lambda}_{nj}-\eta) ,$$

$$D(\lambda_{nj},\bar{\lambda}_{nj}) = \sinh(\bar{\lambda}_{nj}+\eta)\sinh(\lambda_{nj}+\eta) ,$$

$$\lambda_{nj} = \lambda_n - \lambda_j, \quad \bar{\lambda}_{nj} = \lambda_n + \lambda_j .$$

INTEGRAL EQUATION AND FINITE-SIZE EFFECT

In the usual approach of the quantum inverse-scattering method (QISM) one converts Eq. (29) into an integral equation for the density of the eigenvalues λ_i, in the interval $(\lambda_i, \lambda_i + d\lambda_i)$ by letting $L \to \infty$. However, in the present situation since the value of L is not infinite we are to proceed somewhat differently. Several authors have already discussed this question of finite-size correction in the older version of QISM to study the effect of scale and conformal invariance. Taking the logarithim of both sides of Eq. (29) gives

$$iL\lambda_i^2 + \phi(\lambda_i) + \theta_+(\lambda_i) + \sum_{\substack{j=1 \\ (j\neq i)}}^{N} [\theta(\lambda_{ij}) + \theta(\bar{\lambda}_{ij})] = 2\pi\eta_i , \tag{30}$$

where

$$\phi(\lambda_i) = -\ln\sinh(2\lambda_i - \eta) ,$$

$$\theta_+(\lambda_i) = \ln\left[\frac{\sinh(\alpha-\lambda_i)}{\sinh(\alpha+\lambda_i)}\right] , \tag{31}$$

$$\theta(\lambda_{ij}) = -\ln\left[\frac{\sinh(\lambda_{ij}-\eta)}{\sinh(\lambda_{ij}+\eta)}\right] .$$

Now we can rewrite

$$\sum_{\substack{j=1 \\ (j\neq i)}}^{N} \theta(\lambda_{ij})$$

also in the following way:

$$\sum_{\substack{j=1 \\ (j\neq i)}}^{N} \theta(\bar{\lambda}_{ij}) = \sum_{\substack{j=1 \\ (j\neq i)}}^{N} \frac{\sinh(\lambda_i+\lambda_j-\eta)}{\sinh(\lambda_i+\lambda_j+\eta)} = \sum_{j=N}^{0} [\theta(\lambda_{ij}) - \theta(2\lambda_i) - \theta(\lambda_i)] , \tag{32}$$

hence we get

$$iL\lambda_i^2 + \phi(\lambda_i) + \theta_+(\lambda_i) + \sum_{j=-N}^{N} \theta(\lambda_{ij}) - \theta(2\lambda_i) - \theta(\lambda_i) = 2i\pi n_i$$

or

$$\frac{\lambda_i^2}{2\pi} + \frac{1}{2\pi L} - [\phi(\lambda_i)\theta_+(\lambda_i) - \theta(2\lambda_i) - \theta(\lambda_i)] + \frac{1}{2\pi L} \times \sum_{j=-N}^{N} \theta(\lambda_{ij}) = \frac{i}{L} , \tag{33}$$

where

$$\phi(\lambda_i) = -\ln\sinh(2\lambda_i - \eta) ,$$

$$\theta_+(\lambda_i) = \ln\frac{\sinh(\alpha-\lambda_i)}{\sinh(\alpha+\lambda_i)} , \tag{34}$$

$$\theta(\lambda) = \ln\frac{\sinh(\lambda+\eta)}{\sinh(\lambda-\eta)} ,$$

and where $\alpha = iq$ and $\eta = iz$. We now utilize Euler-Maclaurin's sum formulas,

$$\sum_{i=-N}^{N} f(x_i) = \frac{1}{h}\int_a^b f(x)dx + \tfrac{1}{2}[f(a)+f(b)] + \frac{h}{12}[f'(b)-f'(a)] + O(h^3) + \cdots . \tag{35}$$

For convenience, we also define a new variable

$$Z(\lambda_L)=\frac{\lambda_L^2}{2\pi}+\frac{1}{2\pi L}[\phi(\lambda_L)+\theta_+(\lambda_L)-\theta(2\lambda_L)-\theta(\lambda_L)]+\frac{1}{2\pi L}\sum_{j=-N}^{N}\theta(\lambda_L-\lambda_j)\ . \tag{36}$$

