PHYSICAL REVIEW A VOLUME 45, NUMBER 11 1 JUNE 1992

Survival probability and chaos in an open quantum system

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(Received 18 December 1991)

We investigate the quantum survival probability function (SPF) in regular and chaotic systems coupled to a reservoir via a particular quantum-nondemolition coupling. We show that for certain initial conditions the averaged SPF is markedly different in these two kinds of systems. We also relate this SPF to the averaged off-diagonal elements of the density matrix in the pointer basis.

PACS number(s): 03.65.Bz, 05.30.−d, 05.40.+j

I. INTRODUCTION

Consider a total Hamiltonian given by

$$H_T = H + \sum_j \hbar\omega_j b_j^\dagger b_j + V_I \tag{1}$$

where H is any Hamiltonian describing a conservative system, $\sum_j \hbar\omega_j b_j^\dagger b_j$ describes a reservoir consisting of an infinite number of harmonic oscillators, and V_I is an operator coupling system and reservoir variables.

Previous authors have examined certain V_I satisfying $[V_I, H] = 0$ in the context of back-action evading and quantum-nondemolition schemes [1], as well as the decay of the off-diagonal elements in the pointer basis [2]. In this Brief Report we wish to consider interactions satisfying this last equation to determine what differences exist in the decay of the averaged survival probability function (SPF) and off-diagonal elements between chaotic and regular systems. For conservative chaotic systems the only function of the canonical variables that is guaranteed to commute with H is some function of H; we are thus led to consider interactions having H as the only system variable. Perhaps the simplest such interaction V_I is

$$V_I = \hbar H \sum_j [\kappa(\omega_j) b_j + \kappa^*(\omega_j) b_j^\dagger], \tag{2}$$

where $\kappa(\omega_j)$ is a c function.

In the hope of finding a quantum signature of classical chaos, many authors have examined the SPF, $P(t) \equiv \mathrm{Tr}[\rho(0)\rho(t)]$ where Tr denotes the trace and ρ is the density operator, both theoretically (e.g., [3, 4]) and experimentally [5]. Much of the theoretical work has been performed under the assumption that a pure state can be ascribed to the system under investigation, in which case $P(t) = |\langle\psi(0)|\psi(t)\rangle|^2$. Below we examine the averaged SPF for the open system described above and demonstrate that for certain initial conditions there is a marked difference between the regular and chaotic cases.

II. THE MASTER EQUATION AND SURVIVAL PROBABILITY FUNCTION

Using the Hamiltonian H_T, with V_I given by Eq. (2), we derive a master equation following the approach of Louisell [6]. We find

$$\frac{d\rho}{dt} = \frac{1}{\hbar i}[H,\rho] + \frac{\gamma kT}{\hbar}([H\rho, H] + [H, \rho H]), \tag{3}$$

where the spectral density of the reservoir is replaced by a continuous density $g(\omega)$; we denote Boltzmann's constant by k, and $\gamma > 0$ is the limit of $2\pi|\kappa(\omega)|^2 g(\omega)/\omega$ as $\omega \to 0^+$, and is assumed to be finite. Terms analogous to the Lamb shift have been neglected, and the temperature T is assumed high enough so that the usual approximations [6] are justified.

An equation similar to Eq. (3), differing only in the temperature dependence of the coupling term, was considered a number of years ago [7]. Recently, Milburn [8] proposed a modification of quantum mechanics that leads to an equation similar to Eq. (3) under certain approximations.

It is a simple matter to integrate Eq. (3) in the energy representation, and to then calculate $P(t) \equiv \mathrm{Tr}[\rho(0)\rho(t)]$. In what follows we restrict ourselves to cases where the initial state of the system is a linear combination of N energy eigenstates, $\sum_{n=1}^N c_n \mid E_n\rangle$. We will also assume that the system energies are nondegenerate.

Before we examine $P_N(t)$, where the subscript N reminds us of the restriction on the initial state, it is interesting to calculate the transform

$$\begin{aligned} S_N(E) &\equiv \mathrm{Re} \int_0^\infty e^{-iEt/\hbar}\mathrm{Tr}[\rho(0)\rho(t)]\,dt \\ &= \hbar\gamma kT \sum_{n,m=1}^N |c_n|^2|c_m|^2 \frac{(E_m - E_n)^2}{(\gamma kT)^2(E_m-E_n)^4 + (E_n - E_m - E)^2}. \end{aligned} \tag{4}$$

For pure states in the absence of a reservoir, $S_N(E)$ is given by $\pi\hbar\sum_{n,m=1}^N |c_n|^2|c_m|^2\delta(E_n - E_m - E)$, which characterizes a line spectrum with "lines" at the Bohr frequencies $(E_n - E_m)/\hbar$. An examination of Eq. (4) reveals that the effect of the reservoir is to broaden the spectral lines giving them a Lorentzian line shape with half width at half maximum given by $\gamma kT(E_m - E_n)^2$.

