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Reinvestigation of the thermodynamics of blackbody radiation via classical physics

Daniel C. Cole
IBM Corporation, Essex Junction, Vermont 05452
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A key part of the early thermodynamic work on blackbody radiation by Wien, Stéfan, Boltzmann, Planck, and others involved the thermodynamic behavior of a movable piston sliding in a cylinder containing classical electromagnetic thermal radiation. This early work used only classical physics concepts. Here, this analysis is reinvestigated with the change that the implicit assumption is *not* made that the thermal radiation spectrum reduces to zero at the temperature $T=0$. Previous work has shown that this consideration may be an important one to yield agreement, or at least better agreement, between classical physical theory and the actual physical behavior of molecular, atomic, and subatomic systems in nature. Indeed, the present analysis on "cavity thermodynamics" accounts for the thermodynamic behavior of Casimir forces between the walls of the cavity. Using only the traditional thermodynamic definition of $T=0$, the form of the classical electromagnetic zero-point (ZP) radiation spectrum is deduced. From the second law of thermodynamics, two forms of Wien's displacement law are obtained and generalized to include the possibility of ZP radiation. The entropy is then explicitly calculated for the parallel-plate case. Also, the limiting situation of high-temperature radiation between the plates and low temperature outside is examined to recover the early analysis by Wien, Planck, and others. Most of the analysis is carried out for two parallel conducting plates bathed in thermal radiation; the Appendix extends this analysis to a rectangular conducting box with a movable interior wall.

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I. INTRODUCTION

Researchers in the second half of the 1800s and in the early 1900s made extensive investigations on the thermodynamics of blackbody radiation via the use of classical physics. Some of the more significant developments during this time period were carried out by Kirchoff, Boltzmann, Stéfan, Wien, and Planck. Reference [1] reviews the historical developments of this subject. Also, this reference describes the transition from classical to quantum concepts that researchers found necessary to adequately describe the physical properties of thermal radiation within a cavity.

Here, we will attack this problem once more, but we will always remain completely within the context of classical physics. The major difference in the analysis presented here and the much earlier work just mentioned is that an implicit assumption is buried in the earlier analysis that is not a necessary one and may even be a fatally flawed one for classical physics. Specifically, the assumption was generally made that at the temperature of $T=0$, all classical electromagnetic radiation within the cavity must vanish. If the world can indeed be described via classical physics, then atoms consisting of negatively charged particles orbiting about more massive, positively charged nuclei must *continually* be radiating and absorbing electromagnetic energy, even at $T=0$. Hence, just from the standpoint of the question of thermal equilibrium between classical charged particles and classical electromagnetic thermal radiation, the concept of nonzero fluctuating radiation at $T=0$ is an important point to include within the thermodynamic analysis of classical systems. This random radiation is usually referred to as classical electromagnetic zero-point (ZP) radiation. If after including this notion of ZP radiation within the analysis we still obtain that classical physics fails to agree with observation, then we can reasonably conclude that classical physics is inadequate for describing the physical world around us. Making this conclusion before accounting for the presence of ZP radiation may very well be in error.

The idea that classical electromagnetic ZP radiation may be an important concept is not a new one. A number of researchers have made advances along these ideas for more than 30 years; this field of study is often referred to as stochastic electrodynamics (SED). Some fairly early, yet very significant papers on these ideas were by Marshall [2,3] and Boyer [4,5]. For reviews and research papers that serve as good introductions to SED, see Refs. [6–11]. (Much earlier references are discussed in Ref. [8].) Also, Ref. [12] presents a good semipopular account of SED, and the introduction to Ref. [13] contains a recent qualitative overview of SED.

The present article analyzes the behavior of certain thermodynamic operations performed upon a cavity containing blackbody radiation. In particular, here the thermodynamic operations are investigated of (i) quasistatically displacing two conducting parallel plates from each other that are bathed in classical electromagnetic thermal radiation, as well as (ii) slowly changing the temperature of the system. The Appendix (see Ref. [16]) indicates how this analysis can be extended to a hollow box with conducting walls, where one of the walls of the box can be displaced. Thus this analysis is closely connected to the early work by researchers around 1900 involving a closed cylinder at some temperature, with a movable piston at one end.

As we will see, by including the possibility that the thermal radiation spectrum is nonzero at $T=0$, the classical electromagnetic ZP spectrum of $\rho(\omega)=\kappa\omega^3/c^2$ can be deduced strictly from the thermodynamic definition of absolute zero temperature. Other points obtained here include (i) thermodynamically accounting for Casimir forces in the analysis, (ii) derivations of two distinct forms of the Wien displacement law when ZP radiation is present, and (iii) a calculation of the entropy associated with classical electromagnetic thermal radiation between two plates, as a function of both temperature and volume. These last properties have not been found before when ZP radiation is taken into account.

The general ideas, analysis, and results described here complement the recent thermodynamic analysis of a different physical system in Refs. [14] and [15]. There, the operation was investigated of quasistatically displacing simple models of atoms consisting of nonrelativistic simple harmonic electric dipole oscillators bathed in thermal radiation. Again from the thermodynamic definition of absolute zero temperature, the thermal radiation spectrum was shown to reduce at $T=0$ to the classical electromagnetic ZP spectral form of $\rho(\omega)=\kappa\omega^3/c^3$, where κ may be nonzero. This spectrum has previously been deduced by others [3,4] from the demand that the spectral form of classical electromagnetic ZP radiation should appear identical in different inertial frames. Complementing this deduction, Ref. [14] contains the first strictly thermodynamic analysis involving both radiation and charges to specifically require this spectrum at $T=0$. Section X discusses these different restrictions in more detail.

References [14] and [15] also investigated several other properties for linear oscillators and free space that will briefly be shown here to also hold for cavity thermodynamics. In particular, (1) the third law of thermodynamics, (2) one form of Wien's displacement law (both forms will be analyzed here), and (3) a required generalization of the Stéfan-Boltzmann law were deduced. Pushing this work further, other properties unique for a cavity will be obtained here.

An advantage of the thermodynamic analysis here on cavity radiation over the analysis in Refs. [14] and [15] on dipole particles, is that the present analysis much more closely parallels the early 1900s work on blackbody radiation. Hence this article pinpoints more clearly what points were missing in this early work to prevent it from being sufficiently general to include the concept of ZP radiation. Another advantage is that the mathematical calculations for parallel plates, or for a rectangular cavity, are much less involved than for electric dipole oscillators. However, as should also be mentioned, an advantage of the simple harmonic oscillator analysis is that the resonant nature of the oscillators enables more restrictive demands to be imposed on the thermal radiation spectrum than in the case of the cavity.

As for the specific content of this article, Sec. II discusses what aspects of the early thermodynamic analysis around 1900 were insufficiently general to account for nonzero random radiation at $T=0$. Section III then discusses the statistical properties of thermal radiation. The remainder of the article largely deals with the first and second laws of thermodynamics as applied to the case of two conducting parallel plates bathed in thermal radiation. The comments in the Appendix [16] on a rectangular conducting box enable a connection to be made to radiation in a finite-sized cavity.

If we consider the volume $\mathcal{V}$ shown in Fig. 1 that encloses part of two or more infinitely extending parallel plates, energy conservation demands that any operation performed on these plates must result in $\Delta\mathcal{U}_{\text{int}}=\mathcal{Q}+\mathcal{W}$, where $\Delta\mathcal{U}_{\text{int}}$ is the change in internal energy within this volume $\mathcal{V}$, $\mathcal{Q}$ is the electromagnetic energy that flows into $\mathcal{V}$, and $\mathcal{W}$ is the work done by external forces. Operations we will consider here are a displacement of the interior plate and (or) a change in temperature of the plates. We will treat $\Delta\mathcal{U}_{\text{int}}$ as being due entirely to the change in the electromagnetic thermal radiation energy within $\mathcal{V}$. A less idealistic and more realistic analysis would also take into account the nonzero specific heat of the plates; here this quantity is ignored, as we will restrict our attention to thermal radiation bounded by approximately perfectly conducting walls.

All of the quantities $\Delta\mathcal{U}_{\text{int}}$, $\mathcal{Q}$, and $\mathcal{W}$ are actually described by probability distributions, since they depend upon the particular realization of the fluctuation thermal radiation. Typical accessible quantities that can be measured in the laboratory deal with the expectation value of these three quantities. The first law of thermodynamics is just the expectation value of this energy conservation law:

$$\langle \Delta\mathcal{U}_{\text{int}} \rangle = \langle \mathcal{Q} \rangle + \langle \mathcal{W} \rangle \ . \tag{1}$$

Here, $\langle \mathcal{Q} \rangle$ equals the amount of *heat* in the form of classical electromagnetic thermal radiation energy that flows into $\mathcal{V}$ during the thermodynamic operation on the plates.

The quantities $\langle \mathcal{W} \rangle$ and $\langle \Delta\mathcal{U}_{\text{int}} \rangle$ are evaluated in Sec. IV for two cases corresponding to the two volumes illustrated in Figs. 1 and 2. The first case deals with the heat flow into the volume $\mathcal{V}$ enclosed by plates A and C in Fig. 1, while the second case involves the region enclosed by plates A and B in Fig. 2. The key difference between

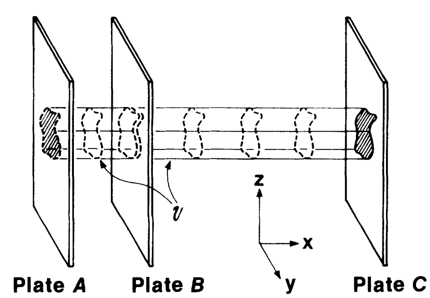

FIG. 1. Three parallel conducting plates that extend infinitely in the y and z directions. Here, the volume $\mathcal{V}$ encloses a part of plate B and has faces at plates A and C that lie in the y-z plane; the remaining surface of $\mathcal{V}$ has its normal component perpendicular to the x direction.

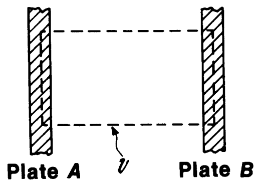

FIG. 2. Here the volume $\mathcal{V}$ is assumed to be of a similar shape to the volume indicated in Fig. 1, but the ends of it now fall halfway between the middle of plates A and B, rather than between the middle of plates A and C.

the two cases is that in case 1 the plate being displaced is entirely enclosed within $\mathcal{V}$, but not in case 2.

To make our calculations tractable, let us assume that the plates are nearly perfect conductors. Let us also assume that the radiation pressure on each side of each plate may be singular, but that the sum of the opposing radiation pressures on any one plate must be nonsingular, as this net pressure must correspond to what is measured in the laboratory for holding the plates apart. Due to the perfect conductor approximation, the thermal radiation fields in the interior of the plates equals zero [17], resulting in thermal radiation impinging on only one side of each plate in the volume $\mathcal{V}$ in Fig. 2. Consequently, upon displacing plate B, the average work $\langle \mathcal{W} \rangle$ required to overcome the force due to the thermal radiation acting upon the left-hand side of plate B in Fig. 2 may be singular, as may be the average change in electromagnetic thermal energy within the volume $\mathcal{V}$ between plates A and B.

Thus, unlike case 1, the quantities $\langle \mathcal{W} \rangle$ and $\langle \Delta\mathcal{U}_{\text{int}} \rangle$ in case 2 do not correspond to physically measurable quantities. Consequently, case 2 may seem rather strange, particularly for researchers closely familiar with Casimir forces [18–22]. Nevertheless, one quantity in case 2 can be extracted that is, in principle, physically measurable: namely, $\langle \mathcal{Q} \rangle$, the flow of heat into the region between the plates. As should be physically demanded, $\langle \mathcal{Q} \rangle$ must be nonsingular for finite temperature changes or for finite displacements of the plates, which requires from Eq. (1) that any singularities in $\langle \Delta\mathcal{U}_{\text{int}} \rangle$ and $\langle \mathcal{W} \rangle$ must cancel.

Besides directly extracting $\langle \mathcal{Q} \rangle$ for a single pair of plates, two other reasons exist for wanting to consider case 2. First, this case involves fewer calculations than the first one for extracting the generalized Wien displacement law. Second, and most important, case 2 corresponds closely to the thermodynamic analysis by Planck and others of displacing a piston in a cylinder [23], since only the radiation pressure on one side of the wall of a plate is considered.

Section V A examines the thermodynamic behavior of the radiation between the plates during quasistatic displacement operations at $T=0$. Here we will deduce that the thermal radiation spectrum must reduce to $\rho(\omega)=\kappa\omega^3/c^3$ at $T=0$ to yield no radiation heat flow. Section V B then finds what κ must be to agree with experiment for the force between conducting parallel plates. Here is where Planck's constant enters the analysis. Section V C then considers the third law of thermodynamics for this system.

In Sec. VI, the demand is made that the second law of thermodynamics must hold. The temperature and frequency form of the generalized Wien displacement law is then deduced. This law is found without the restriction of considering only adiabatic operations, which is a limitation of Wien's original analysis. Section VII then calculates the caloric entropy as a function of temperature and plate separation. Section VIII uses this entropy result to derive the frequency and volume form of the generalized Wien displacement law, which was not treated in Ref. [15]. Section IX then shows that in the regime of high temperature between the plates one can directly recover the early analysis by Wien, Planck, and others. Closing remarks are contained in Sec. X.

