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Transient multimodality for the decay of unstable states

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A possibility of the occurrence of transient multimodality during a decay of an unstable state is analyzed. For the two discussed classes of systems it is found that the appearance of this phenomenon depends only on the sign of the second nonvanishing derivative of the potential at the unstable point. Thus here the transient multimodality is not a noise-induced phenomenon like that for the evolution from an arbitrary state. A scheme of the classification of dynamical systems based on their transient properties is proposed.

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INTRODUCTION

A deterministic approach is usually sufficient for studying an evolution of dynamical systems. The presence of noise acts only as a little disturbance that does not modify the results qualitatively. However, there are some critical situations when the role of fluctuations is very important or even essential for the occurrence of some phenomena, for instance, relaxation from unstable [1–3] or marginal [4] states, noise-induced transitions [5], or the transient bimodality [6–17]. The latter manifests itself as a coexistence of two peaks of the probability distribution for a sizable interval of time, although the initial and final forms of the probability distribution are one peak only. It occurs for systems whose behavior is determined by a potential $U(x)$ with a sufficiently flat plateau. In such a case a long lethargic stage, recognized as a critical slowing down, appears during the deterministic evolution, and consequently the fluctuations become important. If they are strong enough two maxima of the probability distribution located in the region of potential plateau and potential minimum may appear. But, if the fluctuations are too strong, the system is not affected by the potential plateau and its evolution is similar to a case with a potential with a steep slope. The transient bimodality is thus an effect of a delicate balance between critical slowing down and noise.

This phenomenon was first reported in explosive chemical reactions and combustion [6–8]. Later it was identified both experimentally and numerically in optical bistability [9–13], in electronic systems [12], and in a laser with a saturable absorber [13,14].

Almost all the papers mentioned above deal with the evolution from an arbitrary state, for which the transient multimodality arises as a noise-induced phenomenon [9,10]. In the present paper we consider a problem of transient multimodality and a decay of unstable states. It was already discussed in [13] and [14], and it was concluded there that the potential, which possesses a parabolic maximum in such a case, is not flat enough for the occurrence of transient multimodality. There is only the boundary region between the unstable state and the long lethargic state situations, namely, in the vicinity of the marginal stability, where the phenomenon of transient multimodality might appear.

In the following we show that the transient multimodality can appear during a decay of the unstable state even for a quite nonflat potential. As the fluctuations are needed for the initiation of the evolution, the problem must be treated as a stochastic one from the very beginning. The Fokker-Planck equation formalism [18] is very useful here. It allows one to obtain some analytical conditions [19] for the occurrence of transient multimodality. As an illustrative example the evolution in a symmetric sixth-order-polynomial potential is treated numerically. The mechanism of the occurrence of transient multimodality during the decay of an unstable state is discussed, and finally, a classification scheme for systems with a sixth-order-polynomial potential based on their transient properties is presented.

To avoid any misunderstanding, some comments on terminology must be made. Considering stochastic systems one associates the word "multimodality" with the number of maxima (peaks) of the probability distribution, i.e., with the number of the solutions x of the equation $W'(x)=0$ with $W''(x)<0$ (see, e.g., [5]) (let us call these analytic maxima). However, this does not include all the possibilities. First, if the potential $U(x)$ governing the dynamics of the system is not smooth enough the probability distribution function $W(x)$ may have no analytic maxima, nevertheless it is related to a deterministic state (mode) of the system, e.g., for $U(x)=|x|$ the stationary probability distribution $W(x)\sim\exp(-|x|/q)$ (q is a diffusion constant) attains its maximal value at $x=0$ where $W'(x)$ does not exist. Second, if the domain of states x of a system is bounded $W(x)$ may attain its locally maximal values at the boundaries, while $W'(x)\neq 0$ there, e.g., $x\in[0,\infty)$ and $W(x)\sim\exp(-x/q)$. For both examples we can say that the system is in a monomodal (monostable) state despite that there is no solution of $W'(x)=0$ with $W''(x)<0$. Thus, in the following, a mode of a system is meant as a point with a locally maximal value of the probability distribution (locally the most-probable state) and the multiplicity of the multimodality is defined by the number of the probability distribution humps.

