Ubiquitous neutral stability of splay-phase states

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We find that the dynamics of certain dc-driven Josephson-junction arrays is peculiarly weak. The neutral stability of splay-phase states is far more common than previously suspected, even if the junctions are not identical. This has significant consequences for applications of these arrays.

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I. INTRODUCTION

Soon after the discovery of the Josephson effect [1], researchers realized the potential importance of large arrays of Josephson junctions for a variety of technological applications [2–5]. Of particular interest is the ability of these nonlinear oscillators to mutually phase lock or synchronize their oscillations [6]. While their technological interest has been the primary motivation over the past two decades, Josephson-junction arrays more recently have become an archetype in the field of dynamical systems theory, particularly in the study of many-degree-of-freedom systems [7–16].

One important class of Josephson-junction arrays—and the class that concerns us here—is the case of globally coupled junctions. This coupling, in which each element is coupled to all other elements with equal strength, arises quite naturally in many circuit configurations. Simulations of such arrays have uncovered peculiar periodic states called *splay-phase states* (originally termed "antiphase states" by Hadley and Beasley [8]). In this state, each junction has voltage oscillations with precisely the same wave form $V(t)$, but the oscillators are mutually phase shifted:

$$V_k(t)=V_0(t+kT/N) , \tag{1}$$

where all permutations of the relative phases are possible.

Splay-phase states can exist for arrays of N oscillators, for N arbitrarily large. This has significant consequences, since the existence of a single splay-phase state necessarily implies the coexistence of $(N-1)!$ symmetry-related states [given by all permutations of the indices in Eq. (1)]. This can lead to a phenomenon known as attractor crowding [17], where the array becomes increasingly sensitive to noise as N grows [10]. On the other hand, the coexistence of many attractors could be exploited as a memory element, provided one could reliably switch between attractors [18].

In addition to numerical evidence for splay-phase states [7,8,10], Aronson, Golubitsky, and Mallet-Paret [13] and Mirollo [19] have proven their existence for two particular array configurations. Splay-phase states have also been reported in other physical systems [19–25]: Experimental evidence of splay-phase states has been reported in a Nd:YAG (where YAG denotes yttrium-aluminum garnet) laser for $N=3$ modes [20,21], and in an electrical circuit with three oscillators [22]; theoretically, splay-phase states have been found both in models of multimode lasers (for $N=5$) [18,21] and arrays of solid-state lasers (for any N) [23].

Quite recently, this subject took an unexpected turn, when new discoveries surfaced showing that certain "typical" Josephson circuit have decidedly atypical dynamics [11,15]. Ordinarily, a given dynamical state is either stable or unstable: only for special parameter values will a state be at the crossover case of neutral stability. However, for Josephson arrays with pure resistive load and point-contact (zero-capacitance) junctions, it was argued that splay-phase states are *always* neutrally stable, in all N phase-space directions [11]. Such nongeneric behavior requires a deep underlying structure to the dynamics, and was argued [11] to be the result of a reversibility symmetry of the governing equations. However, Tsang and Schwartz then reported numerical evidence that if the resistive load was replaced by an inductor-capacitor load—which removes the reversibility symmetry—that *again* the splay-phase states were neutrally stable, though in "only" $N-2$ directions [15]. No explanation has been forwarded as to why this should be so, and it is an open question whether there might be a single explanation that would cover both types of array.

In this paper, we provide numerical evidence that neutral stability of splay-phase states is far more general than previously suspected. Our main evidence is based on *local* analysis of the splay-phase states. For several array variations, we search for splay-phase states and test their stability by computing the Floquet multipliers. We find neutrally stable splay-phase orbits for four cases. We also find evidence of unusual *global* structure of the phase-space dynamics, again related to neutral stability. In one case only do we find linearly stable splay-phase orbits, namely for arrays subject to a purely capacitive load.