II. COMMENTS ON EARLY CLASSICAL ANALYSIS OF BLACKBODY RADIATION

Parts I and II of Planck's treatise in Ref. [23] give a very detailed and elegant presentation on the physical properties of blackbody radiation, all from within the context of classical physics. Although the thermodynamic analysis described in the present article is largely self-contained, still, a close understanding of Planck's carefully described physical reasoning will help to clarify some of the points made here on including ZP radiation in the analysis. In particular, Part II of Planck's treatise on the Stéfan-Boltzmann law and the Wien displacement law is particularly relevant.

In the present section I will briefly mention some restrictions to the reasoning around 1900 that prevents this early analysis from being sufficiently general to include the case of nonzero random radiation being present at $T=0$. These points are made here not to diminish the important advances made by early researchers, but simply to clarify the difference between the work contained here and this much earlier work. We should note in defense of early researchers that the need for including ZP radiation in the thermodynamic reasoning is not at all obvious. Indeed, once one deduces the form for this $T=0$ spectrum, the opposite is more the case, since one's intuitive reaction to the singular energy of ZP radiation is undoubtedly that the concept of ZP radiation is quite unphysical. Not until after Casimir showed in 1948 [18] that changes in this singular energy were associated with nonsingular forces, which were later experimentally observed [24], did ZP energy begin to acquire a real physical acceptability to researchers.

Section IV A in Ref. [15] already pointed out a few ways in which the early 1900s analysis on blackbody radiation was not sufficiently general to include the possibility of ZP radiation [25]. In particular, if the radiation spectrum does not reduce to zero at $T=0$, the expectation value of the total electromagnetic radiation energy at *any* temperature within the cavity can be shown to be singular [26]. Treating this total singular internal energy $U_{\text{int}} \equiv \langle \mathcal{U}_{\text{int}} \rangle$ as being equal to the volume $\mathcal{V}$ of the cavity, times an energy density u that is independent of the size and shape of the cavity [23,25] will not enable one to properly deduce finite changes in the total electromagnetic energy due to changes in the placement of the cavity walls. To accomplish this aim one must take into account changes in the distribution of the modes of the standing waves in the cavity.

To make the early cavity radiation thermodynamics more general, and to also ensure that the net pressure on any plate is finite and, therefore, physically realistic, requires that the radiation outside the cavity must also be included within the analysis. If the radiation does not reduce to zero at $T=0$, then the pressure of the radiation on *both* the inside and outside of the walls of the cavity will be singular [26].

In the work I am familiar with by early researchers, the radiation outside the cavity walls is not considered. For a movable piston in a cylinder, the force on the piston was treated as arising entirely from the radiation within the cylinder. Although I have not seen the assumption explicitly stated, my guess is that these researchers ignore the outside radiation pressure because they are thinking that the outside temperature is either (i) zero, and no radiation is assumed to be present, or (ii) the outside temperature is much lower than the temperature of the cavity radiation, and consequently the effects of the outside radiation are assumed to be negligible. We can certainly understand why these assumptions would be very natural ones to make; however, they are not valid if the radiation spectrum is nonzero at $T=0$.

Of course, we should also expect there should be a regime where these implicit assumptions of (i) or (ii) *appear* to be true; otherwise, major faults with the analysis would have been uncovered long ago. Section IX discusses this point in some detail. For now, a brief and somewhat loose explanation may be sufficient. When the temperature of the radiation outside the cavity is much lower than the temperature inside the cavity, the pressure due to the difference in the "$T=0$ part of the spectrum" inside and outside the walls, is small compared with the pressure due to the high-temperature spectral part from inside the cavity. The net pressure is then approximately due to the total radiation from *inside* the cavity, minus the corresponding $T=0$ spectral part, which is effectively what was done by early researchers.

In contrast, when the above high-temperature condition is not met and when the standing wave modes in the cavity are considered, then significant changes, as addressed here, are required in many of the deductions made by early researchers. For example, in Sec. 66 in Ref. [23], Planck concludes that the radiation pressure for a perfectly reflecting surface should be the same as the radiation pressure on a black surface. Indeed, at the end of this section he says, ". . . it may be stated as a general law that the radiation pressure depends only on the properties of the radiation passing to and fro [within the cavity], not on the properties of the enclosing substance [i.e., the walls of the cavity]."

However, when the normal modes of radiation within a cavity and the different boundary conditions demanded by different materials are taken into account, this statement is no longer true. For example, if two parallel plates are slightly separated from one another, with thermal radiation at temperature T_{in} existing between them and thermal radiation at temperature T_{out} existing on the outside of both plates, then the pressure one must exert on one plate to keep them apart will certainly be of equal magnitude and opposite direction to the required pressure on the other plate. If we change the material of one or both of the plates, this fact must still always be true, which is essentially the reasoning Planck was using. However, in contrast to what he predicted, the magnitude of the pressure and whether the plates are attracted together or pushed apart by thermal radiation, will vary depending on the material of the plates. For example, when the inside and outside temperatures equal zero, the Casimir force is attractive for two conducting plates, yet the force between a perfectly conducting plate and an infinitely permeable plate is predicted by present theory [27] to not only be of a different magnitude, but to also be repulsive.

Finally, before turning to the quantitative analysis, we should mention that the experimental importance of Casimir-like forces in the thermodynamic behavior of cavity radiation is only evident for small cavities, since only here will such forces be experimentally detectable. Planck and others were explicitly *not* attempting to describe the behavior of very small cavities (see Ref. [23], Sec. 2), which is where the effects of changes in the normal modes of the radiation would be most noticeable [28].

III. CHARACTERIZING CLASSICAL ELECTROMAGNETIC THERMAL RADIATION

A. Mathematical description of fields

The following discussion employs the usual method in SED [6,8,29] for describing random radiation in free space without material boundaries present, but differs in its description of the random field between parallel plates [30]. The differences are subtle, but critically important for our analysis. Electromagnetic fields in free space can be expressed as a sum of plane waves over propagation vectors $\mathbf{k}$ and polarization vectors $\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda}$:

$$\mathbf{E}(\mathbf{x},t)=\sum_{\mathbf{k}}\sum_{\lambda=1,2}\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda}[A_{\mathbf{k},\lambda}\cos(\mathbf{k}\cdot\mathbf{x}-\omega t)+B_{\mathbf{k},\lambda}\sin(\mathbf{k}\cdot\mathbf{x}-\omega t)]\ , \quad (2)$$

$$\mathbf{B}(\mathbf{x},t)=\sum_{\mathbf{k}}\sum_{\lambda=1,2}(\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda})[A_{\mathbf{k},\lambda}\cos(\mathbf{k}\cdot\mathbf{x}-\omega t)+B_{\mathbf{k},\lambda}\sin(\mathbf{k}\cdot\omega-\omega t)]\ . \quad (3)$$

As usual, $\omega=c|\mathbf{k}|$, $\mathbf{k}\cdot\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda}=0$, and $\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda}\cdot\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda'}=0$ for $\lambda\neq\lambda'$. Boundary conditions place limits upon the plane waves included in the above sum. If we choose to describe the fields only within a region of space consisting of a rectangular box region, with one corner at the origin and lying in the positive quadrant with sides L_x, L_y, and L_z, then the usual convenient mathematical simplification is to treat the values of the fields as though they were periodically repeated throughout space. Hence $\mathbf{k}=2\pi[\hat{\mathbf{x}}(n_x/L_x)+\hat{\mathbf{y}}(n_y/L_y)+\hat{\mathbf{z}}(n_z/L_z)]$, where n_x, n_y, and n_z vary in integer values from $-\infty$ to ∞. A point that should be noted, since it is often incorrectly stated in the literature, is that this imaginary box has little to do with a conducting box; the boundary conditions for this periodic box structure are quite different than for the conducting box case analyzed in the Appendix [16].

For thermal radiation, we are largely interested in the statistical properties of the fields. We can imagine an ensemble of similar regions of space described by Eqs. (2) and (3), where each region will have a fixed set of definite $A_{\mathbf{k},\lambda}$ and $B_{\mathbf{k},\lambda}$ values that accurately yield the correct field values at each point $\mathbf{x}$ within the region. The ensemble values of $A_{\mathbf{k},\lambda}$ and $B_{\mathbf{k},\lambda}$ then contain the necessary information to characterize the statistical properties of the thermal radiation.

In what follows, a somewhat more convenient quantity than $A_{\mathbf{k},\lambda}$ and $B_{\mathbf{k},\lambda}$ is

$$\mathcal{A}_{\mathbf{k},\lambda}=(A_{\mathbf{k},\lambda}-iB_{\mathbf{k},\lambda}) , \tag{4}$$

so that

$$A_{\mathbf{k},\lambda}\cos(\mathbf{k}\cdot\mathbf{x}-\omega t)+B_{\mathbf{k},\lambda}\sin(\mathbf{k}\cdot\mathbf{x}-\omega t) =\mathrm{Re}(\mathcal{A}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}) . \tag{5}$$

To compare with previous notations in SED, the quantity $\mathcal{A}_{\mathbf{k},\lambda}$ is most closely related to the quantity $a_{\mathbf{n},\lambda}=\exp(i\theta_{\mathbf{n},\lambda})$ [6,8,29], times the square root of the spectrum and a factor proportional to $1/(L_xL_yL_z)^{1/2}$. Presently, however, we will not specify any particular realization of $\mathcal{A}_{\mathbf{k},\lambda}$, such as is often done in SED [8,29] via the introduction of $\exp(i\theta_{\mathbf{n},\lambda})$, where $\theta_{\mathbf{n},\lambda}$ is assumed to be uniformly distributed from 0 to 2π. Instead, we will only specify the statistical properties of $\mathcal{A}_{\mathbf{k},\lambda}$. An important point to note, which is presently not universally recognized by researchers in SED and which will be clarified shortly, is that the statistical properties of $\mathcal{A}_{\mathbf{k},\lambda}$ are different if we deal with the radiation in, for example, an imaginary box in free space, a box with conducting walls, or a region between conducting parallel plates, even though the associated lengths with these regions may all have the same length L.

The conducting plates will be assumed here to be infinite in size, so that we do not need to worry about edge effects. If one plate is present in the y-z plane at $x=0$, then we can describe the electric field in the $x>0$ region as (1) a sum of plane waves traveling in the minus x direction, plus the reflected waves, as well as (2) a sum of plane waves with $k_x=0$:

$$\begin{aligned}\mathbf{E}_T(\mathbf{x},t)=&\sum_{\substack{\mathbf{k},\lambda\\k_x<0}}\{\boldsymbol{\epsilon}_{\mathbf{k},\lambda}\mathrm{Re}(\mathcal{A}_{T;\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)})+[\hat{\mathbf{x}}(\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda})_x-\hat{\mathbf{y}}(\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda})_y-\hat{\mathbf{z}}(\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda})_z]\mathrm{Re}(\mathcal{A}_{T;\mathbf{k},\lambda}e^{i(-k_xx+k_yy+k_zz-\omega t)})\}\\&+\sum_{\substack{\mathbf{k}\\k_x=0}}\hat{\mathbf{x}}\,\mathrm{Re}(\mathcal{A}_{T;\mathbf{k},x}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)})\\=&\sum_{\substack{\mathbf{k},\lambda\\k_x<0}}\mathrm{Re}(\mathcal{A}_{T;\mathbf{k},\lambda}e^{i(k_yy+k_zz-\omega t)}2\{\hat{\mathbf{x}}(\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda})_x\cos(k_xx)+[\hat{\mathbf{y}}(\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda})_y+\hat{\mathbf{z}}(\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda})_z]\,i\sin(k_xx)\})\\&+\sum_{\substack{\mathbf{k}\\k_x=0}}\mathrm{Re}(\hat{\mathbf{x}}\mathcal{A}_{T;\mathbf{k},x}e^{i(k_yy+k_zz-\omega t)}) .\end{aligned}\tag{6}$$

[In Eq. (6), the added subscript T indicates a thermal radiation field at temperature T.]

The sum of each plane wave traveling along $-\hat{\mathbf{x}}$, plus the reflected wave, satisfies the boundary condition that the y and z components of the electric field equal zero in the conducting plane at $x=0$. Since there are no reflected waves associated with the $k_x=0$ waves, the polarization vector of each wave must lie along $\pm\hat{\mathbf{x}}$; we can arbitrarily choose the $+\hat{\mathbf{x}}$ direction, with the as yet unspecified phase residing in the complex amplitude written here as $\mathcal{A}_{T;\mathbf{k},x}$. Unlike the case with random radiation present in space without a reflecting boundary present, the properties of each plane wave specified by k_x, k_y, k_z, and λ are perfectly statistically correlated to the properties of the corresponding wave specified by $-k_x$, k_y, k_z, and λ.

If we place another conducting plane at $x=L_x$, the new boundary condition is that the tangential components of the electric field equal zero at $x=L_x$. Any single plane wave with $k_x\neq0$ will now undergo an infinite number of reflections between the plates over the course of all time. Nevertheless, we can still use the expression in Eq. (6), since the additional boundary condition can be satisfied by restricting the $k_x\neq0$ waves to have k_x values given by

$$k_x=\frac{\pi n_x}{L_x}, \quad \text{where } n_x=(-\infty,\ldots,-2,-1) . \tag{7}$$

However, the physical significance of $\mathcal{A}_{T;\mathbf{k},\lambda}$ for $k_x<0$ has now changed considerably. A plane wave with $k_x\neq0$ plus a plane wave specified by $-k_x$ now mathematically represent a coalescence of waves internally reflected back and forth between the two plates, all of which are perfectly statistically correlated with each other. For this reason we should expect that moments such as

$\langle \mathcal{A}_{T;\mathbf{k},\lambda}\mathcal{A}^*_{T;\mathbf{k},\lambda}\rangle$ may be different for this two plate case versus what it would be if only one or no plates were present, even though we would be considering the same value of $\mathbf{k}$ in each case.