THEORY

The existence of an unstable state means that there is a local maximum of the potential $U(x)$ describing the system. Localizing the initial state at the maximum one cases that the initial probability distribution (Dirac's δ function) possesses its peak at this point too. When time increases this peak may broaden and/or move away from the unstable point. From among all the possibilities we choose two classes of potentials, which guarantee the knowledge of the position of this peak during the whole time of its existence.

First, let us consider a problem on the positive semiaxis. It may represent, for instance, the evolution of light intensity in a quantum-optical system. The Fokker-Planck equation describing this case reads

$$\frac{\partial W(x,t)}{\partial t}=\left[\frac{\partial}{\partial x}U'(x)+q\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right]W(x,t)\ , \tag{1}$$

where q is a noise strength. We assume that the drift function $U'(x)$ possesses two zeros corresponding to the steady states of the deterministic case: $x_u=0$ the unstable state and $x_s>0$ the stable one. The instability of x_u is guaranteed by a negative value of the first derivative of the drift function at x_u, i.e., $U''(0)<0$.

We are interested in the evolution of the probability distribution $W(x,t)$ given by the half of Dirac's δ function at the initial moment $t=0$. Thus, the probability distribution $W(x,t)$ evolves from the initial one-hump (at $x=0$) form to the final one-hump (at $x=x_s$) form as well. Let us suppose that a one-hump form is conserved during the whole evolution. Hence, for the initial time stage the most probable value of $W(x,t)$ is located at $x=0$, from which it breaks off at a time $t=t_0$ and then tends toward x_s. For a close examination of this moment let us differentiate the Fokker-Planck equation (1) with respect to x at the point $x=0$. Thus we have

$$\dot{W}'=U'''W+2U''W'+2qW''\ . \tag{2}$$

The dot indicates time differentiation, prime x differentiation, and all the functions are taken at $x=0$. Besides, for $t=t_0$ we have

$$W'=0\ , \tag{3}$$

due to the appearance of an analytical extremum at $x=0$,

$$\dot{W}'\geq 0\ , \tag{4}$$

because the slope of the distribution function becomes positive for $t>t_0$,

$$W''\leq 0\ , \tag{5}$$

since the maximal value of W is still located at $x=0$, which is because W is a convex function there. Setting (3)–(5) into (2) one easily concludes that the relation

$$U'''(0)\geq 0 \tag{6}$$

is required for the departure of the maximum of $W(x,t)$ from $x=0$ at $t=t_0$. If (6) is not fulfilled, i.e., if

$$U'''(0)<0\ , \tag{7}$$

we have the following situation. Since the stationary probability distribution $W(x,\infty)$ has a local minimal value at $x=0$ there exists a time t_1 when this minimum appears. At this moment the relations like (3) and (4) are to be fulfilled. So the only possibility for the relation (2) and (7) not to be in a contradiction is

$$W''>0\ . \tag{8}$$

This means nothing but that there already exists a maximum at a point $x>0$ for $t<t_1$. Since for $t<t_1$ there is also a locally maximal value at $x=0$, there is a range of time within which the probability distribution $W(x,t)$ possesses two humps. And this is just the *transient bimodality*.

The second class of systems is defined on the whole x axis and is described by a symmetric potential $U(x)$ with one maximum at $x=0$ and two minima at $x=\pm x_s$. The Fokker-Planck equation for this case reads

$$\frac{\partial W(x,t)}{\partial t}=\left[\frac{\partial}{\partial x}U'(x)+q\frac{\partial^2}{\partial x^2}\right]W(x,t)\ . \tag{9}$$

As before, the initial probability distribution $W(x,0)$ is the Dirac's δ function at $x=0$ which is an unstable point as well, so the probability distribution evolves toward its stationary form with two maxima at the points $x=\pm x_s$ and a minimum at $x=0$.