Obviously, we have

$$\begin{aligned}&Z(\lambda_L=\lambda_i)=i/L\ ,\\&Z(\lambda_L=+(N+r))=(N+r)/L=Z_m+r/L\ ,\end{aligned} \tag{37}$$

so we get, using Eqs. (33), (35), and (36),

$$\begin{aligned}Z(\lambda_L)=&\frac{\lambda_L^2}{2\pi}+\frac{1}{2\pi L}[\phi(\lambda_L)+\theta_+(\lambda_L)-\theta(2\lambda_L)-\theta(\lambda_L)]+\frac{1}{2\pi}\int_{-(Z_m+r/L)}^{Z_m+r/L}\theta(\lambda_L(z)-\lambda_L(Z))dz\\&+\frac{1}{2L}[\theta(\lambda_L(Z)+\lambda_L(Z_m+r/L))+\theta(\lambda_L(Z)-\lambda_L(Z_m+r/L))]\\&+\frac{\lambda_L'(Z_m+r/L)}{12L^2}[\theta'(\lambda_L(Z)+\lambda_L(Z_m+r/L))-\theta'(\lambda_L(Z)-\lambda_L(Z_m+r/L))]\ .\end{aligned} \tag{38}$$

Expanding λ_L around $\pm Z_m$, using

$$\lambda_L(Z)=\lambda_\infty(Z)+\frac{g_1(Z)}{L}+\frac{g_2(Z)}{L^2}+\cdots\ , \tag{39}$$

and expanding $Z(\lambda_L)$ about λ_∞ yields

$$\frac{\lambda_\infty^2(Z)}{2\pi}+\frac{1}{2\pi}\int_{-Z_m}^{Z_m}\theta(\lambda_\infty(Z)-\lambda_\infty(\bar Z))d\bar Z=Z(\lambda_\infty)\ . \tag{40}$$

So, differentiating with respect to λ_∞,

$$\frac{\lambda_\infty}{\pi}+\frac{1}{2\pi}\int_{-Z_m}^{Z_m}\theta'(\lambda_\infty(Z)-\lambda_\infty(\bar Z))d\bar Z=\frac{dZ(\lambda_\infty)}{d\lambda_\infty}\ , \tag{41}$$

and setting

$$\frac{dZ(\lambda_\alpha)}{d\lambda_\infty}=\rho_\infty\ ,$$

we obtain

$$\rho_\infty(\lambda)-\frac{1}{2\pi}\int_{-q}^{q}\theta(\chi-\mu)\rho_\infty(\mu)d\mu=\lambda/\pi\ , \tag{42}$$

where $q=\lambda_\infty(Z_m)$ and everywhere we replace λ_∞. We can rewrite Eq. (42) as

$$(1-\hat K/2\pi)\rho_\infty=\lambda_\infty/2\pi \tag{43}$$

with $\hat K$ standing for the integral operator. The use of the expansion (39) leads also to the following equations for g_1, g_2, etc.:

$$\begin{aligned}g_1\rho_\infty=&\frac{1}{2\pi}\int_{-q}^{q}[\phi(\lambda)+\theta_+(\lambda)-\theta(2\lambda)-\theta(\lambda)]\\&+\frac{1}{2\pi}\int_{-q}^{q}\theta'(\lambda-\mu)\rho_\infty g_1 d\mu\\&-\frac{1+r}{4\pi}[\theta(\lambda+q)+\theta(\lambda-q)]\end{aligned} \tag{44}$$

or

$$\begin{aligned}(1-\hat K/2\pi)g_1\rho_\infty=&\frac{1}{2\pi}[\phi(\lambda)+\theta_+(\lambda)-\theta(2\lambda)-\theta(\lambda)]\\&-\frac{1+r}{4\pi}[\theta(\lambda+q)+\theta(\lambda-q)]\ .\end{aligned} \tag{45}$$

For g_2,

$$\begin{aligned}\rho_\infty g_2=&\tfrac{1}{2}(\rho_\infty g_1^2)+\{\tfrac{1}{2}\}[M(\lambda,q)-M(\lambda,-q)]\\&\times\left[\rho_\infty g_1^2(q)+(2r+1)g_1(q)+\frac{r^2+r+\frac{1}{2}}{\rho_\infty(q)}\right]\ ,\end{aligned} \tag{46}$$

where $1+M$ is the inverse of the integral operator $(1-\hat K/2\pi)$.