Let us restrict our attention to the electromagnetic field values within a confined region of space between the two plates: namely, between $y=0$ and L_y, and $z=0$ and L_z. Again, a convenient mathematical simplification is to imagine that the values of the fields are periodically repeated throughout space in the y and z directions. At the end of our calculations we will let $L_y \to \infty$ and $L_z \to \infty$. Hence

$$k_y=\frac{2\pi n_y}{L_y},\quad k_z=\frac{2\pi n_z}{L_z}\ ,$$
$$n_y \text{ and } n_z=(-\infty,\ldots,-1,0,1,\ldots,+\infty)\ . \qquad (8)$$

The associated magnetic field can easily be obtained from Eq. (6) by writing $\mathbf{B}$ in a Fourier series and using Faraday's law to require that the Fourier frequency components satisfy

$$\mathbf{B}_\omega=\frac{c}{i\omega}\nabla\times\mathbf{E}_\omega\ . \qquad (9)$$

Hence

$$\mathbf{B}_T(\mathbf{x},t)=\sum_{\substack{\mathbf{k},\lambda\\k_x<0}}\mathrm{Re}(\mathcal{A}_{T;\mathbf{k},\lambda}e^{i(k_y y+k_z z-\omega t)}2\{\hat{\mathbf{x}}(\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda})_x i\sin(k_x x)+[\hat{\mathbf{y}}(\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda})_y+\hat{\mathbf{z}}(\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda})_z]\cos(k_x x)\})$$
$$+\sum_{\substack{\mathbf{k}\\k_x=0}}\mathrm{Re}[(\hat{\mathbf{k}}\times\hat{\mathbf{x}})\mathcal{A}_{T;\mathbf{k},x}e^{i(k_y y+k_z z-\omega t)}]\ . \qquad (10)$$

Turning to the statistical properties of thermal radiation, let us assume that the $A_{T;\mathbf{k},\lambda}$ and $B_{T;\mathbf{k}\lambda}$ coefficients in Eqs. (2) and (3) are statistically independent of each other, and of any other coefficient $A_{T;\mathbf{k}',\lambda'}$ and $B_{T;\mathbf{k}',\lambda'}$ for which $\mathbf{k}\neq\mathbf{k}'$ and (or) $\lambda\neq\lambda'$. Also, let us assume that the ensemble average, or expectation value, of each coefficient equals zero, and that for $k_x\neq0$ the second moment is given by

$$\langle (A_{T;\mathbf{k},\lambda})^2\rangle=\langle (B_{T;\mathbf{k},\lambda})^2\rangle=\frac{[\Omega_T(\omega)]^2}{L_xL_yL_z}\ . \qquad (11)$$

When we are discussing the different physical situation of nonreflecting waves, or $k_x=0$, let us assume that the same form of Eq. (11) holds, but with $\Omega_T(\omega)$ replaced by $\Omega_T'(\omega)$, where $\Omega_T'(\omega)$ may be different from $\Omega_T(\omega)$. We could of course generalize the above assumption by simply stating that Ω_T^2 in Eq. (11) depends on $\mathbf{k}$ rather than only on ω, thereby covering both the $k_x=0$ and the $k_x\neq0$ cases just mentioned; however, as we will see, this more general assumption does not appear to be needed to obtain a direct connection with quantum theory and with present experimental results.

The assumption of statistical independence of the $A_{T;\mathbf{k},\lambda}$ and $B_{T;\mathbf{k},\lambda}$ Fourier coefficients, and the demand that the thermal electromagnetic fields follow stationary stochastic processes, yields that $A_{T;\mathbf{k},\lambda}$ and $B_{T;\mathbf{k},\lambda}$ are described by Gaussian distributions [31]. Hence we only need to specify the first and second moments of these coefficients.

From above, we can find all the statistical properties of $\mathcal{A}_{T;\mathbf{k},\lambda}$. In particular,

$$\langle \mathcal{A}_{T;\mathbf{k}_1,\lambda_1}\mathcal{A}_{T;\mathbf{k}_2,\lambda_2}\rangle=\langle \mathcal{A}^*_{T;\mathbf{k}_1,\lambda_1}\mathcal{A}^*_{T;\mathbf{k}_2,\lambda_2}\rangle=0\ , \qquad (12)$$

$$\langle \mathcal{A}_{T;\mathbf{k}_1,\lambda_1}\mathcal{A}^*_{T;\mathbf{k}_2,\lambda_2}\rangle=\delta_{\mathbf{n}_1,\mathbf{n}_2}\delta_{\lambda_1,\lambda_2}\frac{2[\Omega_T(\omega_{\mathbf{n}_1})]^2}{L_xL_yL_z}\ , \qquad (13)$$

where Eqs. (7) and (8) relate $\mathbf{k}$ and $\mathbf{n}$, and where $\Omega_T'(\omega)$ replaces $\Omega_T(\omega)$ in Eq. (13) when $k_x=0$. The main statistical property needed for calculations in this paper is

$$\langle \mathrm{Re}(\mathcal{A}_{T;\mathbf{k}_1,\lambda_1}F_{\mathbf{k}_1,\lambda_1})\mathrm{Re}(\mathcal{A}_{T;\mathbf{k}_2,\lambda_2}F_{\mathbf{k}_2,\lambda_2})\rangle=\delta_{\mathbf{n}_1,\mathbf{n}_2}\delta_{\lambda_1,\lambda_2}|F_{\mathbf{k}_1,\lambda_1}|^2\frac{[\Omega_T(\omega_{\mathbf{n}_1})]^2}{L_xL_yL_z}\ . \qquad (14)$$

Here, $F_{\mathbf{k},\lambda}$ is any complex, nonrandom variable dependent on $\mathbf{k}$ and λ. The analysis in this article will investigate the functional dependence of $\Omega_T(\omega_\mathbf{n})$ upon frequency and temperature.

B. Physical concepts involved in thermodynamic operations on radiation

Let us now discuss certain useful idealized concepts behind the physical operations available to us for analyzing the thermodynamic properties of the radiation existing between conducting plates. Only reversible thermodynamic operations will be considered here. Let us assume, as in Ref. [15], that the temperature of our system is changed quasistatically by the idealized operation of placing the system in contact with an infinite series of heat reservoirs ranging in temperature from T_{I} to T_{II}. (Section III in Ref. [13] contains a discussion that clarifies some of the implicit physical assumptions involving "incident" radiation and an infinite series of heat reservoirs.)

Immediately we are faced with a difficulty not present in Ref. [15]. Earlier we had made the stipulation that the conducting plates be perfect electrical conductors, thereby simplifying the mathematical description of the radiation fields. Now we are faced with the problem of how heat, in the form of electromagnetic thermal radiation, can flow from the region between one pair of plates to the region between another pair. After all, a perfectly conducting plate is also a perfect reflector for the radiation. Of course, if we only analyze adiabatic operations, then we need not concern ourselves with this complication. Otherwise, if we want to consider more general thermodynamic operations such as simply raising the temperature of the thermal radiation without displacing the plates, then this mathematically simplifying assumption of perfectly conducting plates is not a very useful one for physical investigations.

Of course, in reality there are no perfect conductors. If very good, but not perfect conducting plates are displaced, then after a long enough period of time the radiation between one pair of plates will certainly come to equilibrium with the radiation between another pair of plates. Several ways exist for modeling this effect while still retaining our perfect conductor description. One approach is to assume that a small pinhole, or perhaps more appropriately, an infinite set of pinholes to cover the infinite plates, exists in each plate to allow radiation to leak from one pair of plates to the other. If the holes are very far apart in comparison to the distance between plates A and B, then the force per unit area between plates A and B will be largely unaffected. Adiabatic operations can still be accomplished in the following way: during quasistatic displacements of the plates, the temperature of the heat reservoirs in contact with the plates should be slowly changed so that no net heat flows through the pinholes.

An alternative hypothetical, but useful approach, is to imagine that a stopcock, or a set of stopcocks, exist within each plate that can be opened and closed after each infinitesimal displacement of one of the plates. Whether we use this concept, or the pinholes, or some other idea, the main point that should be kept in mind is that our perfect conductor description is meant to only simplify the mathematics. What we really want to discuss are very good, but not perfect conductors, which allow heat to leak if we wait long enough. Indeed, we can expect that the net pressure needed to hold two perfect parallel plate conductors apart is approximately the same as the pressure on two good, but not perfect conductors, provided the skin depth of the not perfect conductors is small compared with the distance between plates.

After each small change performed upon the plates, such as (1) a small displacement, (2) the opening of a stopcock, or (3) an infinitesimal change in temperature of a heat reservoir, we must then wait sufficiently long for the radiation to achieve an approximately stationary state. To ensure that this stationary state corresponds to blackbody thermal radiation, let us assume that near each pinhole or stopcock there exists a very small particle of carbon between the plates, so as to scatter the radiation in all directions. Alternatively, we can imagine that very small patches of the plate walls have a roughness to them, so as to diffusively scatter the radiation [32]. Of course, we must also assume that the carbon particle, or the rough patches, are sufficiently small that they insignificantly influence the measured force on the plates.

Roughly speaking, the radiation will then reflect back and forth off the walls of the plates. In reality, the good, but not perfect conductors, will act to enhance the standing wave modes, but weaken the amplitude of the waves that do not correspond to standing waves. Our analysis of the electromagnetic energy between plates will only deal with the standing waves. More specifically, after each infinitesimal operation has been executed, we assume that the radiation settles into an equilibrium state that is closely represented by the standing waves obtained in the preceding section. To obtain the probability distribution of the amplitudes of these waves, we should imagine an ensemble of such observations of the standing waves, as described in Sec. III A. The average values from this ensemble will then be assumed to equal the corresponding time average, which is the important entity in experimental measurements of quantities like the measured force between the plates. The variances of $A_{T;\mathbf{k},\lambda}$ and $B_{T;\mathbf{k},\lambda}$ are assumed to depend upon the temperature T and the distance L_x between the plates, as given in Eq. (11), where L_x appears in the denominator and in $\omega=|\mathbf{k}|c$ via Eq. (7).

IV. CALCULATION OF $\langle \mathcal{W} \rangle$ AND $\langle \mathcal{U}_{\text{int}} \rangle$

A. Average work and force

Let $\mathbf{F}_{\text{ext},B}(t)$ represent either (1) the externally applied force at time t necessary to hold fixed, with respect to plate A, the region of plate B that is enclosed by volume $\mathcal{V}$ in Fig. 1, or (2) let $\mathbf{F}_{\text{ext},B}(t)$ represent the same quantity for the region of plate B enclosed by $\mathcal{V}$ in Fig. 2. The expectation value of the work by this force during a quasistatic displacement of plate B can be approximated by

$$\langle \mathcal{W} \rangle \equiv \left\langle \int_{t_{\text{I}}}^{t_{\text{II}}} dt\, \dot{\mathbf{Z}}(t)\cdot\mathbf{F}_{\text{ext},B}(t) \right\rangle \approx \int_{t_{\text{I}}}^{t_{\text{II}}} dt\, \dot{\mathbf{Z}}(t)\cdot\langle \mathbf{F}_{\text{ext},B}(t)\rangle \; . \tag{15}$$

Here, we have assumed that $|\dot{\mathbf{Z}}|$, which represents the speed at which plate B moves from plate A, varies very slowly in time in accordance with our quasistatic displacement assumption. In comparison, the fluctuations of $\mathbf{F}_{\text{ext},B}$ are very rapid, since we will make the simplifying assumption, for calculational purposes, that $\mathbf{F}_{\text{ext},B}$ precisely balances the rapidly fluctuating electromagnetic thermal fields. Hence the approximation in Eq. (15) should be excellent.

Focusing attention now on $\langle \mathbf{F}_{\text{ext},B}\rangle$, clearly we expect that

$$\langle \mathbf{F}_{\text{ext},B}(t)\rangle = -\langle \mathbf{F}_{\text{Lor},B}(t)\rangle \; , \tag{16}$$

where $\mathbf{F}_{\text{Lor},B}$ represents the electromagnetic Lorentz force due to the thermal radiation acting on the indicated region of plate B. Specifically,

$$\mathbf{F}_{\text{Lor},B}(t) = \int_{\mathcal{V}'} d^3x \left[\rho_B(\mathbf{x},t)\mathbf{E}_T(\mathbf{x},t) + \frac{1}{c}\mathbf{J}_B(\mathbf{x},t)\times\mathbf{B}_B(\mathbf{x},t) \right] , \tag{17}$$

where $\mathcal{V}'$ is the region of plate B enclosed by volume $\mathcal{V}$ in either Fig. 1 or 2, and ρ_B and $\mathbf{J}_B$ are the charge and current densities on plate B. Due to our perfect conductor assumption, ρ_B and $\mathbf{J}_B$ are nonzero only on the plate surfaces.

