Quantum theory of correlated-emission lasers: Vacuum state for the mode of the relative phase and the relative amplitude

Ning Lu
Program Development Corporation, 300 Hamilton Avenue, Suite 409, White Plains, New York 10601

(Received 12 September 1991)

We present a comprehensive quantum theory of correlated-emission lasers (CEL's) in a combination-mode approach, which is valid for both the quantum-beat laser and the CEL with injected atomic coherence. We first show that, under certain conditions, the linear master equation of the CEL's breaks down into two uncoupled equations in the combination modes, which are linear combinations of two CEL modes. One combination mode that is above threshold represents the average phase and the average amplitude of two CEL modes (called the sum mode), whereas the other that is below threshold represents the relative phase and the relative amplitude of the two CEL modes (called the relative mode). When the relative mode is in its vacuum state, the normally ordered variances of two Hermitian operators corresponding, respectively, to the relative phase and the relative amplitude vanish. Through deriving a nonlinear master equation for the CEL with injected atomic coherence in the combination modes, we show how a set of proper initial atomic conditions can keep the relative mode in its vacuum state. We obtain the Glauber P functions, the mean photon numbers, and the photon-number variances in both the combination modes and the CEL modes, and find the mode pullings of the CEL's and the natural linewidth of the sum mode. We also show a correspondence between the vacuum state of the relative mode and the vanishing of the relative-phase diffusion coefficient.

PACS number(s): 42.50.Lc, 42.50.Ar

I. INTRODUCTION

It is well known that spontaneous-emission events in a laser contribute to the phase (as well as amplitude) fluctuations of the laser field. Recently, Scully [1] showed that, in a correlated-emission laser (CEL) in which V-type three-level atoms interact with two modes of cavity fields, it is possible to eliminate the spontaneous-emission fluctuations in the relative-phase angle of the two laser modes. The quantum-noise quenching in the CEL is achieved by creating atomic coherence between two upper lasing levels, which can be realized in a quantum-beat laser [2,3] and in a CEL with injected atomic coherence [4–6]. The potential application of the CEL's includes a ring-laser gyroscope [7–9] and a gravity-wave detector [10]. Some experiments have been carried out to verify the quantum-noise suppression in the CEL's [11,12].

The linear master equation of the CEL's has been studied extensively. Usually, a Fokker-Planck equation [2,4] in the Glauber-Sudarshan P representation [13,14], or a corresponding Langevin noise equation [15–17], is derived. The quantum-noise quenching is indicated by the vanishing of the P's diffusion coefficient for the relative phase at its phase-locking point under certain conditions. Schleich, Scully, and von Garssen [16] further found that fluctuations in the relative amplitude are also noise quenched. Very recently, Lu [5] has shown that the normally ordered variances of two Hermitian operators which correspond to the relative phase and relative amplitude, respectively, can vanish. Because of the use of the Hermitian operators, Ref. [5] puts the quantum-noise quenching on a more rigorous basis.

There have been some preliminary studies on the nonlinear theories of the CEL's. Bergou, Orszag, and Scully [3] derived a nonlinear master equation of the quantum-beat laser using a "dressed-atom–dressed-mode" approach. Zaheer and Zubairy [6] obtained a nonlinear master equation of the CEL with injected atomic coherence in the original CEL modes. Both papers showed that the vanishing of the relative-phase-diffusion coefficient persists even above threshold.

In this paper we present a comprehensive nonlinear theory of the CEL's, which treats both the quantum-beat laser and CEL with injected atomic coherence in a unified approach. We obtain the Glauber P function and natural linewidth of the CEL's, and calculate the mean photon numbers, mode pullings, and photon-number variances for each of the two CEL modes. We find a "relative mode" and a "sum mode," both of which are linear combination of the two CEL modes. The real and imaginary parts of the annihilation operator of the relative (sum) mode are two Hermitian operators corresponding to the relative (average) amplitude and relative (average) phase, respectively. In a balanced case in which the conditions for the two CEL models are symmetric, the relative and sum modes are decoupled from each other, and the relative (sum) mode is below (above) threshold. While the sum mode gives the total intensity, mode pullings, and natural linewidth (for the average phase) of the CEL's, the relative mode determines the degree of the quantum-noise suppression in the CEL's. The relative mode can be in its vacuum state when a CEL is above threshold. When and only when the relative mode is in its vacuum state, the P's diffusion coefficient for the relative phase vanishes.

The paper is organized as follows. In Sec. II we show how the linear master equation of the CEL's can be separated into two uncoupled equations in terms of two combination modes (called the relative and sum modes) and show the significance of the vacuum state of the relative mode. In Sec. III we derive a nonlinear master equation for the CEL with injected atomic coherence in the combination modes and show how a set of proper initial atomic conditions keeps the relative mode in its vacuum state. In Sec. IV, using a Fokker-Planck approach, we study the operational and noise properties of the CEL's in the combination models. In Sec. V we obtain the Glauber P function and photon statistics in the original CEL modes. Section VI presents a connection between the vacuum state of the relative mode and the vanishing of the relative-phase-diffusion coefficient. Finally, Sec. VII is a summary.

II. SEPARATION OF THE LINEAR MASTER EQUATION

A CEL consists of V-type three-level atoms interacting with two modes of radiation fields in a doubly resonant cavity. The atomic levels of a V-type three-level atom are denoted by $|a\rangle$, $|b\rangle$, and $|c\rangle$ with atomic energies $\hbar\omega_a$, $\hbar\omega_b$, and $\hbar\omega_c$, respectively. Two upper levels $|a\rangle$ and $|b\rangle$ have the same parity, and the lower level $|c\rangle$, with opposite parity, is coupled to the upper levels $|a\rangle$ and $|b\rangle$ by direct electric dipole transitions. The atomic transition a-c (b-c) is near resonant with a cavity field of the bare-cavity frequency Ω_1 (Ω_2) and the oscillation frequency ν_1 (ν_2). The emission from the two upper levels $|a\rangle$ and $|b\rangle$ are correlated as a result of the existence of an atomic coherence between the two levels. Such an atomic coherence can be generated by basically two methods: One method is using a microwave field to couple the two upper levels $|a\rangle$ and $|b\rangle$ (called a quantum-beta laser [2,3]), and the other is preparing the active atoms initially in a coherent superposition of the two upper levels $|a\rangle$ and $|b\rangle$ and then injecting the atoms into the cavity (called a CEL with injected atomic coherence [4–6]). For both the quantum-beat laser and the CEL with injected atomic coherence, the linear master equation for the reduced-field-density operator ρ is of the form [1] (after the contributions from the cavity losses and mode pullings [5] are included)

$$\dot{\rho}=\sum_{l,k=1}^{2}\tfrac{1}{2}\alpha_{lk}e^{i(k-l)\theta}(a_l^\dagger\rho a_k-\rho a_k a_l^\dagger)+\text{H.c.}+\sum_{l=1}^{2}\mathcal{L}(a_l;\gamma_l,\Omega_l-\nu_l)\rho\ . \quad (2.1)$$

Here a_l and $a_l^\dagger$ are the field annihilation and creation operators of mode l ($l=1,2$), respectively; α_{11} and α_{22} are the (self) linear-gain coefficients for modes 1 and 2, respectively; α_{12} and α_{21} are two cross linear-gain coefficients; θ is a time-independent phase constant, and it determines the phase difference of the two CEL modes in steady state;

$$\mathcal{L}(A;\gamma,\Omega-\nu)\rho\equiv\tfrac{1}{2}\gamma(2A\rho A^\dagger-A^\dagger A\rho-\rho A^\dagger A)-i(\Omega-\nu)[A^\dagger A,\rho] \quad (2.2)$$

describes the cavity loss (with a rate γ) and the mode pulling (amount $\nu-\Omega$) of a field mode A ($[A,A^\dagger]=1$); and γ_l is the cavity loss rate of mode l. For definiteness we assume in the following that $\text{Re}\alpha_{12}>0$. [When $\text{Re}\alpha_{12}<0$ in Eq. (2.1), we let $\alpha_{12}\rightarrow-\alpha_{12}$ and $\theta\rightarrow\theta+\pi$.]