CONFORMAL INVARIANCE AND FINITE-SIZE CORRECTION TO ENERGY

Bearing in mind the information obtained earlier, we now proceed to analyze the scaling behavior of our model. It is well known that a physical system near a critical point (or near $m=0$, in the sense of quantum field theory) shows such scaling behavior. This property can be exhibited by studying the behavior of the energy and momentum associated with the excitation spectrum of our system. Mathematically speaking,

$$\begin{aligned}&E_i\approx E_0+(2\pi/L)d_i\quad\text{(energy)}\ ,\\&\rho_i\approx\rho_0+(2\pi/L)s_i\quad\text{(momentum)}\ ,\end{aligned} \tag{47}$$

L being the dimension of the system and $d_i=\Delta+\bar\Delta$ is the scaling dimension and $s_i=\Delta-\bar\Delta$, the spin of the operator associated with the field operator Ψ_i corresponding to the excitation.

In the following we proceed to compute the scaling dimension by calculating the energy of the excitation spectrum. The energy is given as

$$E=-\tfrac{1}{2}\sum_{j=-(N+\gamma)}^{N+\gamma}\lambda_j^2\ . \tag{48}$$

With the aid of Euler-Maclaurin's sum formulas we rewrite this as

$$E=\frac{L}{2}\int_{-(Z_m+r/L)}^{Z_m+r/L}\lambda_L^2(Z)dZ+\tfrac{1}{2}[\lambda_L(Z_m+r/L)]^2$$
$$=\frac{L}{2}\int_{-(Z_m+r/L)}^{Z_m+r/L}\gamma_L^2(Z)dZ+\tfrac{1}{2}\lambda_L^2(Z_m+r/L)$$
$$=\frac{L}{2}\int_{-q}^{q}[\lambda_\infty+q_1(Z)/L+\cdots]^2dZ+\tfrac{1}{2}\{[\lambda_\infty(Z_m)+g_1(Z_m)/L+\cdots]^2+\frac{2r}{L}[\lambda_\infty+g_1(Z_m)/L\cdots]+\cdots\}$$

which can be written as

$$\tfrac{1}{2}\int_{-q}^{q}\lambda^2\rho(\lambda)d\lambda+\tfrac{1}{2}\int_{-q}^{q}\frac{g_1^2(\lambda)}{L}\rho(\lambda)d(\lambda)d\lambda+\int_{-q}^{q}\lambda g_1(\lambda)\rho(\lambda)d\lambda+\frac{1}{L}\int_{-q}^{q}g_2(\lambda)\rho(\lambda)d\lambda$$
$$+\tfrac{1}{2}\left[q^2+\frac{q_1(Z_m)}{L}\lambda(Z_m)+\cdots\right]+\frac{2r}{L}\frac{d\lambda(Z_m)}{dZ}+\cdots=Le_\infty+f+\frac{\pi v_s}{2L}\{[\epsilon(q)+2(r+1)x_p^{1/2}]^2\}\ , \qquad (49)$$

where the notations are as follows:

$$e_\infty=\tfrac{1}{2}\int_{-\infty}^{\infty}\lambda^2\rho(\lambda)d\lambda\ , \qquad (50a)$$

$$f=\tfrac{1}{2}q^2+\int_{-q}^{q}\lambda[\epsilon(\lambda)-r(\lambda)]d\lambda\ , \qquad (50b)$$

$$\epsilon(\lambda)=\frac{1+\hat{M}}{2\pi}[\phi(\lambda)+\theta_+(\lambda)-\theta(2\lambda)-\theta(\lambda)]\ , \qquad (50c)$$

and in Eq. (49) $(\lambda_\infty,\rho_\infty)$ have been replaced by (λ,ρ).

$$V(\lambda)=\frac{1+\hat{M}}{2\pi}[\theta(\lambda+q)+\theta(\lambda-q)]\ , \qquad (50d)$$

$$v_s=\frac{1}{2\pi\rho(q)}(2q+2\int_{-q}^{q}\lambda M(\lambda q)d\lambda)\ , \qquad (50e)$$

$$x_p=\left[\frac{1+\hat{M}}{2\pi}[\theta(\lambda+q)+\theta(\lambda-q)]\right]^2 . \qquad (50f)$$

In Eq. (49) we have displayed the correction to the excitation energy due to the finite dimensionality of the quantization volume. In the following we solve Eqs. (42), (50), and (49) to obtain explicit expressions for them.