From the electromagnetic momentum conservation theorem [33],

$$\langle \mathbf{F}_{\mathrm{ext},B}(t)\rangle = \int_{\mathcal{V}} d^3x \left[\frac{1}{c^2}\frac{\partial}{\partial t}\langle \mathbf{S}_T(\mathbf{x},t)\rangle - \nabla\cdot\langle \overleftrightarrow{\mathbf{T}}_T(\mathbf{x},t)\rangle \right] , \tag{18}$$

where $\mathbf{S}=c/4\pi(\mathbf{E}_T\times\mathbf{B}_T)$ is the Poynting vector and

$$T_{T,ij}=\frac{1}{8\pi}[2(E_{T,i}E_{T,j}+B_{T,i}B_{T,j})-\delta_{i,j}(\mathbf{E}_T^2+\mathbf{B}_T^2)] \tag{19}$$

is the Maxwell stress tensor associated with the thermal radiation field. In Eq. (18) we can assume that $(\partial/\partial t)\langle \mathbf{S}_T(\mathbf{x},t)\rangle$ makes a negligible contribution. If plate B was not displaced from plate A, but held fixed, then this term would identically equal zero, since $\mathbf{S}_T(\mathbf{x},t)$ would be described by a stationary stochastic process in time. Due to our quasistatic approximation, we assume that the term above can be made as small as desired by making $|\dot{\mathbf{Z}}|$ arbitrarily small.

Symmetry demands that the tangential components of $\langle \mathbf{F}_{\mathrm{ext},B}\rangle$ along the plates must equal zero. As for the normal components,

$$\langle \mathbf{F}_{\mathrm{ext},B2}(t)\rangle_x = F_{\ell} = +\int_{\mathcal{S}_{B\ell}} d^2x \frac{1}{8\pi}\langle E_{T,x}^2-E_{T,y}^2-E_{T,z}^2+B_{T,x}^2-B_{T,y}^2-B_{T,z}^2\rangle , \tag{20}$$

$$\langle \mathbf{F}_{\mathrm{ext},B1}(t)\rangle_x = F_{\ell}+F_{r} , \tag{21}$$

$$F_{r} = -\int_{\mathcal{S}_{Br}} d^2x \frac{1}{8\pi}\langle E_{T,x}^2-E_{T,y}^2-E_{T,z}^2+B_{T,x}^2-B_{T,y}^2-B_{T,z}^2\rangle . \tag{22}$$

Equation (20) denotes the required "externally" applied force in Fig. 2 to balance the electromagnetic Lorentz force due to the thermal radiation acting on the left-hand side of plate B. Here, $\mathcal{S}_{B\ell}$ is the left surface of plate B within $\mathcal{V}$. Similarly, Eq. (21) denotes the externally applied force in Fig. 1, which in this case is the force that would be measured in the laboratory. This force equals the sum of the forces F_{ℓ} and F_{r} needed to compensate the thermal radiation pressure on the left and right surfaces, $\mathcal{S}_{B\ell}$ and $\mathcal{S}_{Br}$, within $\mathcal{V}$.

B. Expectation value of force

We need the second moment of the field components to evaluate the expectation value of the forces in Eqs. (20) and (21), as well as to later evaluate the expectation value of the electromagnetic energy. Using the following property satisfied by the polarization vectors:

$$\sum_{\lambda=1}^{2}(\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda})_i(\hat{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda})_j=\sum_{\lambda=1}^{2}(\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}})_i(\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}})_j=\delta_{ij}-\frac{k_ik_j}{k^2} , \tag{23}$$

the second moment of the field components can be calculated:

$$\langle (E_{T,x})^2\rangle = \sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x<0}} 4\left[1-\left(\frac{k_x}{k}\right)^2\right] \times\cos^2\left(\frac{\pi n_x x}{L_x}\right)\frac{[\Omega_T(\omega_{\mathbf{n}})]^2}{L_xL_yL_z} + \sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x=0}} \frac{[\Omega_T'(\omega_{\mathbf{n}})]^2}{L_xL_yL_z} , \tag{24}$$

$$\langle (E_{T,y})^2\rangle = \sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x<0}} 4\left[1-\left(\frac{k_y}{k}\right)^2\right] \times\sin^2\left(\frac{\pi n_x x}{L_x}\right)\frac{[\Omega_T(\omega_{\mathbf{n}})]^2}{L_xL_yL_z} , \tag{25}$$

$$\langle (E_{T,z})^2\rangle = \sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x<0}} 4\left[1-\left(\frac{k_z}{k}\right)^2\right] \times\sin^2\left(\frac{\pi n_x x}{L_x}\right)\frac{[\Omega_T(\omega_{\mathbf{n}})]^2}{L_xL_yL_z} , \tag{26}$$

$$\langle (B_{T,x})^2\rangle = \sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x<0}} 4\left[1-\left(\frac{k_x}{k}\right)^2\right] \times\sin^2\left(\frac{\pi n_x x}{L_x}\right)\frac{[\Omega_T(\omega_{\mathbf{n}})]^2}{L_xL_yL_z} , \tag{27}$$

$$\langle (B_{T,y})^2\rangle = \sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x<0}} 4\left[1-\left(\frac{k_y}{k}\right)^2\right] \times\cos^2\left(\frac{\pi n_x x}{L_x}\right)\frac{[\Omega_T(\omega_{\mathbf{n}})]^2}{L_xL_yL_z} + \sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x=0}} \left(\frac{k_z}{k}\right)^2\frac{[\Omega_T'(\omega_{\mathbf{n}})]^2}{L_xL_yL_z} , \tag{28}$$

$$\langle (B_{T,z})^2\rangle = \sum_{\substack{-\infty<n_y,n_z<\infty \\ n_x<0}} 4\left[1-\left[\frac{k_z}{k}\right]^2\right] \times \cos^2\left[\frac{\pi n_x x}{L_x}\right]\frac{[\Omega_T(\omega_{\mathbf{n}})]^2}{L_xL_yL_z} + \sum_{\substack{-\infty<n_y,n_z<\infty \\ n_x=0}} \left[\frac{k_y}{k}\right]^2\frac{[\Omega_T'(\omega_{\mathbf{n}})]^2}{L_xL_yL_z} . \quad (29)$$

We can now calculate Eq. (20). From above

$$\langle E_{T,x}^2-E_{T,y}^2-E_{T,z}^2+B_{T,x}^2-B_{T,y}^2-B_{T,z}^2\rangle = -\sum_{\substack{-\infty<n_y,n_z<\infty \\ n_x<0}} 8\left[\frac{k_x}{k}\right]^2\frac{[\Omega_T(\omega_{\mathbf{n}})]^2}{L_{AB}L_yL_z} , \quad (30)$$

where L_{AB} is the perpendicular distance between plates A and B, and L_y and L_z are described prior to Eq. (8). Two interesting points should be noted. First, the spatial dependence on x in Eqs. (24)–(29) drops out upon summing the above field moments. Second, the contribution of the sum of plane waves with $k_x=0$ in Eqs. (6) and (10) completely drops out in calculating the x-component force. This result corresponds to our natural expectation, since, roughly speaking, these waves do not "reflect" off the conducting planes.

We obtain,

$$\langle \mathbf{F}_{\text{ext},B2}(t)\rangle_x = -\frac{\mathcal{A}}{\pi L_{AB}L_yL_z} \times \sum_{\substack{-\infty<n_y,n_z<\infty \\ n_x<0}} \left[\frac{k_x}{k}\right]^2[\Omega_T(\omega_{\mathbf{n}})]^2 , \quad (31)$$

where $\mathcal{A}$ is the area of the end face of $\mathcal{V}$ at plate B. The force is negative, which means that an "external" force must be applied in the negative x direction to compensate the pressure of the thermal radiation between plates A and B acting to push these two plates apart.

Equation (31) is closely related to the Maxwell radiation pressure formula $p=u/3$ used by Boltzmann, Planck, and others in analyzing the thermodynamics of cavity radiation [34]. Here, p is the pressure and u is the expectation value of the electromagnetic energy density. Equation (31) reduces to this case if we make the assumption that the sum over normal modes of $(k_x/k)^2$ in Eq. (31) is approximately equal to the same sum, but with k_x replaced by either k_y or k_z. Under that isotropic assumption,

$$\frac{1}{\mathcal{A}}\langle \mathbf{F}_{\text{ext},B2}(t)\rangle_x \approx -\frac{1}{3}\left[\frac{1}{\pi L_{AB}L_yL_z} \times \sum_{\substack{-\infty<n_y,n_z<\infty \\ n_x<0}} [\Omega_T(\omega_{\mathbf{n}})]^2\right] . \quad (32)$$

Provided we ignore the $n_x=0$ contribution in u, the quantity in large parentheses equals u, thereby connecting with Boltzmann and Planck. [Divide Eq. (37) below by $\mathcal{A}L_{AB}$ to obtain u.] However, as we now know from Casimir forces, the above approximation is not necessarily correct. In particular, for a spectrum like zero-point plus Planckian (ZPP) or Rayleigh-Jeans (RJ) radiation [14], the above approximation is not valid. Instead, Eq. (31) is a singular expression and the distribution over normal modes must be carefully taken into account when making a connection with other quantities. Also, the isotropic assumption is undoubtedly not valid for small cavities; rather, the distribution of the normal modes will depend upon the detailed size and shape of the cavity [28].

In Eq. (31),

$$\omega_{\mathbf{n}} = c\left[\left[\frac{\pi n_x}{L_{AB}}\right]^2+\left[\frac{2\pi n_y}{L_y}\right]^2+\left[\frac{2\pi n_z}{L_Z}\right]^2\right]^{1/2} , \quad (33)$$

so that

$$\langle \mathbf{F}_{\text{ext},B2}(t)\rangle_x = +\frac{\mathcal{A}}{\pi L_yL_z}\sum_{\substack{-\infty<n_y,n_z<\infty \\ n_x>0}} \frac{1}{\omega_{AB}}\frac{\partial\omega_{AB}}{\partial L_{AB}} \times [\Omega_T(\omega_{AB})]^2 , \quad (34)$$

where the new label AB has been added to ω and the subscript $\mathbf{n}$ is now suppressed. Since only $(n_x)^2$ appears in Eq. (34) via Eq. (33), the sum over $n_x<0$ was changed to include only positive values of n_x.

If we now turn to the case of Fig. 1, which involves the physically measurable external force needed to hold the region $\mathcal{A}$ of plate B in place, then Eqs. (21) and (22) yield

$$\langle F_{\text{ext},B1}(t)\rangle_x = +\frac{\mathcal{A}}{\pi L_yL_z}\sum_{\substack{-\infty<n_y,n_z<\infty \\ n_x>0}} \left[\frac{1}{\omega_{AB}}\frac{\partial\omega_{AB}}{\partial L_{AB}}[\Omega_T(\omega_{AB})]^2-\frac{1}{\omega_{BC}}\frac{\partial\omega_{BC}}{\partial L_{BC}}[\Omega_T(\omega_{BC})]^2\right] . \quad (35)$$

C. Expectation value of classical electromagnetic thermal energy

The expectation value of the thermal radiation within a volume $\mathcal{V}$ is given by

$$\langle \mathcal{U}_{\text{int}}\rangle = \frac{1}{8\pi}\int_{\mathcal{V}} d^3x\,\langle \mathbf{E}_T^2+\mathbf{B}_T^2\rangle . \quad (36)$$

Changes in this quantity represent the thermodynamic quantity of interest. Applying this formula to the region between plates A and B yields

$$\langle \mathcal{U}_{\text{int},AB}\rangle = \frac{\mathcal{A}}{\pi L_y L_z}\sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x>0}}[\Omega_T(\omega_{AB})]^2 + \frac{\mathcal{A}}{4\pi L_y L_z}\sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x=0}}[\Omega'_T(\omega_{AB})]^2 . \quad (37)$$

The second term on the right-hand side of Eq. (37) is independent of the distance between the plates.

As a side note, when we choose $\mathcal{A}=L_yL_z$, we can immediately see that this expression will agree with the familiar quantum theory (QT) prediction at $T=0$ of

$$\langle \mathcal{U}_{\text{int},AB}\rangle_{\text{QT}}\Big|_{T=0} = \sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x\geq 0}}\sum_{\lambda}\frac{\hbar\omega_{\text{n}}}{2} , \quad (38)$$

when

$$[\Omega_{T=0}(\omega)]^2=\pi\hbar\omega , \quad (39)$$

$$[\Omega'_{T=0}(\omega)]^2=2\pi\hbar\omega . \quad (40)$$

In Eq. (38), the sum is over all normal modes, where λ takes on two values when $n_x\neq 0$ and one value when $n_x=0$. We will indeed obtain that Eq. (39) must hold; however, we will deduce this result in a more solid way than from just the above simple comparison between energy expressions, since we will make use of the fundamental thermodynamic definition of $T=0$ and with experimental measurements.

The expression in Eq. (37) may be singular, as it is for RJ, ZP, and ZPP radiation. Even upon computing a change in this energy due to a displacement of plate B, the result may still be singular. However, upon adding in the energy between plates B and C,

$$\langle \mathcal{U}_{\text{int},BC}\rangle = \frac{\mathcal{A}}{\pi L_y L_z}\sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x>0}}[\Omega_T(\omega_{BC})]^2 + \frac{\mathcal{A}}{4\pi L_y L_z}\sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x=0}}[\Omega'_T(\omega_{BC})]^2 , \quad (41)$$

and upon computing a change in the total energy due to $L_{AB}\rightarrow L_{AB}+dL_{AB}$ and $L_{BC}\rightarrow L_C-dL_{AB}$, then a finite change is obtained for RJ and ZPP radiation [35]. The singular gain in energy between two of the plates is compensated by the singular loss in energy between the other plates, to yield finite change in energy. This result corresponds to the physical demand that the work done as well as the heat transfer during an isothermal displacement of plate B should be finite quantities.