The explicit expressions for α_{lk} are model dependent. For the quantum-beat laser, they were derived in Ref. [2]. For the CEL with injected atomic coherence, we assume that the jth atom is injected into the cavity at time t_j with the initial conditions

$$\begin{aligned}&\rho_{aa}^j(t_j)=\rho_{aa}\ ,\quad \rho_{bb}^j(t_j)=\rho_{bb}\ ,\\&\rho_{ab}^j(t_j)=\rho_{ba}^j(t_j)^*\\&\qquad=\bar{\rho}_{ab}e^{-i(\nu_1-\nu_2)t_j}\ ,\quad \rho_{cc}^j(t_j)=0\ ,\end{aligned} \quad (2.3)$$

in the Schrödinger picture. Note that, when they enter the laser cavity, all atoms have the same initial populations for either of the two upper levels $|a\rangle$ and $|b\rangle$, but have different initial phases between the two upper levels in general (unless $\nu_1=\nu_2$). The initial atomic conditions (2.3) can be generated by injecting the active atoms into the laser cavity through an atomic beam. For each atom, before it reaches the laser cavity, we first pump the atom to one of three transition levels ($|a\rangle$, $|b\rangle$, or $|c\rangle$) and then use two preparation fields of frequencies ν_1 and ν_2, respectively, to create an atomic coherence between levels $|a\rangle$ and $|b\rangle$. It can be shown that this kind of preparation scheme generates the desired form of the initial atomic coherence prescribed in (2.3). The detailed description and proof are similar to those for a two-level laser with injected atomic coherence, which have been discussed in detail in Ref. [18]. Under the initial atomic conditions (2.3), the coefficients α_{lk} are found to be [5]

$$\alpha_{11}=\frac{2r_a g_1^2\rho_{aa}}{\Gamma_a(\Gamma_{ac}+i\Delta_1)}\ , \quad (2.4a)$$

$$\alpha_{22}=\frac{2r_a g_2^2\rho_{bb}}{\Gamma_b(\Gamma_{bc}+i\Delta_2)}\ , \quad (2.4b)$$

$$\alpha_{12}=\frac{2r_a g_1 g_2|\bar{\rho}_{ab}|}{[\Gamma_{ab}+i(\Delta_1-\Delta_2)](\Gamma_{ac}+i\Delta_1)}\ , \quad (2.4c)$$

$$\alpha_{21}=\frac{2r_a g_1 g_2|\bar{\rho}_{ba}|}{[\Gamma_{ab}-i(\Delta_1-\Delta_2)](\Gamma_{bc}+i\Delta_2)}\ . \quad (2.4d)$$

Here r_a is the atomic injection rate; Γ_μ is the decay rate of the atomic level $|\mu\rangle$; $\Gamma_{\mu\mu'}=\frac{1}{2}(\Gamma_\mu+\Gamma_{\mu'})$ is the decay rate of the atomic coherence between levels $|\mu\rangle$ and $|\mu'\rangle$ ($\mu,\mu'=a,b,c$); g_1 and g_2 are the atom-field coupling constants for the a-c and b-c transitions, respectively; and $\Delta_1=\omega_{ac}-\nu_1=\omega_a-\omega_c-\nu_1$ and $\Delta_2=\omega_{bc}-\nu_2=\omega_b-\omega_c-\nu_2$ are the atom-field detunings for the a-c and b-c transitions, respectively. Also in Eq. (2.1), $\theta=\arg\bar{\rho}_{ab}$. Note that, by allowing arbitrary atomic decay rates and arbitrary initial atomic conditions, the expressions (2.4) are more general than those derived in Ref. [4] [see Eqs. (11)

of Ref. [4]].

In a balanced case in which

$$\alpha_{11}=\alpha_{22}\ ,\quad \alpha_{12}=\alpha_{21}\ , \tag{2.5a}$$

$$\gamma_1=\gamma_2\equiv\gamma\ , \tag{2.5b}$$

the linear master equation (2.1) is symmetric about modes a_1 and a_2, and we must have equal mode pullings [5],

$$\Omega_1-\nu_1=\Omega_2-\nu_2\ . \tag{2.6}$$

For the quantum-beat laser, conditions (2.5a) can be met approximately when $g_1=g_2$, $\Delta_1=\Delta_2=\pm\frac{1}{2}\Omega$ (here Ω is the Rabi frequency of the microwave field coupling levels $|a\rangle$ and $|b\rangle$), and Ω is much larger than the atomic decay rates. For the CEL with injected atomic coherence, conditions (2.5a) can be satisfied by choosing

$$\Delta_1=\Delta_2\equiv\Delta\ , \tag{2.7a}$$

$$\Gamma_a=\Gamma_b\ , \tag{2.7b}$$

$$g_1^2\rho_{aa}=g_2^2\rho_{bb}\ . \tag{2.7c}$$

We now introduce two combination modes,

$$B_1=a_1e^{-i\psi_1}\cos\kappa+a_2e^{-i\psi_2}\sin\kappa\ , \tag{2.8a}$$

$$B_2=-a_1e^{-i\psi_1}\sin\kappa+a_2e^{-i\psi_2}\cos\kappa\ , \tag{2.8b}$$

where ψ_1 and ψ_2 are two real constants satisfying $\psi_1-\psi_2=\theta$. The angle κ indicates the degree of mode mixing, and to be specific, we restrict κ to $0<\kappa<\pi/2$. Note that new operators in the combination modes have the properties

$$[B_1,B_1^\dagger]=[B_2,B_2^\dagger]=1\ ,\quad [B_1,B_2]=[B_1,B_2^\dagger]=0\ . \tag{2.9}$$

In other words, the combination modes B_1 and B_2 are two independent field modes.

Using Eqs. (2.5), (2.6), and (2.8), it is easy to find that, in the balanced case, the linear master equation (2.1) becomes

$$\begin{aligned}\dot\rho=&\sum_{l=1}^{2}\tfrac{1}{2}[\alpha_{11}-(-1)^l\alpha_{12}\sin(2\kappa)](B_l^\dagger\rho B_l-\rho B_lB_l^\dagger)\\&+\tfrac{1}{2}\alpha_{12}\cos(2\kappa)(B_1^\dagger\rho B_2-\rho B_2B_1^\dagger+B_2^\dagger\rho B_1-\rho B_1B_2^\dagger)\\&+\text{H.c.}+\sum_{l=1}^{2}\mathcal{L}(B_l;\gamma,\Omega_1-\nu_1)\rho\ ,\end{aligned} \tag{2.10}$$

in the combination modes. When (and only when) $\kappa=\pi/4$, the linear master equation (2.10) separates into two uncoupled equations in the combination modes,

$$\begin{aligned}\dot\rho^{(l)}=&\tfrac{1}{2}[\alpha_{11}-(-1)^l\alpha_{12}](B_l^\dagger\rho^{(l)}B_l-\rho^{(l)}B_lB_l^\dagger)\\&+\text{H.c.}+\mathcal{L}(B_l;\gamma,\Omega_1-\nu_1)\rho^{(l)}\ ,\quad l=1,2\ ,\end{aligned} \tag{2.11}$$

by letting

$$\rho=\rho^{(1)}\rho^{(2)}\ . \tag{2.12}$$

From now on we set $\kappa=\pi/4$. It is easy to see from Eqs. (2.11) that the threshold condition for the combination mode B_1 (and thus for the CEL's) is

$$\text{Re}(\alpha_{11}+\alpha_{12})\geq\gamma\ . \tag{2.13}$$

When $\text{Re}(\alpha_{11}-\alpha_{12})<\gamma$ (we assume this henceforth), the other combination mode B_2 is below threshold and Eq. (2.11) is adequate for the mode B_2. The steady-state solution of the density operator $\rho^{(2)}$ can be obtained by using the detailed balance of the photon-number flux [19,20], yielding

$$\rho^{(2)}_{n_2,m_2}=\delta_{n_2,m_2}\left[1-\frac{\text{Re}(\alpha_{11}-\alpha_{12})}{\gamma}\right]\left[\frac{\text{Re}(\alpha_{11}-\alpha_{12})}{\gamma}\right]^{n_2}\ . \tag{2.14}$$

The below-threshold operation of the mode B_2 implies null amplitude for it, $\langle B_2\rangle=0$. Solving a_1 and a_2 from Eqs. (2.8) with $\kappa=\pi/4$, we arrive at

$$a_1=(B_1-B_2)e^{i\psi_1}/\sqrt{2}\ , \tag{2.15a}$$

$$a_2=(B_1+B_2)e^{i\psi_2}/\sqrt{2}\ . \tag{2.15b}$$

Consequently, we have $\langle a_1\rangle e^{-i\psi_1}=\langle a_2\rangle e^{-i\psi_2}$. While both phases of the CEL modes a_1 and a_2 can be arbitrary, their difference is locked to the phase constant θ since

$$\langle a_1\rangle=\langle a_2\rangle e^{i\theta}\ . \tag{2.16}$$

In addition, both amplitudes $|\langle a_1\rangle|$ and $|\langle a_2\rangle|$ are equal. In other words, the below-threshold operation of the mode B_2 is equivalent to the lockings of the relative phase (to θ) and of the relative amplitude (to 0) for the CEL modes a_1 and a_2.