The whole consideration pertaining to the previous class of potentials may be now repeated, but with two remarks which follow from the symmetry properties of $U(x)$ and $W(x,0)$ with respect to x. First, the probability distribution $W(x,t)$ must be a symmetric function of x during the whole evolution, too. Thus all its odd x derivatives are equal to zero at $x=0$ [as the odd derivatives of $U(x)$]. Hence, instead of a relation like (2), now one ought to analyze the second derivative of (9) at $x=0$, i.e.,

$$\dot{W}''=U''''W+3U''W''+qW''''\ . \tag{10}$$

Second, at $t=t_0$ the two maxima move away from $x=0$ symmetrically. Similarly to (3)–(5) at $t=t_0$ one has

$$W''=0 \tag{11}$$

because the convex function $W(x,t)$ becomes concave as the maximum turns into the minimum,

$$\dot{W}''\geq 0 \tag{12}$$

(due to the same reasons as above),

$$W''''\leq 0\ , \tag{13}$$

because near $x=0$ one has $W(x,t_0)\approx W(0,t_0)+(1/4!)W''''(0,t_0)x^4$ and at $x=0$ there is still a maximum of $W(x,t)$. Setting (11)–(13) into (10) we conclude as before that in order that the maxima of $W(x,t)$ separate from $x=0$ at the time t_0 one needs that

$$U''''(0)\geq 0\ . \tag{14}$$

If not, i.e., if

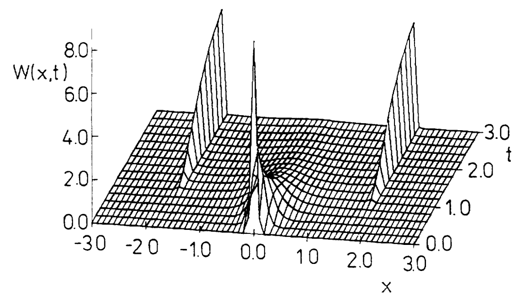

FIG. 1. Time evolution of the probability distribution for the potential (16) for $A=-3$, $B=-1$, $q=0.02$, and the initial Gaussian distribution centered at $x_0=0$ and with the variance $\sigma=0.002$.

$$U''''(0)<0 \tag{15}$$

the only possibility is that before the maximum at $x=0$ disappears there still exist two (because of the symmetry) other maxima at some points $x\neq 0$. Thus there is a *transient trimodality.*

In both cases discussed above the initial probability distribution was the Dirac's δ function. However, the only further needed property of this distribution was the location of its maximal value at the unstable point x_u. One can choose quite a different one-hump function, even a very wide one, as an initial distribution with the only requirement that it has its most probable value at x_u (eventually it must be a symmetric one for the second class of potentials). As none of its characteristics modify the relations (3) and (4) or (11) and (12) the conditions for the occurrence of the transient multimodality [(7) or (15)] concern such a generalization of the problem too.

For an illustration a simple example of the second class was solved numerically. The potential $U(x)$ was chosen as a polynomial

$$U(x)=\tfrac{1}{6}x^6+\tfrac{1}{4}Ax^4+\tfrac{1}{2}Bx^2 \ . \tag{16}$$

One can easily check that the negativeness of B guarantees the unstability of the point $x=0$ [a maximum of $U(x)$] and the negativeness of A guarantees the fulfillment of (15), when one may expect the occurrence of the transient multimodality.

The evolution of $W(x,t)$ for $A=-3$, $B=-1$, $q=0.02$ and the initial Gaussian function with a variance $\sigma=0.002$ is shown in Fig. 1. The range of time for which three maxima exist is clearly seen. Such a behavior was also noticed for other negative values of A. As A became positive the "usual" evolution [1], with moving away of the two maxima from the point $x=0$, was observed.

DISCUSSION

In this paper we have investigated the possibility of the existence of transient multimodality during a decay of an unstable state. For the two classes (1) and (9) of Fokker-Planck equations the analytical conditions (7) and (15) for the appearance of multiple-hump probability distribution have been obtained. Although both conditions differ in their mathematical form, their sense is the same. That is, the new maxima arise if the second nonvanishing derivative of the potential $U(x)$ at the unstable state is negative. And this is the only condition—there is no dependence on the noise level as in the case of the evolution from an arbitrary state [9,10]. Thus the transient multimodality for the decay of the unstable state is *not a noise-induced phenomenon.* The fluctuations are needed only to initiate the evolution as for the "usual" relaxation from the unstable state [1–3]. Naturally the stochastic factors arising in this problem influence the quantitative relations of the evolution as, e.g., the onset time of the multihump distribution. But the fact of the occurrence of the transient multimodality is of the deterministic nature only.