Of course, including plate C in the calculation is clearly very artificial, since experimental setups do not measurably depend upon the presence of some plate C at a large distance from plate B. We include plate C here to simplify the mathematics, while still including the physical concept that displacing plate B from plate A will change the electromagnetic energy between these plates as well as between plate B and the rest of the universe. Here, the rest of the universe will be assumed to be far removed from plate B in comparison with L_{AB}. At the end of our calculations, we will let $(L_{BC}/L_{AB})\rightarrow\infty$. Thus plate C is simply a mathematical aid in modeling this physical concept. An unproven assumption here due to the difficulty of the mathematics, but one that seems physically reasonable, is that the same result occurs if we replace plate C by other objects taken at a large distance from plate B [36].

Finally, before proceeding with our thermodynamic analysis, let us relate $[\Omega_T(\omega)]^2$ to the function $[h_{\text{in}}(\omega,T)]^2$ and the continuous spectral energy density $\rho_{\text{in}}(\omega,T)$ in Refs. [14] and [15] [see Eqs. (15) and (16) in Ref. [15]]. Even though the discrete normal mode distribution of the radiation between two plates cannot in general be ignored when calculating quantities like the force between plates, still we can make a connection between the discrete and continuous distributions, which will enable us to compare the results obtained here to those in Refs. [14] and [15]. For the region between plates A and B, a continuous approximation yields

$$\frac{1}{8\pi}\langle E_T^2+B_T^2\rangle = \sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x>0}}\frac{1}{\pi}\frac{[\Omega_T(\omega_n)]^2}{L_{AB}L_yL_z} + \sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x=0}}\frac{1}{4\pi}\frac{[\Omega'_T(\omega_{\text{n}})]^2}{L_{AB}L_yL_z}$$

$$\approx\int_0^\infty d\omega\frac{\omega^2[\Omega_T(\omega)]^2}{2\pi^3c^3} . \quad (42)$$

Here, possible differences between the forms of the $n_x=0$ and the $n_x>0$ terms were ignored. We identify

$$\rho_{\text{in}}(\omega,T)\approx\frac{\omega^2[\Omega_T(\omega)]^2}{2\pi^3c^3} , \quad (43)$$

$$h_{\text{in}}(\omega,T)\approx\frac{[\Omega_T(\omega)]^2}{2\pi^3} . \quad (44)$$

V. THE THERMAL RADIATION SPECTRUM AS $T\rightarrow 0$

A. Derivation of the $T=0$ spectrum

A system is defined as being at the temperature of absolute zero when no heat can flow into or out of the system during any reversible isothermal process performed on the system [37]. Hence, let us examine under what conditions no heat will flow from between plates A and B upon a quasistatic displacement of plate B, when the entire system is held at a constant temperature. From Eqs. (1), (15), (34), and (37), for an infinitesimal change in L_{AB},

$$\langle Q \rangle|_T = \frac{\mathcal{A}}{\pi L_y L_z} \sum_{\substack{-\infty < n_y, n_z < \infty \\ n_x > 0}} \frac{\partial \omega_{AB}}{\partial L_{AB}} \left[\frac{\partial \Omega_T^2(\omega_{AB})}{\partial \omega_{AB}} - \frac{\Omega_T^2(\omega_{AB})}{\omega_{AB}} \right] \delta L_{AB} . \tag{45}$$

At $T=0$, $\langle Q \rangle|_T$ must equal zero for any length L_{AB} and any arbitrary displacement δL_{AB}.

This condition is analogous to Eq. (67) in Ref. [14] for simple harmonic dipole oscillators. However, in Ref. [14] the simple harmonic oscillator system enabled a resonance approximation to be applied to the analog of Eq. (45) above, to result in the demand that the quantity in large parentheses must equal zero. Here we do not have a resonance approximation to enforce this demand, which exposes one weakness of the present parallel plate analysis. Instead, here we must introduce the additional assumption that each sum of terms in Eq. (45) that involves a single value of ω_{AB}, rather than the entire sum, must equal zero at $T=0$. An argument to support this contention is that the sum of terms for each ω_{AB} in Eq. (45) should represent the heat flow into the region between plates A and B at frequency ω_{AB}. If we make the demand that not only the total heat flow should equal zero at $T=0$, but also that the heat flow at each frequency should equal zero, then we obtain that for $n_x > 0$,

$$[\Omega_{T=0}(\omega)]^2 = 2\pi^3 \kappa \omega , \tag{46}$$

where κ is an arbitrary constant. The factor of $2\pi^3$ was inserted here to correspond with the constant κ used in Ref. [14] [see Eq. (44) here, and Eq. (68) in Ref. [14]]. We note that, as one might expect, displacement operations on the parallel plates do not let us determine $[\Omega'_{T=0}(\omega)]^2$ for unreflected plane waves with $k_x = 0$.

No radiation being present at $T=0$ is satisfied by choosing $\kappa=0$. However, we again see that null radiation is not the only radiation spectrum that can satisfy the fundamental thermodynamic definition of absolute zero temperature. Several qualitative arguments were given in Ref. [14] that a nonzero choice for κ will yield the closest agreement between classical theory and physical observation. Now we turn to one specific experimental means of deducing the appropriate value of κ.

B. Force on plate at $T=0$

The measured Casimir force between conducting parallel plates fits our present needs. The measurements by Sparnaay in Ref. [24] were done at room temperature for a range of gap separations such that the room-temperature correction to measurements near $T=0$ should be negligible [38]. (For larger gap separations the temperature correction is expected to be important [38].) Hence, let us evaluate this force at $T=0$, since we can then use the $T=0$ spectrum already deduced to make a reasonable connection with Sparnaay's experiment.

The measurable force on plate B is given by Eq. (35). Letting $L = L_{AB} + L_{BC}$, then

$$\frac{d\omega_{BC}}{dL_{BC}} = -\frac{d\omega_{BC}}{dL_{AB}} .$$

Hence

$$\langle \mathbf{F}_{\text{ext},B1}(t) \rangle_x = \frac{2\pi^2 \kappa \mathcal{A}}{L_y L_z} \frac{\partial}{\partial L_{AB}} \left[\sum_{\substack{-\infty < n_y, n_z < \infty \\ n_x > 0}} (\omega_{AB} + \omega_{BC}) \right]$$
$$\approx \frac{c\kappa \mathcal{A}}{2} \frac{\partial}{\partial L_{AB}} \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \sum_{n_x=1}^{\infty} \left\{ \left[\left[\frac{\pi n_x}{L_{AB}} \right]^2 + k_y^2 + k_z^2 \right]^{1/2} + \left[\left[\frac{\pi n_x}{L - L_{AB}} \right]^2 + k_y^2 + k_z^2 \right]^{1/2} \right\} . \tag{47}$$

For nonzero κ the first line in Eq. (47) expresses the result familiar from quantum theory: at $T=0$ the force on one of the plates is obtained from a potential function that is just a constant times the sum of the frequencies of the normal modes of the system. The above sums are divergent. However, a number of techniques exist to regularize the sums and arrive at a finite physical answer [39]. For our analysis, which involves the derivative with respect to L_{AB} in Eq. (47), Fierz's method in Ref. [39] is quite convenient. To begin, in the second line above L_y and L_z were taken to infinity so that a continuous spectral distribution was obtained for k_y and k_z. Following Fierz's method, for $L \gg L_{AB}$, one obtains

$$\langle \mathbf{F}_{\text{ext},B1}(t) \rangle_x = \frac{\mathcal{A} \kappa c \pi^4}{120 L_{AB}^4} . \tag{48}$$

Agreement with experimental measurements of this force as a function of the distance between the plates [24,12], as well as with quantum theory predictions [18], is obtained when

$$\kappa = \frac{\hbar}{2\pi^2} . \tag{49}$$

In this way Planck's constant naturally enters into the thermodynamic analysis of classical electromagnetic random radiation. From Eqs. (46) and (43), this spectrum corresponds to what has been termed the classical electromagnetic ZP radiation spectrum [2–11]. Equations (46) and (49) recover our earlier side note of Eq. (39) that was obtained by simply comparing energy expressions between the quantum and classical situations.

C. Third law of thermodynamics

While discussing the $T \to 0$ limit of classical electromagnetic thermal radiation, we should also mention that the analysis in Ref. [14], Sec. VII, for the applicability of the third law of thermodynamics for displacement operations of classical dipole oscillators, also carries over immediately to conducting parallel plates. The calculations are essentially identical. Thus, for displacement operations of the plates, ZPP radiation satisfies the Nernst-Simon form of the third law, while RJ radiation violates it.

Here we should also mention that the further work in Sec. VIII of Ref. [15] shows that (i) the demand of a finite specific heat for classical electromagnetic thermal radiation in free space imposes an even stronger restriction on the thermal radiation spectrum than (ii) the restriction on the spectrum from the Nernst-Simon form of the third law, as applied to displacement operations on dipole oscillators and conducting parallel plates. To obtain (i), a generalized Wien displacement law was used that was derived by applying the second law of thermodynamics to operations involving dipole oscillators. We now turn to show that a similar result holds for thermodynamic operations on conducting parallel plates.

VI. A CONSEQUENCE OF THE SECOND LAW OF THERMODYNAMICS

From the second law of thermodynamics, $dS_{\text{cal}} \equiv d\langle Q \rangle_R / T$ must be an exact differential. Here, the subscript "cal" stands for "caloric," to distinguish this entropy from one calculated according to alternative probabilistic ideas [40], while "R" indicates a thermodynamic reversible operation. We will express dS_{cal} in terms of two independent thermodynamic coordinates: the temperature T and the distance L_{AB}, where $L = L_{AB} + L_{BC}$ is again held fixed.

The change in caloric entropy for the volume $\mathcal{V}$ in Fig. 1 is then given by

$$\begin{aligned} dS_{\text{cal},\,ABC} &= \frac{1}{T}[d(\langle \mathcal{U}_{\text{int},\,AB}\rangle + \langle \mathcal{U}_{\text{int},BC}\rangle) - dL_{AB}\langle \mathbf{F}_{\text{ext},B1}\rangle_x] \\ &= \frac{1}{T}d(\langle \mathcal{U}_{\text{int},\,AB}\rangle - dL_{AB}F_\ell) + \frac{1}{T}d(\langle \mathcal{U}_{\text{int},BC}\rangle - dL_{AB}F_r) \\ &= dS_{\text{cal},\,AB} + dS_{\text{cal},BC}\ . \end{aligned} \tag{50}$$

Thus $dS_{\text{cal},\,ABC}$ equals the sum of the change in caloric entropy associated with the regions between plates A and B and between plates B and C. Here, F_ℓ and F_r in Eqs. (20)–(22) equal the expectation value of the x component of the "external" forces, or the part of the external forces, needed to oppose the pressure on plate B due to radiation between plates A and B, and between plates B and C, respectively [41].

Concentrating now on the entropy $S_{\text{cal},\,AB}$ associated with the volume $\mathcal{V}$ in Fig. 2 [42],

$$\left.\frac{\partial S_{\text{cal},\,AB}}{\partial T}\right|_{L_{AB}} = \frac{1}{T}\frac{\partial}{\partial T}\langle \mathcal{U}_{\text{int},\,AB}\rangle\bigg|_{L_{AB}}\ , \tag{51}$$

$$\left.\frac{\partial S_{\text{cal},\,AB}}{\partial L_{AB}}\right|_{T} = \frac{1}{T}\left[\left.\frac{\partial\langle \mathcal{U}_{\text{int},\,AB}\rangle}{\partial L_{AB}}\right|_T - F_\ell\right]\ . \tag{52}$$

Equating the two expressions that can be obtained above for $\partial^2 S_{\text{cal,AB}}/\partial L_{AB}\partial T$ yields

$$0 = \frac{\mathcal{A}}{\pi L_y L_z T^2} \sum_{\substack{-\infty < n_y, n_L < \infty \\ n_x > 0}} \left[\frac{\partial \omega_{AB}}{\partial L_{AB}}\right]\left[-\frac{T}{\omega_{AB}}\frac{\partial \Omega_T^2(\omega_{AB})}{\partial T} - \frac{\partial \Omega_T^2(\omega_{AB})}{\partial \omega_{AB}} + \frac{1}{\omega_{AB}}\Omega_T^2(\omega_{AB})\right]\ . \tag{53}$$

This condition must be satisfied for any temperature T and for any length L_{AB}. From Eq. (43), Eq. (53) is very analogous to the condition of Eq. (20) in Ref. [15] for simple harmonic dipole oscillators. Reference [15] used the simple harmonic resonance approximation to demand that the analogous quantity in square brackets in Eq. (53) must equal zero. As with Eq. (45), there is no such resonance condition for parallel plates. We must again make an additional assumption. Physically, this assumption seems reasonable if we consider the heat, and consequently the entropy, associated with each frequency, much as what Planck was describing in Ref. [23], Part II, Chap. IV. Demanding that the second law of thermodynamics then holds for the entropy associated with each frequency of the radiation, would then yield that the quantity in square brackets in Eq. (53) must equal zero.