II. SYSTEM AND EQUATIONS OF MOTION

All of the circuit configurations we consider are of the general class shown in Fig. 1. A constant bias current drives a series array of N Josephson junctions, which are coupled together by virtue of a parallel load. The equations of motion are [9]

$$\beta\ddot{\phi}_k+\dot{\phi}_k+\sin\phi_k+\dot{Q}=I_B , \tag{2a}$$

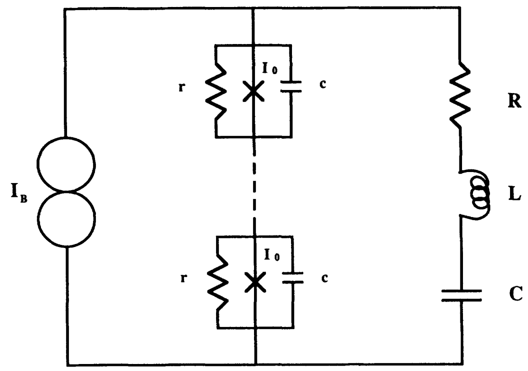

FIG. 1. Circuit schematic for the current biased series Josephson-junction array, with a parallel RLC load.

$$L\ddot{Q}+R\dot{Q}+(1/C)Q=\sum_{j=1}^{N}\dot{\phi}_j\ , \tag{2b}$$

where ϕ_k is the phase difference of the macroscopic wave function across the kth junction, $\dot{Q}$ is the load current, I_B is the bias current, R, L, and C are, respectively, the resistance, inductance, and capacitance of the load, and the overdot denotes differentiation with respect to time. The junction parameter $\beta=2eI_0^2cr^2/\hbar$, where I_0, r, and c are, respectively, the critical current, resistance, and capacitance of each junction, $\hbar$ is Planck's constant divided by 2π, and e is the electron charge. Equations (2a) and (2b) have been rendered in dimensionless form, with time measured in units of $\hbar/2erI_0$ and current in units of I_0. The first of these equations is a statement of current conservation; the second equates the voltage across the array to that across the load. Physically, the supercurrent through the kth junction is given by $\sin\phi_k$, and its voltage is $\dot{\phi}_k$.

In writing Eqs. (2) we have assumed that the Josephson junctions are identical. Almost all analyses to date have considered arrays of identical junctions, and most of what follows will be restricted to this case. We return briefly to this point in Sec. IV.

It is natural to divide this general class into various subcases, defined by two properties. First, the junction capacitance may be negligible, so that $\beta=0$: this is of course a singular limit of Eq. (2a), which is why we consider it as a case separate from $\beta\neq0$. Second, various of the load elements may be missing, which entails any one or more of the following: $R=0$, $L=0$, or $C=\infty$.

While these equations display a rich variety of complex behavior including chaos, our primary focus concerns the synchronization of the elements, with each element displaying periodic behavior. In particular, two kinds of periodic states—in phase and splay phase—have been most studied. In designing applications, it is desirable to know when such time-periodic states are attracting. For junctions having negligible capacitance ($\beta=0$), general results have been discovered for certain circuit configurations, which rule out attracting dynamics. First, for a purely resistive load [see Fig. 2(a)], neither in-phase nor splay-phase periodic orbits are attracting (in fact, simulations suggest that there are no periodic attractors of any kind). For the in-phase state, this result has been rigorously proven [11], and can be traced to a dynamical reversibility symmetry of the governing equations; for the splay-phase state, the proof assumes that the wave form has an extra symmetry, a symmetry which is consistent with numerics. A similar lack of asymptotic stability has been reported for splay-phase states for the inductor-capacitor load [see Fig. 2(b)] [15]. This latter result is based on numerical simulations, which also show that attracting in-phase states coexist with neutrally stable splay-phase orbits. For both kinds of load, there are additional global features which go well beyond these local-stability results, but such issues are not relevant to what follows.

What is peculiar about these results is not that the periodic splay-phase orbits are not attracting, but that they are neutrally stable. Typically, a periodic orbit is neutrally stable only in the tangent direction (i.e., along the trajectory). But for the cases cited above, the splay-phase states are neutrally stable in all N phase-space directions (R load) or in all but four of the $N+2$ phase-space directions (LC load), for N arbitrarily large.

In Sec. III, we consider five different array configurations. For each case, the computations are carried out in two steps: first, we search for a splay-phase state; second, we compute the Floquet multipliers for this orbit. Before turning to the results, we pause to describe some of the details of the numerical procedures.

Our search for a splay-phase state began with a first guess of initial conditions; a good choice was to pick the ϕ_j initially spaced at intervals of $2\pi/N$. The equations of motion were integrated until the first phase variable ϕ_1 advances by 2π. (The ϕ_j are angular variables and so are defined modulo 2π.) The Euclidean distance between this point and the initial point vanishes if the orbit is periodic. Somewhat remarkably, in most cases a periodic orbit was reached simply by integrating for a very long time. (In some cases, a scheme of successive approximations was used to find the orbit more efficiently.) Using this simple technique implies that we never find repelling or saddle-type periodic orbits. Nevertheless, there remains the possibility that the orbits are linearly neutrally stable, which is what we tested. We will return to this point in Sec. IV.