Such a relative-phase locking is closely related to the quantum-noise reduction in the relative phase of the CEL's, since both are due to the presence of the cross-gain coefficients α_{12} and α_{21}. The relative-phase locking also gives the nonvanishing of the beat signal,

$$\begin{aligned}&e^{i(\nu_1-\nu_2)t}\langle a_1^\dagger a_2\rangle\\&\quad=\tfrac{1}{2}e^{i(\nu_1-\nu_2)t+i(\psi_2-\psi_1)}\langle B_1^\dagger B_1-B_2^\dagger B_2\rangle\neq0\ ,\end{aligned} \tag{2.17}$$

in the long-time limit, since the mode B_1 is above threshold and thus $\langle B_1^\dagger B_1\rangle\gg\langle B_2^\dagger B_2\rangle$. Note that the diffusion coefficient for the relative phase, which is important for any short-time measurement of the beat signal [12], does not vanish unless the mode B_2 is in its vacuum state. We will show this connection directly in Sec. VI. In the rest of this section, we reveal the physical meaning of the combination modes B_1 and B_2.

The Hermitian operators corresponding to the relative phase and relative amplitude have been found in Ref. [5] [see Eqs. (5.1) of Ref. [5]]. They correspond to, respectively, the imaginary and real parts of the annihilation operator B_2:

$$\begin{aligned}B_\phi\equiv&\frac{1}{2i}(a_1e^{-i\psi_1}-a_1^\dagger e^{i\psi_1})-\frac{1}{2i}(a_2e^{-i\psi_2}-a_2^\dagger e^{i\psi_2})\\=&-\sqrt{2}\,\text{Im}B_2\ ,\end{aligned} \tag{2.18a}$$

$$B_r \equiv \tfrac{1}{2}(a_1 e^{-i\psi_1}+a_1^\dagger e^{i\psi_1})-\tfrac{1}{2}(a_2 e^{-i\psi_2}+a_2^\dagger e^{i\psi_2})$$
$$=-\sqrt{2}\,\mathrm{Re}B_2 \; . \tag{2.18b}$$

Similarly, the Hermitian operator corresponding to the average phase (amplitude) is proportional to the imaginary (real) part of the annihilation operator B_1:

$$B_\Phi \equiv \frac{1}{4i}(a_1 e^{-i\psi_1}-a_1^\dagger e^{i\psi_1})+\frac{1}{4i}(a_2 e^{-i\psi_2}-a_2^\dagger e^{i\psi_2})$$
$$=\mathrm{Im}(B_1/\sqrt{2}) \; , \tag{2.19a}$$

$$B_R \equiv \tfrac{1}{4}(a_1 e^{-i\psi_1}+a_1^\dagger e^{i\psi_1})+\tfrac{1}{4}(a_2 e^{-i\psi_2}+a_2^\dagger e^{i\psi_2})$$
$$=\mathrm{Re}(B_1/\sqrt{2}) \; . \tag{2.19b}$$

Because of these correspondences, we call the combination mode B_2 the "relative mode" and mode B_1 the "sum mode." The variances of the Hermitian operators B_ϕ and B_r are found to be

$$\langle (\Delta B_{\phi,r})^2\rangle = \tfrac{1}{2}+\langle :(\Delta B_{\phi,r})^2:\rangle$$
$$=\tfrac{1}{2}+\langle B_2^\dagger B_2\rangle$$
$$=\frac{1}{2}+\frac{\mathrm{Re}(\alpha_{11}-\alpha_{12})}{\gamma-\mathrm{Re}(\alpha_{11}-\alpha_{12})} \; . \tag{2.20}$$

Here $\frac{1}{2}$ is the vacuum noise levels of the Hermitian operators B_ϕ and B_r, and the last equality is obtained after using the photon-number distribution (2.14) for the mode B_2. Equation (2.20) shows that the variances of the Hermitian operators B_ϕ and B_r do not drop to their vacuum noise levels unless the relative mode B_2 is in its vacuum state.

Under a further condition of maximum correlation, $\mathrm{Re}\alpha_{11}=\mathrm{Re}\alpha_{12}$ or

$$\alpha_{11}=\alpha_{12} \tag{2.21}$$

[i.e., when all α's are equal; see Eqs. (2.5a)], we find from Eq. (2.14) that the relative mode B_2 is in its vacuum state,

$$\rho^{(2)}_{n_2,m_2}=\delta_{n_2,m_2}\delta_{n_2,0} \; . \tag{2.22}$$

Consequently, $\langle B_2^\dagger B_2\rangle=0$, and the variances of B_ϕ and B_r are at their vacuum-noise levels,

$$\langle (\Delta B_{\phi,r})^2\rangle=\tfrac{1}{2} \; ; \tag{2.23}$$

i.e., we have quantum-noise quenching (down to the vacuum-noise level) for the Hermitian operators B_ϕ and B_r. For the CEL with injected atomic coherence, condition (2.21) is satisfied when, besides the conditions (2.7), we further have a full atomic coherence

$$|\bar{\rho}_{ab}|=\sqrt{\rho_{aa}\rho_{bb}} \; . \tag{2.24}$$

Equation (2.23) agrees with the result obtained in Ref. [5] by using the linear master equation (2.1) and conditions (2.5), (2.6), and (2.21) directly.

III. CEL WITH INJECTED ATOMIC COHERENCE: NONLINEAR MASTER EQUATION

We have studied in Sec. II the linear master equation (2.1) of CEL's, in particular, the properties of the relative mode B_2, which is below threshold. In order to study the properties of the sum mode B_1, which is above threshold, we have to include nonlinear saturation effects on the sum mode B_1. In this section we derive a nonlinear master equation for the CEL with injected atomic coherence in the combination modes B_1 and B_2. In Sec. IV we investigate the properties of the sum mode B_1.

Since the CEL with injected atomic coherence is a coherently pumped, two-mode three-level laser, we use a quantum theory of coherently pumped lasers. This theory was developed in Ref. [18] for a two-level single-mode laser by generalizing the Scully-Lamb laser theory to a form suitable for a coherently pumped laser. The generalization was made by treating the interaction of the laser field with many injected atoms simultaneously: The new theory started from an equation of motion for the *total* density operator of all atoms and the field. The equation of motion for a reduced density operator should be obtained by tracing the rest of atomic and/or field variables on both sides of the starting equation. Two main equations of motion were obtained through this approach in Ref. [18]. Expanding the theory of Ref. [18] to treat the CEL with injected atomic coherence, we give the two main equations of motion in the following. One is for the reduced-field-density operator ρ in the interaction picture,

$$\dot{\rho}=-i\sum_j \Theta(t-t_j)\mathrm{Tr}_{A^j}[\tilde{V}_j,\rho_j^f]+\sum_{l=1}^{2}\mathcal{L}(a_l;\gamma_l,\Omega_l-\nu_l)\rho \; . \tag{3.1}$$

The other is for the reduced-density operator ρ_j^f for the jth atom and fields in the interaction picture,

$$\dot{\rho}_j^f=-i\Theta(t-t_j)[\tilde{V}_j,\rho_j^f]-\tfrac{1}{2}(\Gamma^j\rho_j^f+\rho_j^f\Gamma^j) \; , \tag{3.2}$$

as in the Scully-Lamb theory of lasers. Here t_j is the injection time of the jth atom (assumed to be random), $\Theta(t-t_j)$ is the unit step function [$\Theta(t-t_j)=1$ for $t\geq t_j$ and $\Theta(t-t_j)=0$ for $t<t_j$], and $\Gamma^j=\sum_{\mu=a,b,c}\Gamma_\mu|\mu^j\rangle\langle\mu^j|$ is the decay operator for the jth atom (assuming that all three levels decay to other lower lying levels). In addition, $\tilde{V}_j$ is the interaction Hamiltonian of the jth atom with the laser fields in the interaction picture,

$$\tilde{V}_j=g_1|a^j\rangle\langle c^j|a_1 e^{i(\Delta_1 t-\omega_{ac}t_j)}$$
$$+g_2|b^j\rangle\langle c^j|a_2 e^{i(\Delta_2 t-\omega_{bc}t_j)}+\mathrm{H.c.} \tag{3.3}$$

Equations (3.1) and (3.2) can be solved in the good-cavity limit, in which the cavity-loss rates γ_l are much smaller than the atomic decay rates Γ_μ ($\mu=a,b,c$), so that during atomic lifetimes Γ_μ^{-1}, ρ remains approximately constant. The summation over the injected atoms in Eqs. (3.1) can be replaced by an integral over injection time t_j up to time t, i.e., $\sum_j\Theta(t-t_j)\to r_a\int_{-\infty}^{t}dt_j$. The self- and cross-linear-gain coefficients α_{lk} in Eqs. (2.4) were thus calculated in Ref. [5] up to $g_l g_k$ with the initial conditions (2.3).