Making the substitution of $T = \omega/\Theta$, and using Eqs. (23) and (25) in Ref. [15], we obtain

$$\Omega_T^2(\omega) = 2\pi^3 c^3 \omega f_{\text{in}}\left[\frac{\omega}{T}\right]\ , \tag{54}$$

where $f_{\text{in}}(\Theta) \geq 0$ and $f_{\text{in}}(\Theta)$ is a continuous function of Θ. The factor $2\pi^3 c^3$ was included to yield agreement with Eq. (43) here and with Eq. (27) in Ref. [15]. From Eqs. (46) and (49), one restriction on $f_{\text{in}}(\Theta)$ is that

$$\lim_{\Theta\to\infty} f_{\text{in}}(\Theta)=\frac{\hbar}{2\pi^2c^3} \ . \tag{55}$$

Equation (54), or from Eq. (43) in the near continuum case, $\rho_{\text{in}}(\omega,T)\approx\omega^3 f_{\text{in}}(\omega/T)$, represents one form of what is often called Wien's displacement law. However, our result holds even when nonzero radiation is present at $T=0$, or when $\kappa\neq 0$, thereby providing a generalization of this result. Moreover, even without the inclusion of ZP radiation this result is more general in the following sense. Equation (54) describes how $\Omega_T^2(\omega)$ must depend upon ω and T for the second law of thermodynamics to hold during arbitrary reversible changes in temperature and displacement. Similarly, Wien's original result found the corresponding dependence for $\rho_{\text{in}}(\omega,T)$ upon ω and T during reversible changes in temperature and displacement, but where these changes were restricted to adiabatic ones. This restriction was not made here.

In Sec. VIII we will obtain the second form of Wien's displacement law, which involves frequency and volume. We will see, however, that this second form is not as general as the first since it becomes increasingly less accurate for small distances between the plates.

VII. DETERMINATION OF THE ENTROPY FUNCTION

Now let us find the change in entropy between plates A and B when passing from a thermal equilibrium state specified by $(T_{\text{I}},L_{AB}=L_{\text{I}})$ to one specified by $(T_{\text{II}},L_{AB}=L_{\text{II}})$. For convenience, let us drop all "AB" references, so that $S_{\text{cal},AB}\to S_{\text{cal}}$ and $\omega_{AB}\to\omega$. Choosing any reversible path from $(T_{\text{I}},L_{\text{I}})$ to $(T_{\text{II}},L_{\text{II}})$, one obtains

$$S_{\text{cal}}(T_{\text{II}},L_{\text{II}})-S_{\text{cal}}(T_{\text{I}},L_{\text{I}})=\frac{2\pi^2c^3\mathcal{A}}{L_yL_z}\sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x>0}}\int_{\Theta(\mathbf{n},T_{\text{I}},L_{\text{I}})}^{\Theta(\mathbf{n},T_{\text{II}},L_{\text{II}})} d\Theta\frac{\partial f_{\text{in}}(\Theta)}{\partial\Theta}\Theta+\frac{\mathcal{A}}{4\pi L_yL_z}\sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x=0}}\int_{T_{\text{I}}}^{T_{\text{II}}}\frac{dT}{T}\frac{\partial\Omega'^2_T}{\partial T} \ , \tag{56}$$

where $\Theta=\omega/T$ and ω is given by Eq. (33).

The above result is quite general. As required, the entropy difference appears as a function of state. However, without knowing more about $f_{\text{in}}(\Theta)$ and $\Omega_T'^2$, or without making an approximation, particularly in the summation over n_x, we cannot explicitly evaluate the above quantity.

Of course, one simplification is to convert the sums over n_y and n_z to integrals via

$$\frac{1}{L_y}\sum_{n_y}\to\frac{1}{2\pi}\int dk_y \ , \tag{57}$$

and likewise for n_z, since we can arbitrarily make L_y and L_z as large as we like. The same approximation for the sum over n_x is not in general valid, and it becomes increasingly worse as the plate separation and the temperature become smaller. Shortly we will see an example to illustrate this point.

However, for now let us evaluate Eq. (56) in the regime where Wien, Planck, and others considered cavity radiation, namely, where the wall separation L is large compared to the more critical wavelengths of interest, which are the wavelengths at which most of the heat enters the region between the plates. If we jump ahead of ourselves and use our anticipated result of the ZPP spectrum to explain this point, then although an infinite range of wavelengths are included in the above sums, the major contribution to the change in entropy will largely be due to the sum of waves where the ZPP spectral energy density minus the ZP spectral energy density is large. We know that the wavelength at the peak of this curve satisfies $\lambda_M T=$const. Hence, for $L\gg\lambda_m$, we can anticipate that the continuum approximation of

$$\frac{1}{L}\sum_{n_x}\to\frac{1}{\pi}\int dk_x \tag{58}$$

is a good one.

To connect with Wien and Planck, we need to interchange the operations of integrating over Θ with the summations over $\mathbf{n}$ in Eq. (56). One way to do this is to first break up the integral over Θ into two parts, where T is held fixed and then L is held fixed:

$$S_{\text{cal}}(T_{\text{II}},L_{\text{II}})-S_{\text{cal}}(T_{\text{I}},L_{\text{I}})=\frac{2\pi^2c^2\mathcal{A}}{L_yL_z}\left[\frac{1}{T_{\text{I}}}\int_{L_{\text{I}}}^{L_{\text{II}}}dL\sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x>0}}\frac{\partial\omega}{\partial L}\omega\frac{\partial f_{\text{in}}}{\partial\omega}\left[\frac{\omega}{T_{\text{I}}}\right]+\int_{T_{\text{I}}}^{T_{\text{II}}}\frac{dT}{T}\left[\sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x>0}}\omega\frac{\partial f_{\text{in}}}{\partial T}+\frac{1}{8\pi^3c^3}\sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x=0}}\frac{\partial\Omega_T'^2}{\partial T}\right]\Bigg|_{L_{\text{II}}}\right] . \tag{59}$$

Now making the approximation that we can neglect the $n_x=0$ terms, and using Eqs. (57) and (58), plus

$$\frac{\partial\omega}{\partial L}=-\frac{\omega}{L}\left[\frac{k_x}{k}\right]^2 , \tag{60}$$

one can show that

$$S_{\text{cal}}(T_{\text{II}},L_{\text{II}})-S_{\text{cal}}(T_{\text{I}},L_{\text{I}})\approx\tfrac{4}{3}\sigma'(\mathcal{V}_{\text{II}}T_{\text{II}}^3-\mathcal{V}_{\text{I}}T_{\text{I}}^3) , \tag{61}$$

or

$$S_{\text{cal}}(T,L)\approx\tfrac{4}{3}\sigma'\mathcal{V}T^3+S_0 , \tag{62}$$

where

$$\sigma'=-\tfrac{1}{4}\int_0^\infty d\Theta\,\Theta^4\frac{\partial f_{\text{in}}}{\partial\Theta} , \tag{63}$$

$\mathcal{V}=\mathcal{A}L$, and $S_0=\text{const}$. Thus we have connected with the entropy of radiation in free space [see Eq. (61) in Ref. [15]] by considering the regime of large plate separations compared to the wavelength at the peak of the spectral energy curve above the ZP spectral energy part. Otherwise, one must return to Eq. (56) or (59). Equation (62) yields the $\mathcal{V}T^3$ form of the traditional result for the caloric entropy of blackbody radiation [see Eq. (80) in Ref. [23]]. However, here σ' accounts for the presence of ZP radiation, which was particularly evident in our analysis by the need to take singular forces and singular changes in energy into account. From Eq. (39) in Ref. [15], another form of Eq. (63) above is

$$\sigma'=\int_0^\infty d\Theta\,\Theta^3\left[f_{\text{in}}(\Theta)-\frac{\kappa}{c^3}\right] . \tag{64}$$

In the early analysis of cavity radiation, the second term in large parentheses above was missing [see Eqs. (101)–(104) in Ref. [23]], due to the implicit assumption of no zero-point radiation.

Now let us roughly estimate the ratio of the $n_x=0$ terms to the total terms in Eq. (59) when L is held fixed and when T_{I} and T_{II} differ by a small amount dT. Using the approximations of Eqs. (57) and (58), and assuming that Eq. (54) also holds for the $n_x=0$ terms in order to estimate the following ratio, then

$$\frac{\Delta S_{\text{cal},n_x=0}}{\Delta S_{\text{cal},n_x>0}}\approx\frac{\frac{1}{4}\sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x=0}}\omega\partial f_{\text{in}}/\partial T}{\sum_{\substack{-\infty<n_y,n_z<\infty\\ n_x>0}}\omega\partial f_{\text{in}}/\partial T}$$

$$\approx\left[\frac{1}{LT}\right]\frac{c\pi}{4}\frac{\left[\int_0^\infty d\Theta\,\Theta^3\partial f_{\text{in}}/\partial\Theta\right]}{\left[\int_0^\infty d\Theta\,\Theta^4\partial f_{\text{in}}/\partial\Theta\right]} . \tag{65}$$

Hence, for T and L sufficiently large, this ratio is negligible. However, when a cavity becomes small in size, or if the temperature is quite low, then our continuum approximation in Eq. (58) is not in general valid and we cannot ignore the discrete difference between the $n_x=0$ and the $n_x>0$ terms. Although the $n_x=0$ terms do not influence the force on the plates, they do possess a specific heat content that becomes proportionally larger as LT becomes smaller. Clearly, discrete mode corrections such as this one will be important when analyzing the behavior of very small cavities [26].

VIII. THE $(\omega,\mathcal{V})$ FORM OF WIEN'S DISPLACEMENT LAW

Here we will deduce the frequency and volume form of the Wien displacement law, which holds for adiabatic displacement operations. Moreover, evidently this law only holds when the plate separations are not so small as to focus attention upon the discrete standing wave nature of the cavity radiation. In contrast, the frequency and temperature form of the generalized Wien displacement law, Eq. (54), is not restricted to adiabatic operations, and it holds even for small plate separations.

From Eq. (62),

$$\mathcal{V}T^3\approx\text{const} \tag{66}$$

for adiabatic displacement operations at large plate separations. From Eqs. (43), (54), and (66),

$$\rho_{\text{in}}(\omega,T)|_S\approx\omega^3\mathcal{F}(\omega^3\mathcal{V}) , \tag{67}$$

where $|_S$ indicates constant entropy. This result is the solution for the differential form of the $(\omega,\mathcal{V})$ Wien displacement law:

$$\left.\frac{\partial\rho}{\partial\mathcal{V}}\right|_S=\frac{1}{\mathcal{V}}\left[\frac{\omega}{3}\left.\frac{\partial\rho}{\partial\omega}\right|_{S,\mathcal{V}}-\rho\right] , \tag{68}$$

which enables us to find $d\rho|_S=(d\rho/d\mathcal{V})_S d\mathcal{V}$ due to a small change $d\mathcal{V}$.

As originally noted by Boyer [5], the ω^3 form of the classical electromagnetic ZP radiation spectrum results in $(\partial\rho/\partial\mathcal{V})|_S=0$. We are now in a position to better understand the physical significance of this result. Upon ignoring the discrete nature of the cavity radiation and considering only the effective spectral energy density $\rho(\omega,T)$, the thermodynamic argument of Sec. V A requires that $\rho\propto\omega^3$ at $T=0$. From Eq. (68), we then see that displacements in the walls of the cavity will not alter $\rho(\omega,T)$ at $T=0$. This result is intuitively satisfying because at $T=0$, an adiabatic operation is the same as an isothermal operation [43]; otherwise, these two operations are not generally equivalent. Consequently, when $T\neq0$ we expect the temperature to change as the plates are adiabatically separated, so that $\rho(\omega,T)$ will also change; only at $T=0$ should $\rho(\omega,T)$ remain fixed.

However, we must also note that this effective spectrum $\rho(\omega,T)$ is not the most accurate way to characterize thermal radiation at low temperatures and for small plate separations. A much better way is to calculate the modes, specified here by $\mathbf{n}$ and λ, and to deal directly with the function $\Omega_T^2(\omega_{\mathbf{n}})$ that governs the average electromagnetic energy within each mode. As the plates move apart, the frequencies of the modes shift via Eq. (33), thereby changing the value of $\Omega_T^2(\omega_{\mathbf{n}})$ at a normal mode frequency. Also, the value of T will change for each mode, unless $T=0$. If we reexamine the $T=0$ displacement operation from the point of view of $\Omega_T^2(\omega_n)$, rather than $\rho(\omega,T)$ above, then the change in $\Omega_{T=0}^2(\omega_n)$ will be given by

$$d(\Omega_{T=0}^2(\omega_n))=dL_x\frac{\partial\omega_n}{\partial L_x}\left[\frac{\partial\Omega_{T=0}^2}{\partial\omega_n}\right] . \tag{69}$$

An adiabatic condition is ensured if $\Omega_{T=0}^2(\omega)$ satisfies the ZP form of Eq. (46); no heat then escapes at any mode. Nevertheless, $\Omega_{T=0}^2(\omega_n)$ will change in proportion to $dL_x(\partial\omega_n/\partial L_x)=\delta\omega_n$.

Further important connections and distinctions exist between Wien's original work, as described in Ref. [23], and the present analysis. Wien's analysis began with the $(\omega,\mathcal{V})$ form of his displacement law, in contrast with our (ω,T) form in Sec. VI. He dealt with a single frequency of radiation. Via the conservation of energy, he demanded that the energy of an incident wave on a slowly moving wall in a time dt must be equal to the energy of the wave reflected off the wall in time dt, plus the work done by external forces to move the wall against the pressure due to the reflected wave. A careful account was made of the change in frequency of the reflected wave from the slowly moving wall.

However, the discrete modes of radiation were not addressed. Consequently, Wien was able to average over a continuum of incident angles on the wall to deduce the $\frac{1}{3}$ factor in the second term in Eq. (93) in Ref. [23]. Other than this approximation, however, his $(\omega,\mathcal{V})$ analysis largely *did* account for the possible presence of ZP radiation. Since only one frequency component was considered, no complications occurred due to singular quantities. Of course, one should really supply an additional argument as to why each frequency component could be analyzed separately; the reason lies in a physical assumption, included in our own work as well, involving the statistical independence of electromagnetic waves with different frequencies [6,8].