III. THE AVERAGED SURVIVAL PROBABILITY FUNCTION

Landau and Smorodinsky [9] and Wigner [10] initiated the concept of an ensemble of Hamiltonians. Pechukas [3], and recently Wilkie and Brumer [4], averaged $|\langle\psi(0)|\psi(t)\rangle|^2$ over initial conditions and Hamiltonian ensembles. We do the same for the SPF of our master equation (3).

The average over initial conditions, denoted by a subscript I, is performed by sampling from the distribution $c\,\delta(\sum_{n=1}^{N}|c_n|^2 - 1)$ where c is a normalization constant [3]. We obtain

$$\langle P_N(t)\rangle_I = \frac{2}{N+1} + \frac{1}{N(N+1)}\sum_{\substack{n,m=1\\ n\neq m}}^{N} \cos[(E_m - E_n)t/\hbar]\exp[-\gamma kT(E_m - E_n)^2 t/\hbar]. \tag{5}$$

Since much has been written about the decay of off-diagonal elements of the density matrix when a system is coupled to its environment ([7] and references therein), it is interesting to note that for a certain class of initial states it is possible to relate $\langle P_N(t)\rangle_I$ to the average of the off-diagonal elements in the energy representation. For an initial state of the form $|\psi\rangle = (N)^{-1/2}\sum_{n=1}^{N}|E_n\rangle$, we find, using Eq. (3),

$$C_N(t) \equiv \frac{1}{N(N-1)}\sum_{\substack{n,m=1\\ n\neq m}}^{N}\rho_{mn}(t) = \frac{N+1}{N(N-1)}[\langle P_N(t)\rangle_I - 2/(N+1)]. \tag{6}$$

Even for nondegenerate levels, the states $|E_n\rangle$ are defined only within a phase, but Eq. (6) holds for any such definition, as long as $\rho_{mn}(t)$ are referred to the same states $|E_n\rangle$ appearing in $|\psi\rangle$. We now employ the same averaging procedure used by Wilkie and Brumer [4], which we outline below in a form suitable to ensemble average our $C_N(t)$ and $\langle P_N(t)\rangle_I$.

A sequence of N energy eigenvalues $\{E_1, E_2, ..., E_N\}$, with $E_j > E_k$ if $j > k$, generates a set $\mathcal{S}$ of $N(N-1)/2$ spacings given by $\mathcal{S} = \{(E_j - E_k)|j > k\}$. The order of a spacing D is a function defined on $\mathcal{S}$ by $D(E_j - E_k) = j - k$. Clearly, $\mathcal{S}$ can be partitioned into $N-1$ sets each of whose elements have the same order, $\mathcal{S} = \bigcup_{n=1}^{N-1}\mathcal{P}_n$, where

$$\mathcal{P}_n = \{\Delta_1^{(n)}, \Delta_2^{(n)}, \ldots, \Delta_{N-n}^{(n)} \in \mathcal{S}|D(\Delta_j^{(n)}) = n\}.$$

Now suppose some function A is given as a sum of functions of the $\Delta_j^{(n)}$, and perhaps of other variables not depending on j or n, according to

$$A = \sum_{n=1}^{N-1}\sum_{j=1}^{N-n} a(\Delta_j^{(n)}). \tag{7}$$

The Hamiltonian ensemble average of A is then given by

$$\langle A\rangle_H \equiv \sum_{n=1}^{N-1}\sum_{j=1}^{N-n}\langle a(\Delta_j^{(n)})\rangle \tag{8}$$

where $\langle a(\Delta_j^{(n)})\rangle = \int_0^\infty p(n-1,E)a(E)\,dE$ and $p(n-1,E)$ is the nth nearest-neighbor spacing probability distribution [15].