We now wish to obtain the nonlinear master equation under a set of conditions which lead to the decoupling of the two combination modes B_1 and B_2. Based on the linear-theory results in Sec. II, we continue to consider the balanced case. We assume conditions (2.5b), (2.7a), and (2.7b),

$$g_1=g_2\equiv g \tag{3.4a}$$

and

$$\rho_{aa}=\rho_{bb}=|\bar{\rho}_{ab}|=\tfrac{1}{2} . \tag{3.4b}$$

Note that conditions (3.4) replace the less restrictive conditions (2.7c) and (2.24). Let

$$|a^j\rangle=(|A^j\rangle+|B^j\rangle)e^{-i\psi_1}/\sqrt{2} , \tag{3.5a}$$

$$|b^j\rangle=(|A^j\rangle-|B^j\rangle)e^{-i\psi_2-i(\nu_1-\nu_2)t_j}/\sqrt{2} . \tag{3.5b}$$

We then find that the interaction Hamiltonian (3.3) becomes

$$\tilde{V}_j=g(|A^j\rangle\langle c^j|B_1-|B^j\rangle\langle c^j|B_2)\times e^{i\Delta(t-t_j)-i\nu_1 t_j}+\text{H.c.} , \tag{3.6}$$

in terms of the combination modes and the new atomic states. It follows from Eqs. (2.3), (3.4b), (3.5), and $\psi_1-\psi_2=\theta=\arg\bar{\rho}_{ab}$ that the initial atomic conditions are simply

$$\rho^j_{AA}(t_j)=1 , \quad \rho^j_{BB}(t_j)=\rho^j_{AB}(t_j)=\rho^j_{BA}(t_j)^*=\rho^j_{cc}(t_j)=0 , \tag{3.7}$$

in terms of the new atomic states; i.e., only the state $|A^j\rangle$ is excited initially.

There are two ways to calculate the nonlinear master equation. One is to conduct a two-mode calculation by using the interaction Hamiltonian (3.6) and initial atomic conditions (3.7), which is presented in the Appendix. The other is to realize that there is no gain for the relative mode B_2 (since its upper state $|B^j\rangle$ is not coupled to the other mode B_1) and that the mode B_2 is coupled to a loss reservoir (at zero temperature) only. Thus the relative mode B_2 is always in its vacuum state,

$$\rho^{(2)}_{n_2,m_2}(t)=\delta_{n_2,m_2}\delta_{n_2,0} . \tag{3.8}$$

Since the mode B_2 is in its vacuum state, the interaction Hamiltonian (3.6) can be simplified to

$$\tilde{V}_j=g|A^j\rangle\langle c^j|B_1 e^{i\Delta(t-t_j)-i\nu_1 t_j}+\text{H.c.}\equiv\tilde{V}^{(1)}_j . \tag{3.9}$$

Substituting Eqs. (2.12), (2.5b), (2.6), and (3.9) into Eq. (3.1) and letting

$$\rho^f_j=\rho^{(1)}_j\rho^{(2)} , \tag{3.10}$$

we obtain two uncoupled equations:

$$\dot{\rho}^{(1)}=-i\sum_j\Theta(t-t_j)\text{Tr}_{A^j}[\tilde{V}^{(1)}_j,\rho^{(1)}_j]+\mathcal{L}(B_1;\gamma,\Omega_1-\nu_1)\rho^{(1)} , \tag{3.11a}$$

$$\dot{\rho}^{(2)}=\mathcal{L}(B_2;\gamma,\Omega_1-\nu_1)\rho^{(2)} . \tag{3.11b}$$

Note that the solution (3.8) of the relative mode satisfies its master equation (3.11b), as it should. The equation of motion for the reduced-density operator $\rho^{(1)}_j$ of the jth atom and mode B_1 in the interaction picture is found after substituting Eqs. (2.12), (2.7b), and (3.10) into Eq. (3.2),

$$\dot{\rho}^{(1)}_j=-i\Theta(t-t_j)[\tilde{V}^{(1)}_j,\rho^{(1)}_j]-\tfrac{1}{2}(\Gamma^j\rho^{(1)}_j+\rho^{(1)}_j\Gamma^j) , \tag{3.12}$$

where the decay operator Γ^j can be written as $\Gamma^j=\Gamma_a(|A^j\rangle\langle A^j|+|B^j\rangle\langle B^j|)+\Gamma_c|c^j\rangle\langle c^j|$ in the new atomic bases. It is clear from Eq. (3.12) that the coherently pumped, two-mode three-level laser problem has been reduced to a simple incoherently pumped (to the state $|A^j\rangle$), one-mode two-level laser problem. Thus, in terms of the second method, we immediately know the answer from the Scully-Lamb theory of lasers [19,20]. For equal decay rates $\Gamma_a=\Gamma_c\equiv\Gamma$, the master equation for the sum mode B_1 is

$$\dot{\rho}^{(1)}_{nm}=-\tfrac{1}{2}\alpha[n+m+2+i(n-m)\delta+(n-m)^2g^2/\Gamma^2]\xi^{-1}_{nm}\rho^{(1)}_{nm}+\alpha\sqrt{nm}\,\xi^{-1}_{n-1,m-1}\rho^{(1)}_{n-1,m-1}+\gamma\sqrt{(n+1)(m+1)}\rho^{(1)}_{n+1,m+1}-\tfrac{1}{2}\gamma(n+m)\rho^{(1)}_{nm}-i(\Omega_1-\nu_1)(n-m)\rho^{(1)}_{nm} , \tag{3.13}$$

with

$$\xi_{nm}=1+\frac{\beta}{2\alpha}(n+m+2)+\frac{\beta^2}{16\alpha^2}(1+\delta)^2(n-m)^2 . \tag{3.14}$$

Here $\alpha=2r_a g^2/(\Gamma^2+\Delta^2)$, $\beta=8r_a g^4/(\Gamma^2+\Delta^2)^2$, and $\delta=\Delta/\Gamma$ are the usual linear gain, saturation parameter, and normalized detuning, respectively. Note that α here corresponds to $\text{Re}(\alpha_{11}+\alpha_{12})=2\,\text{Re}\alpha_{11}$ in Sec. II. The two methods, one presented in this section and the other presented in the Appendix, give the same master equation (3.13). It should be noted that, by neglecting nonresonant dressed states (dressed by the driving microwave field), Bergou, Orszag, and Scully [3] have obtained equations similar to (2.12), (3.8), and (3.13) in the quantum-beat laser [21]. Thus the results in the following sections are valid for both the CEL with the injected atomic coherence and the quantum-beat laser.

IV. OPERATION OF THE CEL'S AND THE P FUNCTIONS IN THE COMBINATION MODES

We denote $|\mathcal{E}_1\rangle|\mathcal{E}_2\rangle$ as the coherent state of the combination modes B_1 and B_2, $B_l|\mathcal{E}_1\rangle|\mathcal{E}_2\rangle=\mathcal{E}_l|\mathcal{E}_1\rangle|\mathcal{E}_2\rangle$

($l=1,2$). We now convert two master equations (3.11b) and (3.13) into two Fokker-Planck equations in the Glauber-Sudarshan P representation [13,14]. Using

$$\rho^{(l)}=\int P^{(l)}(\mathcal{E}_l)|\mathcal{E}_l\rangle\langle\mathcal{E}_l|d^2\mathcal{E}_l\ ,\quad l=1,2\ , \tag{4.1}$$

we obtain the Fokker-Planck equations in the P representation,

$$\frac{\partial}{\partial t}P^{(l)}(\mathcal{E}_l,t)=\left[-\frac{\partial}{\partial\mathcal{E}_l}d_{\mathcal{E}_l}+\frac{\partial^2}{\partial\mathcal{E}_l\partial\mathcal{E}_l^*}D_{\mathcal{E}_l^*\mathcal{E}_l}+\frac{\partial^2}{\partial\mathcal{E}_l^2}D_{\mathcal{E}_l\mathcal{E}_l}+\text{c.c.}\right]P^{(l)}(\mathcal{E}_l,t)\ ,\quad l=1,2\ . \tag{4.2}$$

For the relative mode B_2, the drift and diffusion coefficients are derived easily from Eq. (3.11b),

$$d_{\mathcal{E}_2}=\mathcal{E}_2[-\tfrac{1}{2}\gamma+i(\nu_1-\Omega_1)]\ , \tag{4.3}$$

$$D_{\mathcal{E}_2^*\mathcal{E}_2}=D_{\mathcal{E}_2\mathcal{E}_2}=0\ . \tag{4.4}$$