Once a periodic orbit is found, its stability is tested by looking at the dynamical equations linearized about this orbit. For example, if $(\{\phi_{k0}\},Q_0)$ represents the splay-phase solution, linearization of Eq. (1) yields

$$\beta\ddot{\eta}_k+\dot{\eta}_k+\cos\phi_{k0}\eta_k+\dot{q}=0\ , \tag{3a}$$

$$L\ddot{q}+R\dot{q}+(1/C)q=\sum_j\dot{\eta}_j\ , \tag{3b}$$

where $\eta_k=\phi_k-\phi_{k0}$ and $q=Q-Q_0$. This can be written as a system of $M=2N+2$ first-order equations of the form

$$\dot{\mathbf{X}}=\mathbf{A}(t)\mathbf{X}\ , \tag{4}$$

where $\mathbf{X}$ is an M-dimensional vector and $\mathbf{A}(t)$ is an $M\times M$ matrix with period T. A floquet solution $\mathbf{F}(t)$ of Eq. (4) is defined by the property

$$\mathbf{F}(t+T)=\mu\mathbf{F}(t) , \quad (5)$$

where T is the period of the underlying orbit. The (possibly complex) constant μ is called a Floquet multiplier. In general, there are M linearly independent Floquet solutions: stability of the orbit requires that none of the corresponding multipliers lie outside the unit disk. One can find the multipliers as follows. For the initial condition $\mathbf{X}(0)$, Eq. (4) is integrated for one period and the vector $\mathbf{X}(T)$ is used to form a column of a new matrix $\mathbf{E}$. This process is repeated for the M independent initial conditions $\mathbf{X}(0)=(1,0,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,0,\ldots,1)$. The eigenvalues of the $M\times M$ matrix $\mathbf{E}$ are then the Floquet multipliers [26].

An important check for the numerics comes from rigorously known results. Aronson, Golubitsky, and Mallet-Paret have proven the existence of splay-phase states for the cases $\beta\neq 0$ and either purely capacitive or purely resistive load. We measured how close an orbit was to being periodic by the Euclidean distance in phase space between successive piercings of the ϕ_1 plane (i.e., when $\phi_1=0$). Using a fourth-order Runge-Kutta integrator with fixed step size of 0.000 039 062 5, we found orbits which were periodic to within $\pm 10^{-7}$ in the Euclidean distance. The step size was chosen as a compromise between desired accuracy and computational expediency. The numerical accuracy for the Floquet multipliers was calibrated by using the fact that, for any periodic orbit, there must be (at least) one multiplier equal to $+1$, corresponding to the tangent direction of the periodic orbit.