When proceeding from the $(\omega,\mathcal{V})$ to the (ω,T) form of Wien's law, here is where the possibility of ZP radiation was not taken into account. The problem largely lies with the manner in which the adiabatic curves $\mathcal{V}T^3=$const were derived (see Ref. [23], Chap. II), which involved the use of $p=u/3$, where the pressure p and the energy density u were treated as nonsingular quantities. The following section helps identify some of the problems here.

IX. HIGH-TEMPERATURE REGIME BETWEEN PLATES

We can recover the analysis of early researchers by considering the case when the temperature of thermal radiation inside a large cavity is much greater than the temperature outside. Let T_{AB} be the "inside" radiation temperature between plates A and B in Fig. 1, and let T_{BC} be the "outside" temperature between plates B and C. Using the continuum approximations, we obtain, from Eqs. (35), (54), and (60),

$$\langle F_{\text{ext},B1}(t)\rangle_x \approx \frac{\mathcal{A}}{4\pi^4}\int_0^\infty dk_x\int_{-\infty}^\infty dk_y\int_{-\infty}^\infty dk_z\left[\frac{k_x}{k}\right]^2\{-[\Omega_{T_{AB}}(\omega)]^2+[\Omega_{T_{AB}}(\omega)]^2\}$$
$$=-\frac{\mathcal{A}}{3}\int_0^\infty d\omega\,\omega^3\left[f_{\text{in}}\left[\frac{\omega}{T_{AB}}\right]-f_{\text{in}}\left[\frac{\omega}{T_{BC}}\right]\right] . \tag{70}$$

From Eqs. (54) and (43), Eq. (70) equals $\mathcal{A}/3$ times the difference in the electromagnetic energy densities on either side of plate B. The analysis of early researchers would certainly agree with this result. However, if $T_{BC}\ll T_{AB}$, their approximation would consist of assuming the second term to be negligible. We cannot make this approximation because of Eq. (55), which implies that each integral taken separately above is singular. Instead, if we subtract out the singular part of each term via Eqs. (49) and (55), then

$$\frac{\langle F_{\text{ext},B1}(t)\rangle_x}{\mathcal{A}}\approx-\tfrac{1}{3}\int_0^\infty d\omega\,\omega^3\left\{\left[f_{\text{in}}\left[\frac{\omega}{T_{AB}}\right]-\frac{\kappa}{c^3}\right]-\left[f_{\text{in}}\left[\frac{\omega}{T_{BC}}\right]-\frac{\kappa}{c^3}\right]\right\}$$
$$=-\tfrac{1}{3}(T_{AB}^4-T_{BC}^4)\sigma' , \tag{71}$$

via Eq. (64). If $T_{BC}\ll T_{AB}$, now we can drop the second term. The total pressure is then approximately equal to the first term in square brackets in the first line of Eq. (71), which represents the pressure due to the amount by which the T_{AB} spectral contribution exceeds the $T=0$ spectral contribution to the total pressure. In the case of the ZPP spectrum,

$$\int_0^\infty d\omega\,\omega^3\left[f_{\text{in}}\left[\frac{\omega}{T}\right]-\frac{\kappa}{c^3}\right]\Longrightarrow\int_0^\infty d\omega\frac{\omega^2}{\pi^2c^3}\times\frac{\hbar\omega}{[\exp(\hbar\omega/k_BT)-1]}$$
$$=T^4\left[\frac{\pi^2k_B^4}{c^3\hbar^3 15}\right] \tag{72}$$

equals the familiar energy density due to the Planckian spectrum alone, without any $T=0$ spectral contribution.

With this consideration for the pressure, now let us reexamine the conventional deduction of the Stéfan-Boltzmann law starting from

$$TdS_{\text{cal}\,AB} = \left[\frac{\partial \langle \mathcal{U}_{\text{int},AB} \rangle}{\partial T} \right] \Bigg|_{L_{AB}} dT + \left[\frac{\partial \langle \mathcal{U}_{\text{int},AB} \rangle}{\partial L_{AB}} \Bigg|_T - \mathcal{F}_\ell \right] dL_{AB} . \quad (73)$$

The approach around 1900 involved approximating $\langle \mathcal{U}_{\text{int},AB} \rangle$ by $u_{AB}\mathcal{A}L_{AB}$, with u_{AB} assumed to be nonsingular and independent of $\mathcal{A}L_{AB}$. Assuming that $\mathcal{F}_\ell = -\mathcal{A}u_{AB}/3$ results in the coefficient of dL_{AB} above equaling $\frac{4}{3}\mathcal{A}u_{AB}$. Equating the two expressions for $\partial^2 S_{AB}/\partial T \partial L_{AB}$ then leads to the Stéfan-Boltzmann law that $u_{AB} \propto T^4$.

When $\langle \mathcal{U}_{\text{int},AB} \rangle$ and $\mathcal{F}_\ell$ are allowed to be singular, then the appropriate analysis is as given in Sec. VI, resulting not immediately in the Stéfan-Boltzmann law, but rather in the (ω, T) form of Wien's displacement law. By next following the continuum approximations in Sec. VII, the generalized Stéfan-Boltzmann relationship in Ref. [15] for free space was shown to approximately hold for the region between two conducting parallel plates. Specifically, from Eqs. (51) and (62),

$$\frac{\partial}{\partial T} \langle \mathcal{U}_{\text{int},AB} \rangle |_{L_{AB}} \approx \frac{\partial}{\partial T} [\mathcal{A}L_{AB}(\sigma' T^4)] , \quad (74)$$

which fits in with the ideas of physicists around 1900.

A few points are important to note regarding Eq. (73). The change in heat $TdS_{\text{cal},AB}$ between plates A and B must be nonsingular for any arbitrary small changes of dT or dL_{AB}, so the coefficients of dT and dL_{AB} in Eq. (73) must be nonsingular. Hence the specific heat $\partial \langle \mathcal{U}_{\text{int},AB} \rangle / \partial T$ must be finite, and the singular part of $\mathcal{F}_\ell$ must cancel with the singular part of $\partial \langle \mathcal{U}_{\text{int},AB} \rangle / \partial L_{AB}$. To connect with early 1900s analysis for large values of L_{AB}, L_y, and L_z, the continuum limits of Eqs. (57) and (58) can be used in approximating $\partial \langle \mathcal{U}_{\text{int},AB} \rangle / \partial T$, $\mathcal{F}_\ell$, and $\partial \langle \mathcal{U}_{\text{int},AB} \rangle / \partial L_{AB}$. However, with regard to this last quantity, one must first take the derivative of $\langle \mathcal{U}_{\text{int},AB} \rangle$ with respect to L_{AB} and *then* make the continuum approximation of Eq. (58), rather than vice versa. If $\langle \mathcal{U}_{\text{int},AB} \rangle$ was nonsingular, the order of these operations would not matter. More specifically, the latter *incorrect* procedure yields

$$\frac{\partial \langle \mathcal{U}_{\text{int},AB} \rangle}{\partial L_{AB}} = \frac{\partial}{\partial L_{AB}} \left[\frac{\mathcal{A}}{\pi L_y L_z} \sum_{\substack{-\infty < n_y, n_z < \infty \\ n_x > 0}} \Omega_T^2 \right]$$
$$\approx \frac{\partial}{\partial L_{AB}} \left[(\mathcal{A}L_{AB}) \left[\frac{1}{2\pi^3 c^3} \int_0^\infty d\omega\, \omega^2 \Omega_T^2 \right] \right]$$
$$= \mathcal{A} \left[\frac{1}{2\pi^3 c^3} \int_0^\infty d\omega\, \omega^2 \Omega_T^2 \right] . \quad (75)$$

This method corresponds to the treatment of $\langle \mathcal{U}_{\text{int},AB} \rangle$ around 1900, since the quantity in square brackets in the second line down is just the volume times an energy density that is independent of L_{AB}.

In contrast, if we now correctly make the continuum approximation after differentiating, we can explicitly see how the implicit assumption of no ZP radiation was contained in the early-1900s analysis:

$$\frac{\partial \langle \mathcal{U}_{\text{int},AB} \rangle}{\partial L_{AB}} = \frac{\mathcal{A}}{\pi L_y L_z} \sum_{\substack{-\infty < n_y, n_z < \infty \\ n_x > 0}} \frac{\partial \Omega_T^2}{\partial \omega_{AB}} \left[\frac{\partial \omega_{AB}}{\partial L_{AB}} \right]$$
$$\approx -\frac{1}{3} \frac{\mathcal{A}}{2\pi^3 c^3} \int_0^\infty d\omega\, \omega^3 \frac{\partial \Omega_T^2}{\partial \omega}$$
$$= +\frac{1}{3} \frac{\mathcal{A}}{2\pi^3 c^3} \left[-\omega^3 \Omega_T^2 |_0^\infty + 3 \int_0^\infty d\omega\, \omega^2 \Omega_T^2 \right] . \quad (76)$$

The second term in the last line above is equivalent to the last line of Eq. (75). Due to Eqs. (54) and (55), the first term above is not only nonzero, but also singular. Only if the thermal radiation reduces to zero at $T=0$ in such a way that $f_{\text{in}}(\Theta)$ in Eq. (54) goes to zero faster than $1/\Theta^4$ as $\Theta \to \infty$, will the first term above equal zero. Only then are the approximations in Eqs. (75) and (76) equivalent. Hence this implicit assumption of no ZP radiation was buried in the early analysis around 1900.

Applying the continuum limit to $\mathcal{F}_\ell$ given by Eq. (34), we obtain

$$\mathcal{F}_\ell \approx -\frac{1}{3} \frac{\mathcal{A}}{2\pi^3 c^3} \int_0^\infty d\omega\, \omega^2 \Omega_T^2 . \quad (77)$$

Combining Eqs. (76) and (77), along with Eqs. (49), (54), and (55), yields

$$\frac{\partial \langle \mathcal{U}_{\text{int},AB} \rangle}{\partial L_{AB}} - \mathcal{F}_\ell \approx \tfrac{4}{3}\mathcal{A} \int_0^\infty d\omega\, \omega^3 \left[f_{\text{in}} \left[\frac{\omega}{T_{AB}} \right] - \frac{\kappa}{c^3} \right]$$
$$= \tfrac{4}{3}\mathcal{A}(\sigma' T^4) . \quad (78)$$

In this way we recover the form of $\frac{4}{3}\mathcal{A}u_{AB}$ for the coefficient of dL_{AB} in Eq. (73), where $u_{AB} = \sigma' T^4$, as well as seeing explicitly where the singularities enter into the analysis.

X. CONCLUDING REMARKS

The analysis discussed here involved the thermodynamic behavior of two parallel conducting plates, immersed in classical electromagnetic thermal radiation. Changes in separation of the plates and changes in the temperature were investigated. The Appendix [16] extended this work to a conducting box with a movable interior conducting wall. Throughout the analysis, the possibility that the thermal radiation spectrum may not reduce to zero at $T=0$ was taken into account. With this, quite possibly, critically important physical consideration, the thermodynamic analysis of blackbody radiation compression by Wien, Planck, and others was reinvestigated here, as well as extended to nonadiabatic thermodynamic operations.

Two particularly noticeable results of this analysis were the deduction of (i) the (ω, T) form of Wien's displacement law and of (ii) the thermal spectrum at $T=0$. These results were deduced, respectively, from the demands of (i) the second law of thermodynamics and (ii) the thermodynamic definition of $T=0$. Both of these results bring us a bit closer toward understanding the full set of requirements imposed by thermodynamic equilibrium conditions on the thermal radiation spectrum found in nature. We now have two electrodynamic systems namely, simple harmonic electric dipole oscillators [14,15] and conducting parallel plates, that agree in their predictions of the appropriate functional form of the thermal spectrum needed to satisfy (i) and (ii).

The ZP radiation field has a number of interesting properties, with two of the more important ones being that (a) different inertial reference frames possess the same radiation energy spectrum [3,4] and (b) no heat will flow during reversible isothermal operations, at least for the two systems mentioned above. The first condition directly involves only the radiation, while the second condition directly involves *both* radiation and systems of matter. Either of these properties enable the functional form of the classical electromagnetic ZP radiation spectrum to be deduced. However, the two properties do not appear to be equivalent. Instead, these properties are results of separate conditions that must be met by the ZP radiation field if it is to have an acceptable physical behavior, where here we are assuming that nature does indeed obey rules of Lorentz invariance and of conventional thermodynamics. Thus we could certainly conceive of the situation where either (a) or (b) might be satisfied, but not the other, in which case we would have to conclude either that the classical physical framework of SED does not correctly describe nature, or that we were mistaken about nature exactly obeying the imposed conditions. The former being the much more likely situation, then one immediate conclusion is that a minimum condition for the classical electromagnetic ZP field to be acceptable as a real part of nature is that (b) must hold for all physical systems of matter found in nature. At present we do not know to what extent this condition is true, since (b) has only been verified for two very specific physical systems [44].