With such drift and diffusion coefficients, the steady-state solution for $P^{(2)}$ is simply a δ function which peaks at $\mathcal{E}_2=0$,

$$P^{(2)}(\mathcal{E}_2)=\delta(\mathcal{E}_2)\ . \tag{4.5}$$

For the sum mode B_1, the drift and diffusion coefficients can be derived from Eq. (3.13) under the assumption that the average photon number in the mode is much larger than unity. The derivation has been presented in Ref. [18] for more general initial atomic conditions. Specializing to the pumping situation here [pump to the state $|A^j\rangle$ only; see Eq. (3.7)], the drift coefficient is

$$d_{\mathcal{E}_1}=\frac{\mathcal{E}_1}{2}\left[\frac{\alpha(1-i\delta)}{1+|\mathcal{E}_1|^2\beta/\alpha}-\gamma+2i(\nu_1-\Omega_1)\right]\ , \tag{4.6}$$

and the diffusion coefficients are

$$D_{\mathcal{E}_1^*\mathcal{E}_1}=\frac{4\alpha+\beta(1+\delta^2)|\mathcal{E}_1|^2}{8(1+|\mathcal{E}_1|^2\beta/\alpha)}-\frac{\beta|\mathcal{E}_1|^2}{4(1+|\mathcal{E}_1|^2\beta/\alpha)^2}\ , \tag{4.7a}$$

$$D_{\mathcal{E}_1\mathcal{E}_1}=-\frac{\beta(1+\delta^2)\mathcal{E}_1^2}{8(1+|\mathcal{E}_1|^2\beta/\alpha)}-\frac{\beta(1-i\delta)\mathcal{E}_1^2}{4(1+|\mathcal{E}_1|^2\beta/\alpha)^2}\ . \tag{4.7b}$$

In order to study the intensity and phase properties of the sum mode B_1, we rewrite the Fokker-Planck equation (4.2) for the sum mode B_1 in terms of intensity and phase variables I and Ψ through the relation $\mathcal{E}_1=\sqrt{I}e^{i\Psi}$,

$$\frac{\partial}{\partial t}P^{(1)}(I,\Psi,t)=\left[-\frac{\partial}{\partial I}d_I-\frac{\partial}{\partial\Psi}d_\Psi+\frac{\partial^2}{\partial I^2}D_{II}+\frac{\partial^2}{\partial\Psi^2}D_{\Psi\Psi}+2\frac{\partial^2}{\partial I\,\partial\Psi}D_{I\Psi}\right]P^{(1)}(I,\Psi,t)\ , \tag{4.8}$$

where

$$d_I=\left[\frac{\alpha}{1+I\beta/\alpha}-\gamma\right]I\ , \tag{4.9a}$$

$$d_\Psi=\nu_1-\Omega_1-\frac{\alpha\delta}{2(1+I\beta/\alpha)} \tag{4.9b}$$

are the intensity- and phase-drift coefficients, respectively, and

$$D_{II}=2I[D_{\mathcal{E}_1^*\mathcal{E}_1}+\text{Re}(D_{\mathcal{E}_1\mathcal{E}_1}e^{-i2\Psi})]=\frac{\alpha I}{(1+I\beta/\alpha)^2}\ , \tag{4.10a}$$

$$D_{\Psi\Psi}=[D_{\mathcal{E}_1^*\mathcal{E}_1}-\text{Re}(D_{\mathcal{E}_1\mathcal{E}_1}e^{-i2\Psi})]/2I=\frac{\alpha+\frac{1}{2}I\beta(1+\delta^2)}{4I(1+I\beta/\alpha)}\ , \tag{4.10b}$$

$$D_{I\Psi}=\text{Im}(D_{\mathcal{E}\mathcal{E}}e^{-i2\Psi})=\frac{\beta I\delta}{4(1+I\beta/\alpha)^2} \tag{4.10c}$$

are the intensity-, phase-, and cross-diffusion coefficients, respectively.

Above threshold $\alpha>\gamma$, the sum mode B_1 first builds up spontaneously in the cavity and then reaches its steady-state value. Recall that we are considering the case in which the steady-state mean photon number $\langle\hat{N}_1\rangle=I_0$ is much larger than unity, where $\hat{N}_1=B_1^\dagger B_1$ is the photon-number operator for the sum mode B_1. Consequently, it is easy to show that I_0 satisfies the "semiclassical" equation

$$d_I(I_0)=0\ . \tag{4.11}$$

Such an I_0 is also the position at which the P function peaks in steady state. Substitution of Eqs. (4.9a) into Eqs. (4.11) gives the mean photon number in the sum mode B_1,

$$I_0=\frac{\alpha}{\gamma}\left[\frac{\alpha-\gamma}{\beta}\right]\ . \tag{4.12}$$

Solution (4.12) is stable, since the "locking strength"

$$A_I\equiv\frac{\partial d_I(I_0)}{\partial I}=-\frac{\gamma}{\alpha}(\alpha-\gamma) \tag{4.13}$$

is negative ($A_I<0$). Another "semiclassical" equation $d_\Psi(I_0)=0$ leads to the mode pulling $\nu_1-\Omega_1=\frac{1}{2}\gamma\delta$, where use has been made of Eq. (4.12). The CEL oscillation frequencies are found with the help of Eq. (2.6),

$$\nu_1=\frac{\Gamma\Omega_1+\frac{1}{2}\gamma\omega_{ac}}{\Gamma+\frac{1}{2}\gamma}\ , \tag{4.14a}$$

$$\nu_2=\frac{\Gamma\Omega_2+\frac{1}{2}\gamma\omega_{bc}}{\Gamma+\frac{1}{2}\gamma}\ . \tag{4.14b}$$

In steady state the diffusion coefficients take their values at $I=I_0$. Making use of Eq. (4.12), we find from Eqs. (4.10) the steady-state intensity- and phase-diffusion coefficients

$$D_{II}(I_0)=I_0\gamma\frac{\gamma}{\alpha}\ , \tag{4.15a}$$

$$D_{\Psi\Psi}(I_0)=\frac{\alpha+\gamma+(\alpha-\gamma)\delta^2}{8I_0}\ . \tag{4.15b}$$

Expression (4.15b) reduces to the result of Ref. [22] when $\delta=0$. The quantity $D_{\Psi\Psi}(I_0)$ is half the natural linewidth [23] of the CEL's in the sum mode B_1; i.e., it is half width at half maximum for the Fourier transform of the stationary two-time correlation function $\langle B_1^\dagger(t)B_1(0)\rangle$. Equation (4.15b) shows that the detuning δ increases phase fluctuations and the natural linewidth and that the effect of the detuning δ becomes significant when a CEL (or an ordinary laser) is far above threshold ($\alpha\gg\gamma$).

Unlike the phase-diffusion coefficient $D_{\Psi\Psi}(I_0)$, the intensity-diffusion coefficient $D_{II}(I_0)$ itself represents only a part of intensity fluctuations. We calculate the photon-number variance in the remaining of this section. Since the drift coefficients (4.9) and diffusion coefficients (4.10) are independent of the phase variable Ψ, the steady-state solution of the Fokker-Planck equation (4.8) must be Ψ independent too. Consequently, it satisfies the equation

$$\frac{\partial}{\partial I}\left[d_I-\frac{\partial}{\partial I}D_{II}\right]P^{(1)}(I)=0\ . \tag{4.16}$$

The detailed-balance solution of Eq. (4.16) is

$$P^{(1)}(I)=\frac{C}{D_{II}(I)}\exp\left[\int_0^I\frac{d_I(x)}{D_{II}(x)}dx\right] \tag{4.17a}$$

$$=\frac{C}{D_{II}(I)}\exp\left[\left[1-\frac{\gamma}{\alpha}\right]I+\left[1-\frac{2\gamma}{\alpha}\right]\frac{\beta I^2}{2\alpha}-\frac{\gamma\beta^2I^3}{3\alpha^3}\right]$$

$$\equiv f(I)\ , \tag{4.17b}$$

where C is a normalization constant, and Eq. (4.17b) is obtained after using Eqs. (4.9a) and (4.10a). As an approximation, we now expand d_I and D_{II} in Eq. (4.17a) around $I=I_0$ up to first and zeroth order in $\delta I=I-I_0$, respectively, and obtain the linearized steady-state solution

$$P^{(1)}(I)=\frac{1}{\pi}\left[\frac{|A_I|}{2\pi D_{II}(I_0)}\right]^{1/2}\exp\left[-\frac{|A_I|(I-I_0)^2}{2D_{II}(I_0)}\right]\ . \tag{4.18}$$

As a function of I, $P^{(1)}(I)$ is a Gaussian distribution peaked at $I=I_0$ with a "variance" [by Eqs. (4.13) and (4.15a)]

$$\langle(\delta I)^2\rangle=D_{II}(I_0)/|A_I|=\alpha/\beta\ . \tag{4.19}$$

Since the P function is a normal-ordering function, the photon-number variance is found by using Eqs. (4.19) and (4.12),

$$\langle(\Delta\hat{N}_1)^2\rangle=\langle(B_1^\dagger)^2B_1^2\rangle-\langle\hat{N}_1\rangle^2+\langle\hat{N}_1\rangle$$

$$=\langle(\delta I)^2\rangle+I_0=\frac{\alpha I_0}{\alpha-\gamma}\ , \tag{4.20}$$

which is the same as that of an ordinary laser with a linear gain α, a cavity loss γ, and a mean photon number I_0.

