Steady-state analysis of a two-mode laser with multiplicative white noise

Shiqun Zhu
*China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100 080 People's Republic of China and Department of Physics, Suzhou University, Suzhou, Jiangsu 215 006, People's Republic of China**
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The steady-state fluctuations of an inhomogeneously broadened two-mode laser with both additive and multiplicative white noise are investigated theoretically. A Fokker-Planck equation of the intensity-distribution function is presented. Analytic expressions of the autocorrelation and cross correlation of the steady-state two-mode intensities are derived for equal pump parameters of the two modes. Large intensity fluctuations occur when the laser is operated below threshold. However, when the laser is operated far above threshold, there is almost no difference between the laser models with and without multiplicative noise.

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I. INTRODUCTION

The statistical properties of two-model ring lasers have attracted a great deal of attention both experimentally and theoretically in recent years [1–5]. Some of these investigations were concerned with the intensity correlations of inhomogeneously broadened gas [1] and homogeneously broadened dye ring lasers [2,4]. Others were concerned with the first-passage-time problems [3] and the backscattering in a laser gyro [5]. In these treatments, the laser model including additive quantum noise was solved analytically [1,5]. While both additive and multiplicative noise were included in the laser model, only numerical simulations were available [2–4].

In this paper, the steady-state fluctuations of an inhomogeneously broadened two-mode laser with both additive and multiplicative white noise are investigated theoretically. A general Fokker-Planck equation of the distribution function for the two-mode intensities is presented in Sec. II. In Sec. III analytic expressions of the equal time auto-correlation and cross correlation of the steady-state intensities are derived for equal pump parameters of the two modes. The effects of multiplicative white noise are presented in Sec. IV. A discussion of the results concludes the paper.

II. EQUATIONS OF MOTION

The dimensionless coupled complex electric fields $E_1(t)$ and $E_2(t)$ of a two-mode laser operating at line center are well described by the following Langevin equations:

$$\frac{dE_1}{dt}=(a_1-|E_1|^2-\xi|E_2|^2)E_1+p(t)E_1+q_1(t)\ ,$$
$$\frac{dE_2}{dt}=(a_2-|E_2|^2-\xi|E_1|^2)E_2+p(t)E_2+q_2(t)\ , \tag{1}$$

where a_1 and a_2 are the pump parameters of the two modes and ξ is the mode-coupling constant, $q_1(t)$ and $q_2(t)$ are the independent quantum noise, and $p(t)$ is the pump fluctuation. These random noise terms are taken to be zero mean and correlation

$$\langle q_i^*(t)q_j(t')\rangle=2P\delta_{ij}\delta(t-t')\ , \tag{2}$$

and

$$\langle p^*(t)p(t')\rangle=2P'\delta(t-t') \tag{3}$$

where P and P' are the strength of the additive and multiplicative noise, respectively. The additive noise is interpreted as a result of the spontaneous emission (i.e., internal fluctuations) and the multiplicative noise as a result of the pump fluctuation (i.e., external disturbances). These may have different origins and thus be independent of each other [2,3]. However, in some situations both noises may have a common origin and also have cross correlations [10]. The statistical fluctuations of a laser with a certain correlation between the quantum and the pump noises will be discussed in a forthcoming paper.

The corresponding Fokker-Planck equation for the probability function $Q(I_1,I_2,t)$ of the two-mode laser intensities $I_1=|E_1|^2$ and $I_2=|E_2|^2$ is given by [6]

$$\frac{\partial Q}{\partial t}=-\frac{\partial}{\partial I_1}\left[(2a_1I_1-2I_1^2-2\xi I_1I_2+2P'I_1+2P)Q-2P'\frac{\partial}{\partial I_2}(I_1I_2Q)-2\frac{\partial}{\partial I_1}[(P+P'I_1)I_1Q]\right]$$
$$-\frac{\partial}{\partial I_2}\left[(2a_2I_2-2I_2^2-2\xi I_1I_2+2P'I_2+2P)Q-2P'\frac{\partial}{\partial I_1}(I_1I_2Q)-2\frac{\partial}{\partial I_2}[(P+P'I_2)I_2Q]\right] \tag{4}$$

where the complex electric fields E_1 and E_2 have been transformed to polar coordinates by writing

$$E_1=(I_1)^{1/2}e^{i\theta_1}\ ,\quad E_2=(I_2)^{1/2}e^{i\theta_2} \tag{5}$$

and the probability density function $Q(I_1,I_2,t)$ has been taken to be independent of the phase variables θ_1 and θ_2. The detailed derivation of Eq. (4) is given in the Appendix.