An examination of Eq. (5) reveals that $\langle P_N(t)\rangle_I$ can be put in the form of A [see Eq. (7)] with

$$a(\Delta_j^{(n)}) = \frac{2}{N(N+1)}\left(\frac{2}{N-1} + \cos(\Delta_j^{(n)}t/\hbar) \times \exp[-\gamma kT(\Delta_j^{(n)})^2 t/\hbar]\right). \tag{9}$$

We may then calculate $\langle\langle P_N(t)\rangle_I\rangle_H$, using the technique just outlined, provided we have appropriate spacing distributions; $\langle C_N(t)\rangle_H$ then follows immediately from Eq. (6).

It is known [4] that $|\langle\psi(0)|\psi(t)\rangle|^2$, when averaged over initial conditions and Hamiltonian ensembles, does or does not fall below its long-time limit for spacing distributions which lead to level repulsion or level clustering respectively. We will now show that for the *special cases* considered below, this property continues to hold for $\langle\langle P_N(t)\rangle_I\rangle_H$ of our open system.

A. Regular systems

1. *Generic*

For the spacing distributions of generic regular systems we use [11]

$$p(n-1,E) = \left(\frac{E}{\langle\Delta E\rangle}\right)^{n-1}\frac{\exp(-E/\langle\Delta E\rangle)}{\langle\Delta E\rangle(n-1)!}$$

where $\langle\Delta E\rangle$ is the average nearest-neighbor spacing. We find

$$\langle\langle P_N^{\rm reg}(t)\rangle_I\rangle_H = \frac{2}{N+1} + \frac{2}{N(N+1)} \times \sum_{n=1}^{N-1}\frac{N-n}{\langle\Delta E\rangle^n}\left(\frac{\hbar}{2\gamma kTt}\right)^{n/2} \times \mathrm{Re}\left[e^{z^2/4}D_{-n}(z)\right] \tag{10}$$

where $z \equiv (\hbar/\langle\Delta E\rangle + it)/\sqrt{2\gamma kT\hbar t}$ and $D_n(z)$ are parabolic cylinder functions [12]. Here, and for the chaotic case treated below, we will consider the special case $N = 2$. We get

$$\langle\langle P_2^{\text{reg}}(t)\rangle_I\rangle_H = \frac{2}{3} + \frac{1}{6\langle\Delta E\rangle}\sqrt{\frac{\pi\hbar}{\gamma kTt}}\,\text{Re}\,w\!\left(\frac{t + i\hbar/\langle\Delta E\rangle}{2\sqrt{\gamma kT\hbar t}}\right)$$

where $w(u) \equiv \exp(-u^2)\,\text{erfc}(-iu)$ [12]. But $\text{Re}\,w(u)$ is non-negative in the first quadrant. Therefore, we conclude that $\langle\langle P_2^{\text{reg}}(t)\rangle_I\rangle_H$ does not fall below its long-time limit of 2/3. In Fig. 1, we plot this last quantity as a function of time using a solid line. (See caption for the parameters used.)

2. Harmonic oscillator

Although not generic, the harmonic oscillator is worthy of examination because a similar Hamiltonian to Eq. (1) with $H = \hbar\omega a^\dagger a$ may be used as a model for four-wave mixing [13] and is thus of physical interest. The energy spacing probability distributions for the harmonic oscillator are $p(n-1, E) = \delta(E - n\langle\Delta E\rangle)$. Note that although the harmonic oscillator is a (nongeneric) regular system, the spacing distributions lead to level repulsion; in this sense, the harmonic oscillator is more akin to chaotic systems (see below). And indeed, we now show that the averaged SPF falls below its long-time limit for all N. We obtain

$$\langle\langle P_N^{\text{HO}}(t)\rangle_I\rangle_H = \frac{2}{N+1} + \frac{2}{N(N+1)} \times \sum_{n=1}^{N-1}(N-n)\cos(n\langle\Delta E\rangle t/\hbar) \times \exp(-\gamma kTn^2\langle\Delta E\rangle^2 t/\hbar). \quad (11)$$

For $t = \pi\hbar/\langle\Delta E\rangle$, the averaged SPF given by Eq. (11) is less than its long-time limit of $2/(N+1)$ since the first term in the resulting alternating sum is negative and the absolute value of the terms is strictly decreasing.