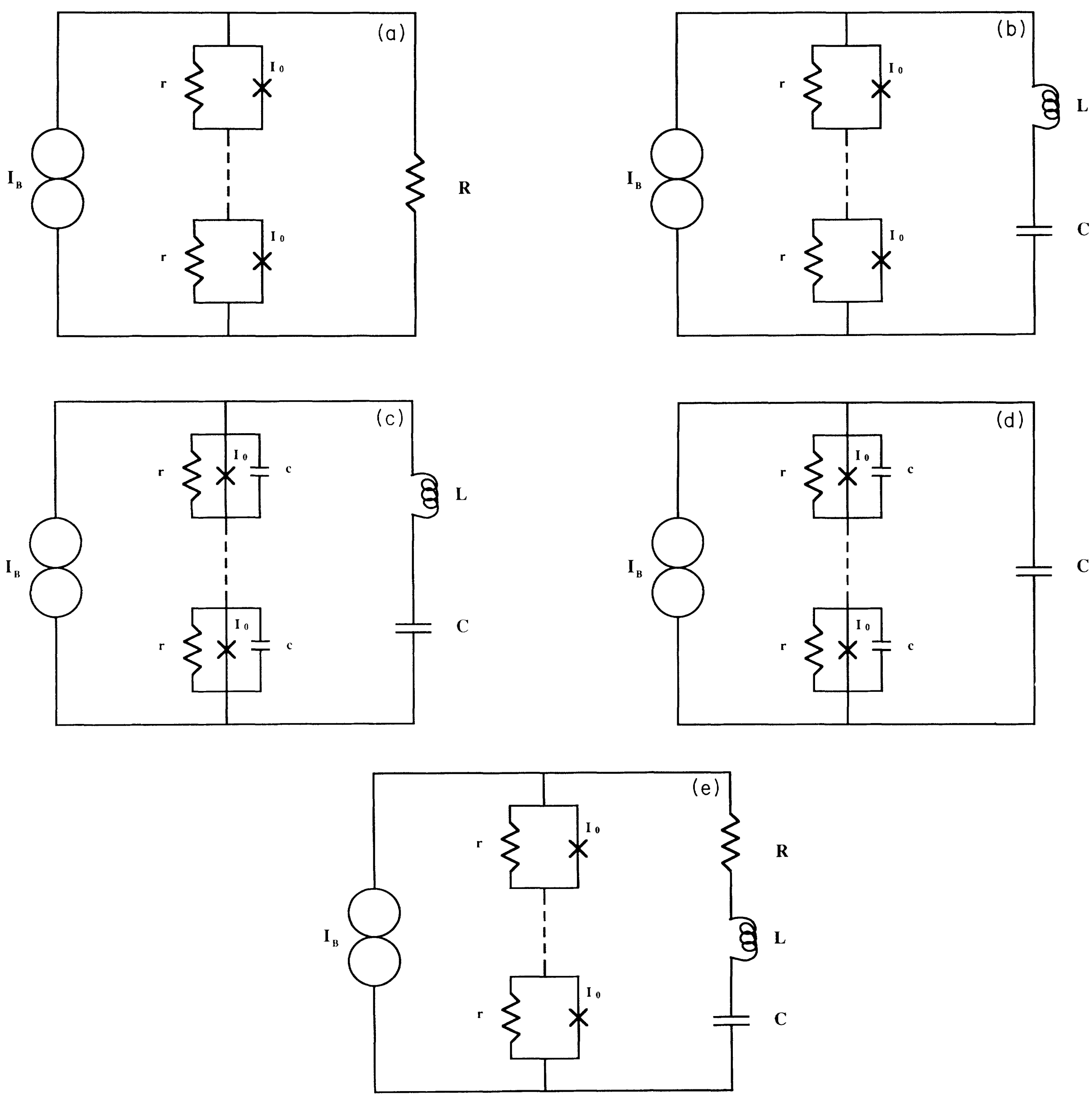

FIG. 2. Circuit schematics for various array configurations: (a) $\beta=0$, R load; (b) $\beta=0$, LC load; (c) $\beta\neq 0$, LC load; (d) $\beta\neq 0$, C load; (e) $\beta=0$, RLC load.

In all results quoted below, this multiplier was computed to be 1.0 ± 10^{-5}.

Owing to the finite resolution of our numerics, we can never find an "exactly" periodic orbit. We cannot rule out the possibility, therefore, that these orbits are quasiperiodic, or even mildly chaotic. These are important distinctions from a rigorous point of view, and resolving this issue would require far more careful numerics than we have done. (The task is subtle, since the orbits involved are neutrally stable, or very nearly so.) From a physical point of view these issues are less important: the difference between a periodic orbit and one which "just misses" by one part in 10^8 is negligible.

III. RESULTS

In each of the cases below, we tried various values of the load parameters and bias current. We considered arrays as large as $N=10$. Typical numerical results for the Floquet multipliers are listed in Table I.

Case I: $\beta=0$, LC load. The governing equations are [see Fig. 2(b)]

$$\dot{\phi}_k + \sin\phi_k + \dot{Q} = I_B \ , \tag{6a}$$

$$L\ddot{Q} + (1/C)Q = \sum_j \dot{\phi}_j \ . \tag{6b}$$

The dimension of the phase space is $N+2$. Tsang and Schwartz have reported numerical values for the multipliers [15]; our numbers are consistent with those values. In particular, there are $N-2$ multipliers equal to $+1$; the other four multipliers lie inside the unit disk. For $N=3$, the splay-phase orbit is (linearly) asymptotically stable.