Comparing the present results with those of previous papers [6–17] one can say that the phenomenon of transient multimodality for the decay of an unstable state is a limiting case of the evolution from an arbitrary state. It was noticed [9,13] that the flatter the potential, the lower the noise level required for the appearance of transient multimodality. And here, at the unstable state, the slope of the potential is zero, so for each small noise level the phenomenon of transient multimodality exists. Hence, in this case, it is not a noise-induced phenomenon. Nevertheless there is one more difference between the two discussed cases. That is, for the occurrence of transient multimodality during the evolution from an unstable state a long flat part of the potential is needed. Here we start from a top of a potential, but since the only requirement on $U(x)$ is the negativeness of $U''(0)$ and the next nonvanishing derivative, the potential may possess an arbitrary curvature at that point. Thus the flat part of the potential reduces to one point only. Therefore, what is the sense of the conditions (7) and (15) for the appearance of the transient multimodality? To discuss this let us find the velocity $\dot{x}$ of the deterministic motion in the vicinity of the unstable state $x_u=0$. It reads

$$\dot{x}=-U'(x)\simeq|U''(0)|[1-\alpha U^{(p)}(0)x^{p-2}]x \ , \tag{17}$$

where $\alpha=[(p-1)!|U''(0)|]^{-1}$ is positive and $U^{(p)}(0)$ is the second nonvanishing at $x=0$ derivative of $U(x)$. Obviously the linear term in (17) is the dominant one. It causes the velocity $\dot{x}$ to grow linearly with the distance from the unstable state. The next term on the right-hand side of (17) expresses the relation between the linear approximation and the real evolution. The negativeness of $U^{(p)}(0)$ means that the growth of the velocity $\dot{x}$ is in fact stronger than linear. Because of the appearance of the transient multimodality for the negative value of $U^{(p)}(0)$ one can say that such a velocity allows a trajectory that is just thrown out from the unstable state to leave the region of the initial probability maximum and to reach the vicinity of the stable attractor sooner than the initial probability peak is emptied.

Nonlinear dynamical systems are usually classified with respect to the types of the steady states they may reach. One need not know the transient behavior for this purpose. For the one-dimensional case the stationary states are the only steady states, so one deals with mono-, bi-, or multistability situations. As for the stochastic problems the distinction between the one-dimensional systems is connected with the number of the maxima of the stationary probability distribution. They are usually related to the deterministic stable steady states, but some other maxima induced by the noise [5] may appear, too. However, the above criterion does not exhaust all the possible qualitative differences between the systems—in the presence of transient bimodality the differences occur only during the evolution. Thus one can propose a classification scheme for the dynamical systems defined by their transient properties. In the following we present it for the potential (16). The possible situations are shown in Fig. 2 depending on the values of the parameters A and B. There are four kinds of regions of the probability distribution behavior:

I. The region of one maximum during the whole evolution;

II. The region of initially one and then two maxima;

III. The region of initially one and then three maxima;

IV. The region of initially one, then three, and finally two maxima.

As one considers the stationary situation only region I corresponds to the monostable case, regions II and IV to the bistable case, and region III to the tristable case. The most interesting region of the transient trimodality (IV) lies in the bistability region (II+IV) just by the tristability (III) one. For a given negative value of A, by lowering the value of B we can change the stationary behavior of a system from tristable to bistable with a bifurcation point at $B=0.0$. However, after crossing this point the system is not completely unaffected by the tristability. The transient trimodality is a lingering effect of it.