V. P FUNCTION IN THE CEL MODES

We denote $|v_1;v_2\rangle$ as the two-mode coherent state for the CEL modes a_1 and a_2, $a_l|v_1;v_2\rangle=v_l|v_1;v_2\rangle$ ($l=1,2$). Now let a_l act on the B_l's coherent state $|\mathcal{E}_1\rangle|\mathcal{E}_2\rangle$ and use Eqs. (2.15); we find that $|\mathcal{E}_1\rangle|\mathcal{E}_2\rangle$ is also the coherent state of a_1 and a_2, with the relation

$$|\mathcal{E}_1\rangle|\mathcal{E}_2\rangle=|(\mathcal{E}_1-\mathcal{E}_2)e^{i\psi_1}/\sqrt{2};(\mathcal{E}_1+\mathcal{E}_2)e^{i\psi_2}/\sqrt{2}\rangle\ . \tag{5.1}$$

Representing the density matrix ρ of the fields by the Glauber-Sudarshan P function in the CEL modes, we have

$$\rho=\int\int P(v_1,v_2)|v_1;v_2\rangle\langle v_1;v_2|d^2v_1d^2v_2\ . \tag{5.2}$$

By making a linear coordinate transformation [c-number version of Eqs. (2.15)]

$$v_1=(\mathcal{E}_1-\mathcal{E}_2)e^{i\psi_1}/\sqrt{2}\ , \tag{5.3a}$$

$$v_2=(\mathcal{E}_1+\mathcal{E}_2)e^{i\psi_2}/\sqrt{2}\ , \tag{5.3b}$$

in Eq. (5.2), and using Eq. (5.1), we arrive at

$$\rho=\int\int P(v_1,v_2)|\mathcal{E}_1\rangle|\mathcal{E}_2\rangle\langle\mathcal{E}_1|\langle\mathcal{E}_2|d^2\mathcal{E}_1d^2\mathcal{E}_2\ , \tag{5.4}$$

where two sets of coordinates $\mathcal{E}_1,\mathcal{E}_2$ and v_1,v_2 are related through Eqs. (5.3). Substituting Eqs. (2.12) and (4.1) into Eq. (5.4), we find the steady-state P function in the CEL modes,

$$P(v_1,v_2)=P^{(1)}(\mathcal{E}_1)P^{(2)}(\mathcal{E}_2) \tag{5.5a}$$

$$=f(|\mathcal{E}_1|^2)\delta(\mathcal{E}_2) \tag{5.5b}$$

$$=2f(\tfrac{1}{2}|v_1e^{-i\psi_1}+v_2e^{-i\psi_2}|^2)\delta(v_2e^{-i\psi_2}-v_1e^{-i\psi_1})$$

$$=2f(2|v_l|^2)\delta(v_2e^{-i\psi_2}-v_1e^{-i\psi_1})\ ,\quad l=1\text{ or }2\ , \tag{5.5c}$$

where Eq. (5.5b) is obtained after using Eqs. (4.5) and (4.17).

It follows from Eq. (5.5c) that

$$\begin{aligned}\langle (a_l^\dagger)^m a_l^m \rangle &= \int\int P(v_1,v_2)|v_l|^{2m} d^2v_1 d^2v_2 \\ &= \int 2f(2|v_l|^2)|v_l|^{2m} d^2v_l \\ &= \langle (B_1^\dagger)^m B_1^m \rangle /2^m \ , \quad l=1,2 \ , \end{aligned} \tag{5.6}$$

which can be obtained alternatively by using Eqs. (2.15) and the fact that the mode B_2 is in its vacuum state [see Eq. (3.8)]. Let $\hat{n}_l = a_l^\dagger a_l$; we get the mean photon number in each of the CEL modes a_l,

$$\begin{aligned}\langle \hat{n}_l \rangle &= \langle a_l^\dagger a_l \rangle \\ &= \tfrac{1}{2}\langle B_1^\dagger B_1 \rangle = \tfrac{1}{2} I_0 \ , \quad l=1,2 \ . \end{aligned} \tag{5.7}$$

This result can be understood by noting Eq. (4.5): $\delta(\mathcal{E}_2)$ means $\mathcal{E}_2=0$. Thus Eqs. (5.3) reduce to

$$v_l = \mathcal{E}_1 e^{i\psi_l}/\sqrt{2} \ , \quad l=1,2 \ . \tag{5.8}$$

Consequently, the photon number in the mode a_l is only half of that in the mode B_1. The normalized photon-number variance in each of the CEL modes a_l is

$$\begin{aligned}\langle (\Delta\hat{n}_l)^2 \rangle &= \langle (a_l^\dagger)^2 a_l^2 \rangle - \langle \hat{n}_l \rangle^2 + \langle \hat{n}_l \rangle \\ &= \tfrac{1}{4}\langle (B_1^\dagger)^2 B_1^2 \rangle - \tfrac{1}{4} I_0^2 + \tfrac{1}{2} I_0 \\ &= \langle \hat{n}_l \rangle \frac{\alpha - \frac{1}{2}\gamma}{\alpha-\gamma} \ , \quad l=1,2 \ , \end{aligned} \tag{5.9}$$

which approaches the mean photon number $\langle \hat{n}_l \rangle$ when the ratio α/γ increases to infinity. It follows from Eqs. (5.9) and (4.20) that the normalized photon-number variance in each of the CEL modes a_l is smaller than that in the sum mode B_1,

$$\frac{\langle (\Delta\hat{n}_l)^2 \rangle}{\langle \hat{n}_l \rangle} < \frac{\langle (\Delta\hat{N}_1)^2 \rangle}{\langle \hat{N}_1 \rangle} = \frac{\alpha}{\alpha-\gamma} \ . \tag{5.10}$$

Recall that, for an ordinary laser with a linear gain α and a cavity loss γ, its normalized photon-number variance is also $\alpha/(\alpha-\gamma)$. Thus, compared with an ordinary laser, we find quantum-noise reduction in each of the CEL modes. The relative photon-number fluctuations can also be represented by the Mandel Q parameter [24] ($\hat{n}$ is a photon-number operator):

$$Q = \frac{\langle (\Delta\hat{n})^2 \rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle} = \langle \hat{n} \rangle [g^{(2)}(0) - 1] \ , \tag{5.11}$$

which has a lower bound, -1, corresponding to a pure number state. Here $g^{(2)}(0)$ is the normalized second-order correlation function. In terms of the Mandel Q parameter and $g^2(0)$, the fluctuations in the CEL modes a_l and those in the sum mode B_1 are related by the following simple relations ($l=1,2$):

$$\begin{aligned} Q_{a_l} &= \tfrac{1}{2} Q_{B_1} \ , \\ g_{a_l}^{(2)}(0) &= g_{B_1}^{(2)}(0) \ . \end{aligned} \tag{5.12}$$

VI. VACUUM STATE OF B_2 AND THE VANISHING OF THE RELATIVE-PHASE-DIFFUSION COEFFICIENT

In most of previous studies on the CEL's, the focus was on the vanishing of the relative-phase-diffusion coefficient under certain conditions. Those studies began with the introduction of the polar coordinates in the CEL modes,

$$v_l = r_l e^{i\phi_l} \ , \quad l=1,2 \ , \tag{6.1}$$

and of the average and relative phases of the two CEL fields,

$$\Phi = \tfrac{1}{2}(\phi_1+\phi_2) \ , \tag{6.2a}$$

$$\phi = \phi_1 - \phi_2 \ . \tag{6.2b}$$

Then the master equations of CEL's were converted [through Eq. (5.2)] into the Fokker-Planck equations expressed in terms of the variables Φ, ϕ, r_1, and r_2,

$$\begin{aligned}\frac{\partial}{\partial t} P(\Phi,\phi,r_1,r_2,t) = \Bigg[&-\frac{\partial}{\partial\Phi} d_\Phi - \frac{\partial}{\partial\phi} d_\phi + \frac{\partial^2}{\partial\Phi^2} D_{\Phi\Phi} + \frac{\partial^2}{\partial\phi^2} D_{\phi\phi} + 2\frac{\partial^2}{\partial\Phi\,\partial\phi} D_{\Phi\phi} + \frac{2}{r_1 r_2}\frac{\partial^2}{\partial r_1 \partial r_2} r_1 r_2 D_{r_1 r_2} \\ &+ \sum_{l=1}^{2} \left[-\frac{1}{r_l}\frac{\partial}{\partial r_l} r_l d_{r_l} + \frac{1}{r_l}\frac{\partial^2}{\partial r_l^2} r_l D_{r_l r_l} + \frac{2}{r_l}\frac{\partial^2}{\partial r_l \partial\Phi} r_l D_{r_l\Phi} + \frac{2}{r_l}\frac{\partial^2}{\partial r_l \partial\phi} r_l D_{r_l\phi} \right] \Bigg] P(\Phi,\phi,r_1,r_2,t) \ . \end{aligned} \tag{6.3}$$

Last, it was shown that, under certain conditions, the diffusion coefficient $D_{\phi\phi}$ in the relative phase can vanish in steady state,

$$D_{\phi\phi}^0 = 0 \ , \tag{6.4}$$

where the superscript 0 indicates a steady-state value.