Case II: $\beta>0$, LC load. The governing equations are [see Fig. 2(c)]

$$\beta\ddot{\phi}_k + \dot{\phi}_k + \sin\phi_k + \dot{Q} = I_B \ , \tag{7a}$$

$$L\ddot{Q} + (1/C)Q = \sum_j \dot{\phi}_j \ . \tag{7b}$$

The phase-space dimension is $2N+2$. For $N=3$, there is one multiplier equal to $+1$, and seven multipliers inside the unit disk, so the splay-phase state is a limit cycle. However, for $N>3$, we find $N-2$ multiplers equal to $+1$.

Case III: $\beta>0$, C load. The governing equation is [see Fig. 2(d)]

$$\beta\ddot{\phi}_k + \dot{\phi}_k + \sin\phi_k + C\sum_j \ddot{\phi}_j = I_B \ . \tag{8}$$

The phase-space dimension is $2N$. For $N=3,4,5$, we find just one multiplier equal to $+1$, with all others inside the unit disk. This is the *only* case where we found (linearly) stable splay-phase orbits for $N>3$.

Case IV: $\beta=0$, RCL load. The governing equations are [see Fig. 2(e)]

$$\dot{\phi}_k + \sin\phi_k + \dot{Q} = I_B \ , \tag{9a}$$

$$L\ddot{Q} + R\dot{Q} + (1/C)Q = \sum_j \dot{\phi}_j \ . \tag{9b}$$

The phase-space dimension is $N+2$. Qualitatively, the results here are the same as those for case I: we find $N-2$ multipliers equal to $+1$, with the four others inside the unit disk.

Case V: $\beta>0$, RLC load. This is the case depicted in Fig. 1. The governing equations are given by Eqs. (2). The phase-space dimension is $2N+2$. Qualitatively, the results here are the same as for case II: we find $N-2$ multipliers equal to $+1$, with the other $N+4$ multipliers inside the unit disk. Two of these other $N+4$ form a single complex-conjugate pair lying just inside the unit circle.

IV. DISCUSSION

Our numerical results show that, in all cases but one, the splay-phase states are linearly neutrally stable for $N>3$. Moreover, the degree of neutral stability—i.e., the number of multipliers equal to $+1$—increases linearly with array size N. The exceptional case is the purely capacitive load, where we located stable splay-phase limit cycles.

By measuring the Floquet multipliers, we have investigated only local properties of the dynamics. The two previous studies which motivated the present work each indicated interesting global structure as well [11,15]. In fact, our simulations indicate that global structure may well be present in these other array configurations. We base this conjecture on the ease with which we were able to locate periodic solutions. Typically, it was sufficient to let the dynamics evolve from initial conditions that were only approximately splay phase, since the subsequent evolution settled down to a periodic orbit. This is behavior typical of *attracting* orbits, and must somehow be reconciled with our Floquet analysis. One possibility is that the splay-phase orbits are linearly neutrally stable, but (nonlinearly) attracting. On the other hand, we were able to start with different initial conditions, and thereby settle down to different periodic orbits, which is consistent with the existence of a higher-dimensional invariant torus, foliated by a continuous family of periodic orbits. This echoes other recent results reported for globally-coupled-oscillator systems [11,15,16,23–25]. Whether or not these Josephson-junction arrays are ultimately found to have special global phase-space structure, in most practical instances the local stability is just as important, and the neutral stability of the splay-phase states has a number of ramifications, as we now discuss.

Attractor crowding is a phenomenon wherein the number of attractors increases factorially while the phase-space volume increases "only" exponentially with array size N. The result is that the distance between attractors in phase space diminishes with increasing N. As the attractors are crowded ever more closely, even small levels of noise can induce diffusive hopping of the system between attractors. This effect has been seen in studies of discrete time dynamical systems, and in simulations of Josephson-junction arrays [10,17]. If, however, the splay-phase states are not attractors, the underlying picture of attractor crowding must be modified. In particular, one can expect diffusive behavior even for small values of N. Whether or not there remain systematic effects as N grows large is an open question.

A second issue concerns the control of globally-coupled-dynamical systems. It has been suggested that the ability to select any desired splay-phase state can be viewed as a new type of multistate switch, in this case a switch having $(N-1)!$ states. (In numerical simulations of a multimode laser system, Otsuka [18] has shown one way selection could work.) If the splay-phase states are

TABLE I. Numerical values for the floquet multipliers. Asterisks denote the absolute magnitude of multipliers with complex values.