EXPLICIT SOLUTIONS

Let us denote the Fourier transform of a function $F(\alpha)$ as $\bar{F}(y)$,

$$\bar{F}(y)=\int d\alpha\, e^{-i\alpha y}F(\alpha)\ ,$$

hence from Eq. (42) we obtain

$$\bar{\rho}_\infty(y)=\frac{1}{2\pi y[1+\tilde{K}(y)]}\ , \qquad (51)$$

where $\tilde{K}(y)$ is the Fourier transform of the function $\theta'(\lambda)=\coth(\lambda+iZ)-\coth(\lambda-iZ)$. It is not difficult to show that

$$\bar{K}(y)=\int_{-\infty}^{\infty}[\coth(\lambda+iZ)-\coth(\lambda-iZ)]e^{-i\lambda y}dy$$
$$=2\pi i\cosh(\pi y/2)\sinh Z-(4\pi/y)\sinh(yZ)\ . \qquad (52)$$

So, we obtain

$$\bar{\rho}_\infty(y)=2\pi y[1+2\pi i\cosh(\pi y/2)\times\sinh Z-(4\pi/y)\sinh(yZ)]^{-1}\ ,$$

which gives the density of the excitation when L is infinite.

We now recall Eq. (50c), which when Fourier transformed leads to

$$\bar{\epsilon}(y)=\frac{\bar{\theta}(y)+\bar{\theta}_+(y)-\bar{\theta}(2y)-\bar{\theta}(y)}{2\pi[1+\bar{K}(y)]} \qquad (53)$$

with the following expressions occurring in the numerator:

$$\bar{\theta}(y)=\frac{1}{iy}[2\pi i\cosh(\pi y/2)\sinh Z-(4\pi/y)\sinh(yZ)]\ , \qquad (54a)$$

$$\bar{\theta}_+(y)=\frac{2\pi}{y}\frac{\sinh(\pi-2q)y/2}{\sinh(\pi y/2)}\ , \qquad (54b)$$

$$\bar{\phi}(y)=\frac{1}{iy}\left[-i\pi\cosh(y\pi/4)\sinh Z-(4\pi/y)\frac{\sinh(yZ/2)+\sinh[(\pi+Z)/2]y}{\cos(\pi y/2)-1}\right]\ , \qquad (54c)$$

and finally x_p can be obtained from Eq. (50f), again by a Fourier transform. For this purpose note that Eq. (50f) actually can be written as

$$(1-\hat{K}/2\pi)x_p^{1/2}=\frac{1}{2\pi}[\theta(\lambda+q)+\theta(\lambda-q)]\ . \qquad (54d)$$

However, the Fourier transform of $\theta(\lambda+q)+\theta(\lambda-q)$ is given as

$$\frac{2}{iy}\cosh(qy)[2\pi i\cosh(\pi y/2)\sinh Z-(4\pi/y)\sinh(yZ)]\ ,$$

so that we get immediately

$$x_p^{1/2}(y)-1+\frac{\cosh(qy)N(y)}{2\pi^2 iy[1+\bar{K}(y)]}\ ,$$

where

$$N(y)=2\pi i\cosh(\pi y/2)\sinh Z-(4\pi/y)\sinh yZ \qquad (55)$$

so finally we have obtained an explicit expression of all the functions occurring in the expression for the correction of $O(1/L)$ to the excitation energy. It may be noted that though some of the inverse Fourier transforms may be computed analytically, yet their expressions are quite complicated and it is wiser to treat them numerically. Expression (49) gives the value of the scaling dimension $\Delta+\bar{\Delta}$ explicitly, when combined with Eq. (47).

DISCUSSION

In our above computation we have shown how it is possible to develop the quantum inverse problem for nonlinear Alfvén waves in a finite region with boundary conditions other than the periodic one, and subsequently how the finite-size correction to the excitation energy can be explicitly evaluated. Incidentally, it can be mentioned that many of the previous attempts to calculate this correction relied mainly on the energy spectrum calculated on the basis of quantum inverse method formulated in the $L \rightarrow \infty$ limit. That method used the standard commutation rules of the scattering data rather than the modified rules due to Sklyanin. On the other hand, it seems more appropriate to keep L finite from the very beginning, which is possible only in Sklyanin's approach, and which was adopted in this formulation.

ACKNOWLEDGMENTS

One of the authors (N.D.) is grateful to Council for Scientific and Industrial Research (Government of India) for support which made this work possible.

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