It is the purpose of this section to present a correspondence between the vacuum state of the relative mode B_2 [Eq. (2.22) or (3.8)] and the vanishing of the relative-phase-diffusion coefficient [Eq. (6.4)]. The following study is quite general, and the only assumptions made are (i) that the sum mode B_1 and relative mode B_2 are uncoupled, i.e., Eq. (2.12) and, consequently, Eq. (5.5a) holds; and (ii) that the relative mode B_2 is below threshold.

Using Eqs. (5.3), we obtain the relations

$$\frac{\partial^2}{\partial\mathcal{E}_l\partial\mathcal{E}_l^*}=\frac{1}{2}\left[\frac{\partial^2}{\partial v_1\partial v_1^*}+\frac{\partial^2}{\partial v_2\partial v_2^*}\right]-(-1)^l\frac{1}{2}\left[e^{i(\psi_1-\psi_2)}\frac{\partial^2}{\partial v_1\partial v_2^*}+\text{c.c.}\right], \tag{6.5a}$$

$$\frac{\partial^2}{\partial\mathcal{E}_l^2}=\frac{1}{2}\left[e^{i2\psi_1}\frac{\partial^2}{\partial v_1^2}+e^{i2\psi_2}\frac{\partial^2}{\partial v_2^2}-2(-1)^l e^{i(\psi_1-\psi_2)}\frac{\partial^2}{\partial v_1\partial v_2}\right], \quad l=1,2\ . \tag{6.5b}$$

Following Eqs. (6.2), we find that

$$\frac{\partial^2}{\partial\phi_l^2}=\frac{1}{4}\frac{\partial^2}{\partial\Phi^2}+\frac{\partial^2}{\partial\phi^2}-(-1)^l\frac{\partial^2}{\partial\Phi\,\partial\phi}, \quad l=1,2\ , \tag{6.6a}$$

$$\frac{\partial^2}{\partial\phi_1\partial\phi_2}=\frac{1}{4}\frac{\partial^2}{\partial\Phi^2}-\frac{\partial^2}{\partial\phi^2}\ . \tag{6.6b}$$

Substituting Eqs. (5.5a), (4.2), (6.5), and (6.6) into the left-hand side of Eq. (6.3), we find that the average- and relative-phase-diffusion coefficients are related to the diffusion coefficients in the combination modes by the relations

$$D_{\Phi\Phi}=\frac{1}{32}\sum_{l=1}^{2}\left[\frac{1}{r_1^2}+\frac{1}{r_2^2}-(-1)^l\frac{2\cos(\phi+\psi_2-\psi_1)}{r_1r_2}\right]D_{\mathcal{E}_l^*\mathcal{E}_l}$$
$$-\frac{1}{32}e^{-i2(\phi_1-\psi_1)}\sum_{l=1}^{2}\left[\frac{1}{r_1^2}+\frac{e^{i2(\phi+\psi_2-\psi_1)}}{r_2^2}-(-1)^l\frac{2e^{i(\phi+\psi_2-\psi_1)}}{r_1r_2}\right]D_{\mathcal{E}_l\mathcal{E}_l}+\text{c.c.}\ , \tag{6.7a}$$

$$D_{\phi\phi}=\frac{1}{8}\sum_{l=1}^{2}\left[\frac{1}{r_1^2}+\frac{1}{r_2^2}+(-1)^l\frac{2\cos(\phi+\psi_2-\psi_1)}{r_1r_2}\right]D_{\mathcal{E}_l^*\mathcal{E}_l}$$
$$-\frac{1}{8}e^{-i2(\phi_1-\psi_1)}\sum_{l=1}^{2}\left[\frac{1}{r_1^2}+\frac{e^{i2(\phi+\psi_2-\psi_1)}}{r_2^2}+(-1)^l\frac{2e^{i(\phi+\psi_2-\psi_1)}}{r_1r_2}\right]D_{\mathcal{E}_l\mathcal{E}_l}+\text{c.c.}\ . \tag{6.7b}$$

Since the relative mode B_2 is below threshold, we have Eqs. (2.16) and (5.8). In terms of the amplitudes and phases, they become [by Eqs. (6.1) and (6.2b)]

$$r_1^0=r_2^0=\sqrt{I_0/2}\ ,$$
$$\phi^0=\theta=\psi_1-\psi_2\ ,\quad \Psi=\phi_1-\psi_1\ . \tag{6.8}$$

Substituting Eqs. (6.8) into Eqs. (6.7), we find that the steady-state diffusion coefficient $D_{\Phi\Phi}^0$ ($D_{\phi\phi}^0$) is related only to the diffusion coefficients in the sum mode B_1 (relative mode B_2),

$$D_{\Phi\Phi}^0=[D_{\mathcal{E}_1^*\mathcal{E}_1}^0-\text{Re}(D_{\mathcal{E}_1\mathcal{E}_1}^0e^{-i2\Psi})]/2I_0$$
$$=D_{\Psi\Psi}(I_0)\ , \tag{6.9a}$$

$$D_{\phi\phi}^0=2[D_{\mathcal{E}_2^*\mathcal{E}_2}-\text{Re}(D_{\mathcal{E}_2\mathcal{E}_2}e^{-i2\Psi})]/I_0\ , \tag{6.9b}$$

where use has been made of Eq. (4.10b) in obtaining the second equality in Eq. (6.9a). Equation (6.9a) reveals the physical meaning of the average-phase-diffusion coefficient $D_{\Phi\Phi}^0$: It is half the natural linewidth of the CEL's in the sum mode B_1. When the relative mode B_2 is in its vacuum state [i.e., when Eq. (2.22) or (3.8) holds], we have Eq. (4.4). Substitution of Eq. (4.4) into Eq. (6.9b) leads to the vanishing of the relative-phase-diffusion coefficient [Eq. (6.4)]. On the other hand, when the relative mode B_2 is not in its vacuum state [which occurs when $\text{Re}\alpha_{11}>\text{Re}\alpha_{12}$; see Eq. (2.14)], we find from Eqs. (2.11), (4.1), and (4.2) that

$$D_{\mathcal{E}_2^*\mathcal{E}_2}=\tfrac{1}{2}\text{Re}(\alpha_{11}-\alpha_{12})\ , \tag{6.10a}$$

$$D_{\mathcal{E}_2\mathcal{E}_2}=0\ . \tag{6.10b}$$

Consequently, the relative-phase-diffusion coefficient does not vanish in steady state ($D_{\phi\phi}^0>0$).

VII. SUMMARY

We have presented a comprehensive quantum theory of correlated-emission lasers (CEL's) in a combination-mode approach, which is valid for both the quantum-beat laser and the CEL with injected atomic coherence. In Sec. II we showed that, in the balanced case (2.5), the linear master equation (2.1) breaks down into two uncoupled equations (2.11) for two combination modes B_1 and B_2 defined in Eqs. (2.15). The real and imaginary parts of the annihilation operator B_1 (B_2) are proportional to the Hermitian operators corresponding to the average (relative) phase and the average (relative) amplitude of two CEL modes, respectively, and the mode B_1 (B_2) is called the sum (relative) mode. With a further condition (2.21), we found that the relative mode B_2 is in its vacuum state and the normally ordered variances of the Hermitian operators corresponding to the relative phase and relative amplitude vanish. In Sec. III we derived a nonlinear master equation for the CEL with injected atomic coherence in the combination modes and showed how a set of proper initial atomic conditions can keep the relative mode B_2 in its vacuum state when the CEL is above threshold. In Secs. IV and V, we obtained the Glauber P functions, mean photon numbers, and photon-number variances in both the combination and CEL modes, and calculated the mode pullings of the CEL's and natural linewidth of the sum mode. We found quantum-noise reduction in each of the CEL modes. In Sec. VI we showed that the physical meaning of the average-phase-diffusion coefficient $D^0_{\Phi\Phi}$ is half the natural linewidth of the sum mode B_1. We showed that, under the decoupling condition (2.12), the relative-phase-diffusion coefficient $D_{\phi\phi}$ vanishes in steady state when and only when the relative mode B_2 is in its vacuum state.