Case	N	R	L	C	I_B	β	Floquet multipliers	
I	4		2.0	0.125	2.1		1.000 000	1.000 000
							0.987 557*	0.987 557*
							0.033 708 1*	0.033 708 1*
I	4		2.0	0.125	2.0		1.000 005	0.999 999
							0.935 776*	0.935 776*
							0.028 403 6*	0.028 403 6*
I	4		2.0	0.125	1.9		1.000 015	0.999 990
							0.870 140*	0.870 140*
							0.023 514 9*	0.023 514 9*
I	10		5	0.02	1.9		1.000 022	1.000 014
							1.000 014	1.000 006
							0.999 998	0.999 992
							0.999 984	0.999 977
							0.870 218*	0.870 218*
							0.023 513*	0.023 513*
II	4		3.0	5.0	2.5	1.1	0.999 992	0.999 946
							0.985 275*	0.985 275*
							0.886 015	0.335 306*
							0.335 306*	0.101 048*
							0.101 048*	0.099 582 3
II	4		3.1	5.0	2.5	1.0	1.000 000	0.999 957
							0.984 178*	0.984 178*
							0.885 641	0.298 201*
							0.298 201*	0.079 966*
							0.079 966*	0.078 722
II	10		3.1	5.0	2.5	1.0	1.000 000	1.000 000
							1.000 000	1.000 000
							0.999 998	0.999 994
							0.999 943*	0.999 943*
							0.959 745*	0.959 745*
							0.952 550	0.287 833*
							0.287 833*	0.081 917 5*
							0.081 917 5*	0.078 723 1*
							0.078 723 1*	0.078 719 1
							0.078 718 8	0.078 718 7
							0.078 718 5	0.078 718 3
III	4			0.25	1.5	0.1	0.999 982	0.940 388
							0.713 993*	0.713 993*
							0.016 877 7	$<10^{-6}$
							$<10^{-6}$	$<10^{-6}$
III	5			0.2	1.5	0.1	1.000 00	0.966 842*
							0.966 842*	0.715 857*
							0.715 857*	0.016 894 4
							$<10^{-6}$	$<10^{-6}$
							$<10^{-6}$	$<10^{-6}$
IV	4	0.1	2.0	0.125	1.9		1.000 01	0.999 994
							0.874 533*	0.874 533*
							0.021 229*	0.021 229*
V	4	0.1	3.0	5.0	2.5	1.0	0.999 888	0.999 997
							0.983 835*	0.983 835*
							0.888 256	0.285 165*
							0.285 165*	0.080 026*
							0.080 026*	0.078 699 9

typically not attracting, however, then the dynamics would be fundamentally altered by external noise: in particular, a given dynamical state would have a finite lifetime, destroying the reliability of the system as a memory device.

A third point concerns the potential for analytic progress on these systems. Under ordinary circumstances, analytic headway for nonlinear systems is notoriously difficult, without some special properties on which to capitalize. Since the neutral stability of orbits is decidedly nongeneric behavior, there must be some deep explanation for it. Moreover, since it is common to many different circuit configurations, the origin of the underlying structure must be quite general, extending to still other Josephson array circuits not of the general class indicated in Fig. 1. This structure, in turn, may make the Josephson-junction arrays tractable, analogous to a Hamiltonian system being integrable. Some progress in this direction has been reported in the weak-coupling limit, based on averaging methods [25].

In addition to the work detailed in Sec. III, we have made preliminary observations on arrays of nonidentical junctions. In particular, we introduced a small spread of 10^{-4} in the β parameters in Eq. (9a), which serves to break the permutation symmetry of the underlying dynamics. Nevertheless, the neutral stability of (nearly) splay-phase orbits was preserved. Similar results were found in other cases as well. Of course, such levels of symmetry breaking are very small from the physical point of view—present tolerances in Josephson-junction fabrication are closer to 10%—but some subtleties arise in trying to find periodic orbits via simulations. Further work along these lines is in progress.

Finally, one can ask whether the neutral stability of splay-phase orbits holds in contexts other than that of Josephson-junction arrays. Recent work on a model for solid-state-laser arrays [23], for which the stability calculation can be done exactly, shows that the answer to this question is yes. Similar results have been reported in a certain class of globally-coupled phase oscillators [24]. This supports the intriguing conjecture that neutral stability of splay-phase states rests on some fairly general—and possibly simple—property of the underlying equations.

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