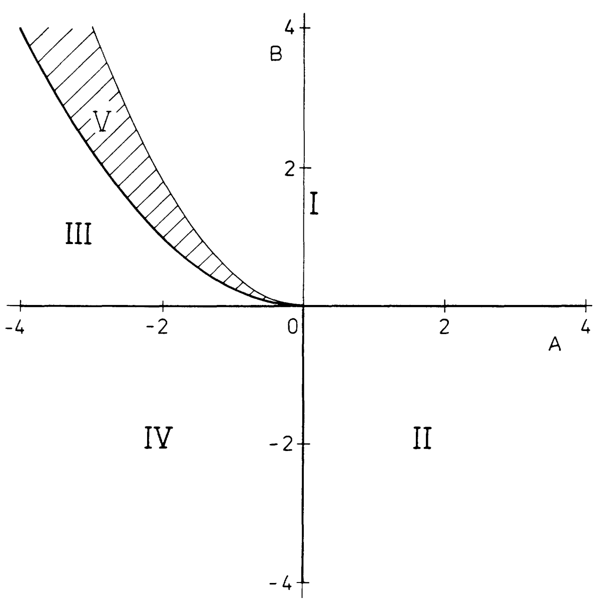

FIG. 2. The division of the parameter space (A,B) of the potential (16) into the regions of different transient behavior of the probability distribution (bold lines). The marked area V means the region of the existence of the inflection points of the one-well potential. The bold parabola $B=0.25A^2$ is given as a condition for the existence of nonzero solutions of $U'(x)=0$, the thin parabola $B=0.45A^2$ is given as a condition for the existence of solutions of $U''(x)=0$.

Figure 2 concerns the decay of the unstable state $x_u=0$ only. If one considers all the possible initial states x_0 $[W(x,0)=\delta(x-x_0)]$ the parameter space must be enlarged to a three-dimensional one with a parameter x_0 on the third axis. And it is obvious that the transient tristability is possible only for x_0 in the vicinity of x_u, and for the same values of A and B, but for x_0 located near x_s we have bimodality during the evolution only.

In this place it is worth noting, that for a potential (16) a second region (V) of transient multimodality could appear. This corresponds to the transient multimodality for the initial point at the slope of the potential (an evolution from an arbitrary state), the case discussed previously [6–17]. This region is included in region I and lies to the right from the tristability region (III) (Fig. 2). It is bordered by a curve $B=0.45A^2$ given from the condition for the existence of an inflection point, i.e., $U''(x)=0$. Obviously, in this case, the occurrence of transient multimodality depends on the value of the initial state x_0 [7,15,17]. As an example, in Fig. 3 an evolution of a probability distribution is shown for $A=-3$, $B=2.3$, and $q=0.02$ and an initial distribution given by two Gaussian functions centered at $x_0=\pm 1.15$ and with widths $\sigma=0.002$. Because for such conditions $U(x)$ possesses two symmetrical flat regions, allowing for the appearance of transient bimodality [8,9,13], so effectively, in fact, one can observe the transient quadromodality. This example shows that also after crossing the second bifurcation curve $B=0.25A^2$ from region III, a system still exhibits evidence that it is near this region. However, since this transient multimodality is now a noise-induced phenomenon [9,10], the parameter space should be enlarged with the fourth axis of a noise intensity parameter q.

After this paper was sent for publication P. Colet, F. de Pasquale, and M. San Miguel published a paper [20] that deals with the relaxation from an unstable steady state for the same (generic for the subcritical pitchfork bifurcation discussed in that paper) potential as in the present example (16). They noticed the occurrence of transient trimodality too, and investigated it quantitative-

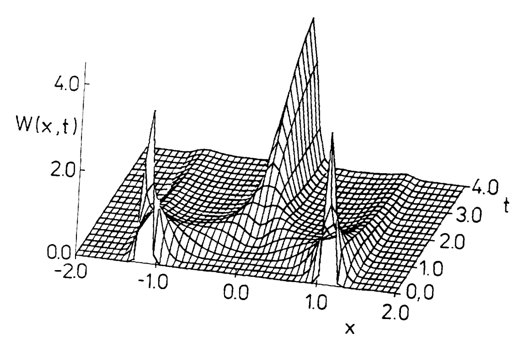

FIG. 3. Time evolution of the probability distribution for the potential (16) for $A=-3$, $B=2.3$, $q=0.02$, and the initial distribution given by two Gaussian functions centered at $x_0=\pm 1.15$ and both with a width $\sigma=0.002$.

ly. However, we cannot agree with one ascertainment of that article, namely, that far above the bifurcation point ($B \ll 0$ in our notation) the decay process for the subcritical pitchfork bifurcation ($A < 0$) is essentially the same as for the supercritical pitchfork bifurcation ($A > 0$), which means that for $B \ll 0$ the transient trimodality disappears. These two types of bifurcation deal with regions IV and II in Fig. 2, respectively, and the present considerations show analytically that the phenomenon of transient trimodality exists always (even for $B \ll 0$) in the whole region IV, though it may persist for a very short time.

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