APPENDIX: MASTER EQUATION OF THE CEL WITH INJECTED ATOMIC COHERENCE

We derive here the master equation of the CEL with injected atomic coherence directly form the interaction Hamiltonian (3.6). For the purpose of identification, we rewrite the interaction Hamiltonian (3.6) as

$$\tilde{V}_j=(g_1|A^j\rangle\langle c^j|B_1-g_2|B^j\rangle\langle c^j|B_2)e^{i\Delta(t-t_j)-i\nu_1 t_j}+\text{H.c.}\ , \tag{A1}$$

which describes the interaction of two modes of laser fields (B_1 and B_2) with V-type three-level active atoms (two upper levels are $|A^j\rangle$ and $|B^j\rangle$). Zaheer and Zubairy [6] have derived a master equation for such a two-mode laser system when all three levels have the same decay rate Γ and the active atoms are pumped to a coherent superposition of the two upper levels. Corresponding to the initial conditions (3.7) (i.e., incoherent pumping to level $|A^j\rangle$ only), we should set $C_b=0$ and $|C_a|^2=1$ in Eq. (21) of Ref. [6] and thus obtain the master equation for the CEL with injected atomic coherence in the combination modes B_1 and B_2,

$$\begin{aligned}\langle n_1,n_2|\dot{\rho}|m_1,m_2\rangle=&-r_a g_1^2 D^{-1}_{n_1+1,n_2}[\tfrac{1}{2}g_1^2(n_1-m_1)^2+\tfrac{1}{2}\Gamma^2(n_1+m_1+2)+i\Gamma\Delta n_1\\&+g_2^2(n_1+1)n_2+g_2^2(m_1+1)n_2N_{n_1+1,n_2}L^{-1}_{n_1+1,n_2}]\langle n_1,n_2|\rho|m_1,m_2\rangle\\&+r_a D^{-1}_{n_1,n_2+1}g_1^2g_2^2\sqrt{n_1m_1(n_2+1)(m_2+1)}(1+N_{n_1,n_2+1}L^{-1}_{n_1,n_2+1})\\&\quad\times\langle n_1-1,n_2+1|\rho|m_1-1,m_2+1\rangle\\&+r_a\Gamma^2D^{-1}_{n_1,n_2}g_1^2\sqrt{n_1m_1}\langle n_1-1,n_2|\rho|m_1-1,m_2\rangle+(\text{c.c.})_{n_1\leftrightarrow m_1,n_2\leftrightarrow m_2}\\&+\sum_{l=1}^{2}\langle n_1,n_2|\mathcal{L}(B_l;\gamma_l,\Omega_l-\nu_l)\rho|m_1,m_2\rangle\ ,\end{aligned} \tag{A2}$$

with

$$D_{n_1,n_2}=\Gamma^4+\Gamma^2\Delta^2+2\Gamma^2[g_1^2(n_1+m_1)+g_2^2(n_2+m_2)]+[g_1^2(n_1-m_1)+g_2^2(n_2-m_2)]^2\ , \tag{A3a}$$

$$L_{n_1,n_2}=\Gamma^2+i\Gamma\Delta+g_1^2m_1+g_2^2m_2\ , \tag{A3b}$$

$$N_{n_1,n_2}=2\Gamma^2+g_1^2(n_1-2m_1)+g_2^2(n_2-2m_2)\ . \tag{A3c}$$

In the above equations, whenever n_1 (n_2) is shifted, it is implicit that m_1 (m_2) is also shifted. The quantity $(\text{c.c.})_{n_1\leftrightarrow m_1,n_2\leftrightarrow m_2}$ denotes complex-conjugate terms with n_1 and m_1 interchanged as well as n_2 and m_2 interchanged. When $g_2=0$, the gain part of Eq. (A2) correctly reduces to that of the familiar Scully-Lamb master equation for a single-mode laser.

Trying solution (2.12) and (3.8), i.e.,

$$\langle n_1,n_2|\rho|m_1,m_2\rangle=\rho^{(1)}_{n_1,m_1}\rho^{(2)}_{n_2,m_2}=\rho^{(1)}_{n_1,m_1}\delta_{n_2,m_2}\delta_{n_2,0}\ , \tag{A4}$$

in Eq. (A2), we succeed in obtaining an equation for $\rho^{(1)}_{n_1,m_1}$. Using the relation

$$D_{n_1,0}=\Gamma^2(\Gamma^2+\Delta^2)\xi_{n_1-1,m_1-1}\ , \tag{A5}$$

where ξ_{n_1-1,m_1-1} is defined in Eq. (3.14), we find that the equation for $\rho^{(1)}_{n_1,m_1}$ is nothing else but the Scully-Lamb master equation (3.13) ($n_1=n$ and $m_1=m$ used).

[1] M. O. Scully, Phys. Rev. Lett. **55**, 2802 (1985).

[2] For a linear theory of the quantum-beat laser, see M. O. Scully and M. S. Zubairy, Phys. Rev. A **35**, 752 (1987).

[3] For a nonlinear theory of the quantum-beat laser, see J. Bergou, M. Orszag, and M. O. Scully, Phys. Rev. A **38**, 754 (1988).

[4] For linear theories of the CEL with injected atomic coherence, see J. Bergou, M. Orszag, and M. O. Scully, Phys. Rev. A **38**, 768 (1988) and Ref. [5].

[5] N. Lu, J. Opt. Soc. Am. B **7**, 2025 (1990).

[6] For a nonlinear theory of the CEL with injected atomic coherence, see K. Zaheer and M. S. Zubairy, Phys. Rev. A **38**, 5227 (1988).

[7] For a review on ring-laser gyroscopes, see W. W. Chow, J. Gea-Banacloche, J. M. Pedrotti, V. E. Sanders, W. Schleich, and M. O. Scully, Rev. Mod. Phys. **57**, 61 (1985).

[8] M. O. Scully, Phys. Rev. A **35**, 452 (1987).

[9] J. Krause and M. O. Scully, Phys. Rev. A **36**, 1771 (1987).

[10] M. O. Scully and J. Gea-Banacloche, Phys. Rev. A **34**, 4043 (1986).

[11] For long-time measurements, see P. E. Toschek and J. L. Hall, in *Proceedings of the XV International Quantum Electronics Conference,* Vol. 21 of *Tehcnical Digest Series* (Optical Society of America, Washington, D.C., 1987), p. 102 [J. Opt. Soc. Am. B **4**, P124 (1987)]; M. Ohtsu and K.-Y. Liou, Appl. Phys. Lett. **52**, 10 (1988).

[12] For a short-time measurement, see M. P. Winters, J. L. Hall, and P. E. Toschek, Phys. Rev. Lett. **65**, 3116 (1990).

[13] R. J. Glauber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963).

[14] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).

[15] W. Schleich and M. O. Scully, Phys. Rev. A **37**, 1261 (1988).

[16] W. Schleich, M. O. Scully, and H.-G. von Garssen, Phys. Rev. A **37**, 3010 (1988).

[17] R. G. K. Habiger, H. Risken, M. James, F. Moss, and W. Schleich, Phys. Rev. A **41**, 3950 (1990).

[18] N. Lu and J. A. Bergou, Phys. Rev. A **40**, 237 (1989).

[19] M. O. Scully and W. E. Lamb, Jr., Phys. Rev. **159**, 208 (1967); M. O. Scully, D. M. Kim, and W. E. Lamb, Jr., Phys. Rev. A **2**, 2529 (1970).

[20] M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974), Chap. 17.

[21] The master equation (27) of Ref. [3], which has a separable solution (29)–(31), is valid only when the two coupling constants are equal, $g_1=g_2$. The reason for this restriction is that (in the notation of Ref. [3] with $\psi=0$), when using $A=(g_1a_1+g_2a_2)/(g_1^2+g_2^2)^{1/2}$ and $B=(g_2a_1-g_1a_2)/(g_1^2+g_2^2)^{1/2}$ (see Eqs. (12a) and (26a) of Ref. [3]), the elements V_{ac} and V_{bc} of the interaction Hamiltonian V_{II} become $V_{ac},V_{bc}=\frac{1}{2}e^{i[\Delta-(\Omega/2)]t}(g_1^2+g_2^2)^{1/2}A \pm\frac{1}{2}e^{i[\Delta+(\Omega/2)]t}(g_1^2+g_2^2)^{-1/2}[(g_1^2-g_2^2)A+2g_1g_2B]$. Unless $g_1=g_2$, mode B (corresponding to mode B_2 in the present paper) is coupled to mode A (corresponding to mode B_1). The requirement $g_1=g_2$ is consistent with the requirement $\kappa=\pi/4$ obtained in Sec. II of the present paper and is also consistent with the finding from the linear theory (Ref. [2]) of the quantum-beat laser.

[22] J. Gea-Banacloche, M. O. Scully, and M. S. Zubairy, Phys. Scr. T **21**, 81 (1988).

[23] M. Lax, in *Statistical Physics, Phase Transitions and Superconductivity,* edited by M. Chretien, E. P. Gross, and S. Deser (Gordon and Breach, New York, 1968), Vol. II; M. Lax and W. H. Louisell, Phys. Rev. **185**, 568 (1969); W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).

[24] L. Mandel, Opt. Lett. **4**, 205 (1979).