B. Generic chaotic systems

No rigorous result for the level spacing distributions of chaotic systems is known. However, for Hamiltonians with time-reversal symmetry, the spacing statistics of the Gaussian orthogonal ensemble have been shown numerically to be valid for several systems [14]. Here we examine the averaged SPF with $N = 2$ for which a convenient approximation to the first-order spacing distribution is available in the form of the Wigner surmise [10],

$$p(0, E) = \frac{\pi E}{2\langle\Delta E\rangle^2}\exp\left(\frac{-\pi E^2}{4\langle\Delta E\rangle^2}\right).$$

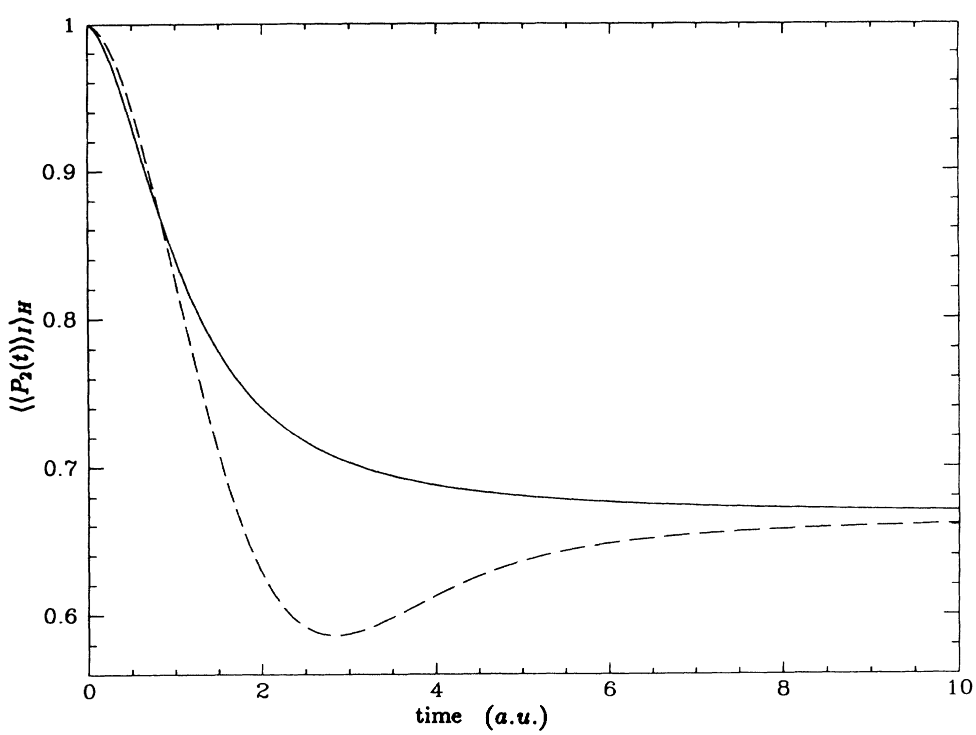

FIG. 1. The regular (solid line) and chaotic (dashed line) averaged survival probability function vs time for $N = 2$. The parameters used, all in atomic units, are $\langle\Delta E\rangle = 1$ and $\gamma kT = 5 \times 10^{-2}$.

Using this result, and defining $F(x)$ to be Dawson's integral [12] and $u \equiv \gamma kTt/\hbar + \pi/4\langle\Delta E\rangle^2$, we obtain

$$\langle\langle P_2^{\rm irreg}(t)\rangle_I\rangle_H = \frac{2}{3} + \frac{\pi}{12\langle\Delta E\rangle^2 u}\left(1 - \frac{t\,F[t/(2\hbar\sqrt{u})]}{\hbar\sqrt{u}}\right). \qquad (12)$$

This last quantity falls below its long-time limit of 2/3; in particular, for the time given implicitly by $t/[2\hbar\sqrt{u(t)}] = 1$, $1 - 2F(1) < 0$. In Fig. 1 we plot the expression appearing in Eq. (12) using a dashed line. For the sample parameters used (see figure caption), we see that the difference between the averaged SPF in regular and chaotic systems is quite substantial.

IV. DISCUSSION

It is impossible to isolate a system completely from its environment. For the interaction V_I discussed above, we have seen that the environment is inextricably intertwined with the system of interest; the exponentially decaying factor in Eq. (5), for example, cannot be taken outside the summation. We have found that significant differences may exist in the averaged SPF and off-diagonal elements in the pointer basis, although we expect that these differences may become less appreciable for values of γT much larger than that used in our sample calculations.

We are currently investigating actual physical systems for which the coupling to the reservoir may be modeled by an interaction of the form explored in this paper.

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[15] Note that $\langle a(\Delta_j^{(n)})\rangle$ is independent of j, so that we also have $\langle A\rangle_H = \sum_{n=1}^{N-1}(N-n)\langle a(\Delta_j^{(n)})\rangle$.