Finally, one point we should emphasize here is the important physical significance of the generalized (ω,T) form of Wien's displacement law, which might easily be overlooked. The (ω,T) form of Wien's law is deceptively very simple: specifically, that the thermal spectrum in the near continuum case, or the variance of the electromagnetic mode amplitudes in the discrete case, should be proportional to $\omega^3 f_{\text{in}}(\omega/T)$ and $\omega f_{\text{in}}(\omega/T)$, respectively. Consequently, one might suspect that the main content of Wien's displacement law can be deduced by a means as simple as dimensional analysis. Indeed, Sommerfeld in Ref. [45], Sec. 20.C, uses a dimensional analysis argument to deduce this spectral form [46,47]. Nevertheless, we can see that this sort of argument is largely a plausibility argument, as Sommerfeld does indeed suggest on p. 141 in Ref. [45], rather than a substitute for the thermodynamic reasoning. Afterall, standard dimensional analysis deals only with assumptions about the physical constants and dimensions that might enter into a problem and does not, otherwise, directly touch on the governing laws of nature. Hence this simple approach cannot address the key content of the (ω,T) form of Wien's displacement law, which is that a condition for the second law of thermodynamics to be obeyed is that the thermal spectrum must possess this spectral form. Indeed, as discussed here, the (ω,T) form of Wien's displacement law should be derived *before* the Stéfan-Boltzmann law, which previously had been the point where researchers thought the second law entered the analysis via the requirement that entropy be a function of state.

Along a similar line of thought, one might suspect that the classical electromagnetic ZP spectrum can be deduced from the generalized (ω,T) Wien's law derived here of $\rho_{\text{in}}(\omega,T)=\omega^3 f_{\text{in}}(\omega/T)$. After all, if we let $T\to 0$ here and *if* $\lim_{\Theta\to\infty} f(\Theta)$ is a *nonzero* constant, then we do indeed obtain the correct ω^3 form of the ZP spectrum [48]. Certainly, this argument is helpful to note since it makes this nonzero spectral form at $T=0$ plausible. However, this argument does not address the key point we examined in some detail here as to whether this ω^3 spectrum will satisfy the demand that no heat will flow during reversible isothermal operations, and thereby serve as a thermodynamically acceptable $T=0$ spectrum.

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[15] D. C. Cole, Phys. Rev. A **42**, 7006 (1990).

[16] See AIP document no. PAPS PLRAA-45-8471-7 for 7 pages of the Appendix entitled *Appendix: Extension to a Conducting Box.* Here, much of the analysis on conducting parallel plates described in the present article is extended to a hollow rectangular box with thin conducting walls, one of which can be displaced. Order by PAPS number and journal reference from American Institute of

Physics, Physics Auxliliary Publication Service, 335 E. 45 St., New York, NY 10017. The price is $1.50 for each microfiche (60 pages) or $5.00 for photocopies of up to 30 pages and $0.15 for each additional page over 30 pages. Airmail additional. Make checks payable to the American Institute of Physics.

[17] Zero electromagnetic fields inside a conducting material cannot really occur in SED, due to the singular forces that would arise and due to the fact that any real plate is composed of atoms with fluctuating charge distributions. This nonphysical result of zero fields is due to our macroscopic model of a plate consisting of continuous material with infinite conductivity at all frequencies, rather than a material microscopically composed of atoms with electrons that can flow fairly freely through the material. Nevertheless, our model is useful because it is an idealization that predicts the important frequency components of the radiation fields between pairs of plates.

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[24] The first quantitative experimental work on forces between uncharged conducting parallel plates, as analyzed by Casimir in Ref. [18], was reported by M. J. Sparnaay in Physica **24**, 751 (1958). The forces existing between a variety of other materials and shapes of materials, due to the presence of the matter within the electromagnetic ZP radiation field, have been calcualted since Casimir's initial work. Experiments investigating these forces were reported in B. V. Derjaguin, I. I. Abrikossova, and E. M. Lifshitz, Q. Rev. Chem. Soc. **10**, 295 (1956) and B. V. Derjaguin and I. I. Abrikossova, Zh. Eksp. Teor. Fiz. **30**, 993 (1956) [Sov. Phys.—JETP **3**, 819 (1957)], which describe experiments involving fused quartz and K-8 glass; W. Black, J. G. V. De Jongh, J. Th. G. Overbeek, and M. J. Sparnaay, Trans. Faraday Soc. **56**, 1597 (1960), which discusses results for pairs of quartz plates and between a flat and a spherically curved plate; D. Tabor and R. H. S. Winterton, Nature (London) **219**, 1120 (1968) and Proc. R. Soc. London, Ser. A **312**, 435 (1969), which reports on measurements of unretarded and retarded forces between mica sheets; E. S. Sabisky and C. H. Anderson, Phys. Rev. A 7, 790 (1973), which describes experiments involving the thickness of liquid-helium films on cleaved crystal surfaces, where the thick films are due to van der Waals forces; and W. Arnold, S. Hunklinger, and K. Dransfeld, Phys. Rev. B **19**, 6049 (1979), which reports on the measurements of van der Waals forces between a variety of different materials, including crystalline quartz, borosilicate glass, and silicon.

[25] Incidentally, this analysis deduced nearly a century ago is still the traditional analysis reported in standard textbooks on thermodynamics. See, for example, F. W. Sears and G. L. Salinger, *Thermodynamics, Kinetic Theory, and Statistical Thermodynamics,* 3rd ed. (Addison-Wesley, Reading, MA, 1975), Sec. 8.7. Also see T. S. Kuhn, Ref. [1], pp. 5 and 6.

[26] This result can be deduced from Ref. [15] (see Secs. IV A and V). We should note, though, that if a cutoff is assumed in the spectrum, as some researchers believe is necessary, then the radiation pressure will not be singular. Nevertheless, it will still be enormous and completely out of range of what is observed experimentally. This large pressure arises because the cutoff frequency needs to be quite large to agree with appropriate physical cutoff mechanisms. See, for example, L. de la Peña, in Ref. [8], Sec. 2.3.

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[28] See the discussion on experiments performed with small cavities, by S. Harouche and D. Kleppner in Phys. Today **42** (1), 24 (1989). This field of study has become known as cavity quantum electrodynamics.

[29] D. C. Cole, Found. Phys. **20**, 225 (1990).

[30] As for characterizing the thermal radiation between two perfectly conduting parallel plates, the method in Ref. [27] will be roughly followed here as it is convenient and physically clear. The main differences between the two approaches, besides treating thermal radiation versus only ZP radiation, are that a few fairly important points need correcting in Ref. [27], as well as some misprints. As for the more significant points, first, one cannot make the same assignment of Eq. (11) in Ref. [27] to the variance of the plane-wave amplitude for the cases of the ZP fields, for example, (i) in free space, (ii) between two conducting plates, or (iii) within a conducting box. After all, there is an effective infinite number of reflections that one plane wave will make between two conducting plates over the course of time, but not so in the case of a plane-wave decomposition of fields within an imaginary box in free space, where periodic boundary conditions are used to specify the waves. The amplitudes of the plane-wave contributions can be different in these cases. Second, the integral expressions for the fields in Eqs. (7) and (8) of Ref. [27] need correcting, as well as the computations and relationships in Eqs. (20) and (21); the point made on p. 232 in Ref. [29] needs to be taken into account. Finally, the $k_x=0$ waves were not treated in Ref. [27], which can be important in the case of specific heat. As for simple misprints in Ref. [27]: (1) the k_y and k_z terms in Eqs. (26), (30), and (34) should contain factors of 2π rather than just π; (2) a minus sign belongs on the left-hand side of the first line of Eq. (25) and on the right-hand side of the first equal sign in Eq. (29) (i.e., right before $\int_0^\infty dk_x \cdots$); (3) the last lines in Eqs. (25) and (29) should have the factor of 4, not $\frac{1}{2}$, while the earlier lines do have the correct numerical factor (as a quick check, having two polarizations present, each with a contribution of $\hbar\omega/2$, plus accounting for the

$-n_y$, $+n_y$, $-n_z$, and $+n_z$ contributions, yields the $4\hbar\omega$ term); (4) a factor of $1/2d$ is missing in Eq. (39); (5) a factor of L^2 is missing in the second line in Eq. (41); (6) $\sin(k_x x)$ should be replaced by $\sin(-k_x x)$ in Eq. (16) due to the comment following Eq. (13); (7) the E_{xx}^2, E_{yy}^2, etc., terms in Eq. (19) should be written simply as E_x^2, E_y^2, etc; (8) Eq. (33) needs an additional factor of 4; and (9) $Y=2n_y/L$ and $Z=2n_z/L$ are the correct substitutions just prior to Eq. (35).

[31] M. S. Barlett, *An Introduction to Stochastic Processes,* 3rd ed. (Cambridge Univeristy Press, Cambridge, 1978), p. 207.

[32] Planck discussed the concept of a black carbon particle in Ref. [23], as well as the use of a "stopcock" to transfer radiation energy out of a cavity. Sections 51, 52, and 68–70 in Ref. [23] contain particularly interesting comments on these concepts. Section 10 discussed the idea of a "white" surface to diffusively scatter radiation; p. 60 in Ref. [23] also contained some interesting comments.

[33] See, for example, J. D. Jackson, *Classical Electrodynamics,* 2nd ed. (Wiley, New York, 1975), Sec. 6.8.

[34] See, for example, Ref. [23], Part II, Chaps. I and II, or R. Becker, *Electromagnetic Fields and Interactions* (Dover, New York, 1982), Vol. II, pp. 279 and 280.

[35] For the change in internal energy in the ZP case, see, for example, Refs. [18] and [19]. For other related calculations, such as the force in the ZPP and RF cases, see T. H. Boyer, Phys. Rev. A **11**, 1650 (1975), Sec. V. [However, in this reference Eq. (116) should be replaced by Eq. (33) in the present article, and an additional factor of 8 is needed in the last line of Eq. (115).]

[36] As a check on this idea, we could replace the infinitely conducting plate C by an infinitely permeable plate and still handle the mathematics without significant difficulty. As can be shown in the case of ZPP radiation, the same net force does occur on plate B with this replacement, and the same change still exists in total electromagnetic energy, provided that $L_{BC}/L_{AB}\rightarrow\infty$. Reference [27] contains relevant calculations that one needs to carry out this demonstration (also see the notes in Ref. [30]).

[37] See, for example, M. W. Zemansky and R. H. Dittman, *Heat and Thermodynamics,* 6th ed. (McGraw-Hill, New York, 1981), Sec. 7-10.

[38] See the end of the article by J. Mehra in Physica **37**, 145 (1967).

[39] See, for example, Casimir's fairly general method in Ref. [18]; M. Fierz's specific, but convenient cutoff method in Helv. Phys. Acta **33**, 855 (1960), which involves the cutoff function $\exp(-\alpha\omega)$, where the limit of $\alpha\rightarrow 0$ is taken at the end of the calculation; the general remarks by Boyer in Ref. [19], and the further elaboration of Fierz's method in Sec. IIa of Ref. [19], as well as in the first part of p. 2083 in Ref. [27]; the ζ-function regularization method discussed by E. Elizalde, Nuovo Cimento B **104**, 685 (1989); and the general overview by Plünien *et al.* in Ref. [20], Sec. 4.1.

[40] See Ref. [5] for the first recognition of the need for this distinction. Reference [14], Sec. VII, makes a few additional comments.

[41] As a check on the signs of the work terms $d\mathcal{W}_{AB}\equiv dL_{AB}\mathcal{F}_\ell$ and $d\mathcal{W}_{BC}\equiv dL_{AB}\mathcal{F}_r$, we note that $\mathcal{F}_\ell=-|\mathcal{F}_\ell|$ and $\mathcal{F}_r=+|\mathcal{F}_r|$. Hence, for a contraction of L_{AB}, or $dL_{AB}<0$, then $d\mathcal{W}_{AB}>0$, since positive work is being done by external forces to compress the "AB" radiation. Similarly, for a contraction of L_{BC}, or $dL_{AB}=-dL_{BC}>0$, then $d\mathcal{W}_{BC}>0$.

[42] If we were to directly consider $S_{\text{cal},ABC}$, the reasoning would be essentially identical to what follows. Note that we demand that $dS_{\text{cal},AB}$ and $dS_{\text{cal},BC}$ be nonsingular, although the terms comprising them are generally singular.

[43] M. W. Zemansky, *Temperatures Very Low and Very High* (Dover, New York, 1964) , p. 23.

[44] Also see the discussion in D. C. Cole, in Ref. [13], Sec. I, on properties (i) and (ii) for the ZP field, as well as some other properties.

[45] A. Sommerfeld, *Thermodynamics and Statistical Mechanics,* Lectures on Theoretical Physics Vol. V (Academic, New York, 1956).

[46] Sommerfeld's argument has been cited by others, in particular with regards to SED. See A. M. Cetto and L. de la Peña, Found. Phys. **19**, 419 (1989), on p. 425.

[47] However, we should note that even Sommerfeld's dimensional analysis leading to Wien's law needs to be modified if it is to include the case of nonzero radiation at $T=0$. On p. 142 in Ref. [45], Sommerfeld is correctly concerned about a divergent spectrum, but does not touch on how differences in the total internal energy are the important thermodynamic quantities, instead of the total energy. Sommerfeld proposes to overcome this divergent spectrum difficulty by introducing another universal constant, which is how Planck's constant eventually enters into his analysis. In the process he chooses $u=\int_0^\infty d\omega\,\rho(\omega,T)$ to be proportional to T^4, so as to agree with the Stéfan-Boltzmann law. However, he deduced the Stéfan-Boltzmann law in Sec. 20.B via the traditional method, which does not take a possible ZP radiation field into account, thereby making this particular dimensional argument inappropriate for the case where the spectrum is nonzero as $T\rightarrow 0$.

[48] Apparently P. Braffort, M. Spighel, and C. Tzara, C. R. Acad. Sci. **239**, 157 (1954), first gave this argument, which has since been cited by other workers in SED.