III. STEADY-STATE SOLUTION

The system will reach a steady state after a sufficiently long time. In steady state, the intensity probability function $Q(I_1,I_2,t)$ is independent of the time t. Then Eq. (4) reads

$$\frac{\partial}{\partial I_1}\left[(2a_1I_1-2I_1^2-2\xi I_1I_2+2P'I_1+2P)Q_s-2P'\frac{\partial}{\partial I_2}(I_1I_2Q_s)-2\frac{\partial}{\partial I_1}[(P+P'I_1)I_1Q_s]\right]$$
$$+\frac{\partial}{\partial I_2}\left[(2a_2I_2-2I_2^2-2\xi I_1I_2+2P'I_2+2P)Q_s-2P'\frac{\partial}{\partial I_1}(I_1I_2Q_s)-2\frac{\partial}{\partial I_2}[(P+P'I_2)I_2Q_s]\right]=0\ . \tag{6}$$

In general, no analytic solution exists for Eq. (6) with arbitrary values of the pump parameters and the mode-coupling constant [7] except approximate numerical calculations. However, for an inhomogeneously broadened laser with equal pump parameters of the two modes, i.e., $\xi=1$ and $a_1=a_2=a$, the analytic solution of Eq. (6) can be obtained immediately,

$$Q_s(I_1,I_2)=\frac{\exp(-\alpha P/P')}{B}\left[\frac{\alpha}{P'}\right]^{\beta+1}(P+P'I_1+P'I_2)^{\beta}\times\exp[-\alpha(I_1+I_2)]\ , \tag{7}$$

where

$$B=\left[\frac{a}{P'}-1\right]\Gamma(\beta+1)+\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha P/P')^{\beta+2+n}}{n!(\beta+1+n)(\beta+2+n)}\ , \tag{8}$$

$$\alpha=\frac{1}{P'}\ ,\quad \beta=\frac{a}{P'}+\frac{P}{(P')^2}-2\ , \tag{9}$$

and $\Gamma(x)$ is the gamma function which is given by [8]

$$\Gamma(x)=\int_0^{\infty}e^{-t}t^{x-1}dt\ . \tag{10}$$

After straightforward calculations from Eqs. (7)–(9), the mean light intensity, the equal time autocorrelation and cross correlation of the two-mode intensities can be expressed as

$$\langle I_1\rangle=\langle I_2\rangle=\frac{1}{2\alpha B}\left[\frac{a}{P'}\Gamma(\beta+2)+\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha P/P')^{\beta+3+n}}{n!(\beta+2+n)(\beta+3+n)}\right]\ , \tag{11}$$

$$\begin{aligned}\lambda_{11}(0)=\lambda_{22}(0)&=\langle(\Delta I_1)^2\rangle/\langle I_1\rangle^2=\langle I_1^2\rangle/\langle I_1\rangle^2-1\\ &=\frac{1}{\langle I_1\rangle^2}\left[\frac{1}{B}\left[\frac{\Gamma(\beta+4)}{3\alpha^2}-P\Gamma(\beta+3)+\frac{3\alpha P^3\Gamma(\beta+1)}{3(P')^3}\right.\right.\\ &\quad+\frac{1}{\alpha^2}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha P/P')^{\beta+4+n}}{n!(\beta+1+n)(\beta+4+n)}\\ &\quad\left.\left.-2P\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha P/P')^{\beta+3+n}}{n!(\beta+1+n)(\beta+3+n)}\right]+\left[\frac{P}{P'}\right]^2\right]-1\ ,\end{aligned} \tag{12}$$

$$\begin{aligned}\lambda_{12}(0)&=\frac{\langle\Delta I_1\Delta I_2\rangle}{\langle I_1\rangle\langle I_2\rangle}=\frac{\langle I_1I_2\rangle}{\langle I_1\rangle\langle I_2\rangle}-1\\ &=\frac{1}{2\langle I_1\rangle^2}\left\{\frac{1}{\alpha^2B}\left[\left[\frac{a}{P'}+1\right]\Gamma(\beta+3)+\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha P/P')^{\beta+4+n}}{n!(\beta+3+n)(\beta+4+n)}\right]-\left[\frac{P}{P'}\right]^2\right\}\\ &\quad-\lambda_{11}(0)-\frac{2P}{P'\langle I_1\rangle}-2\ .\end{aligned} \tag{13}$$

In the next section, the effects of multiplicative noise are discussed and these results are compared with those from a laser model without multiplicative noise.

IV. EFFECTS OF MULTIPLICATIVE NOISE

For a conventional two-mode laser model without multiplicative noise, Eqs. (7)–(13) reduce to very simple forms which have already been derived in Ref. [1]. Here those equations have been written down for comparison with Eqs. (7)–(13). The steady-state distribution function $W_s(I_1,I_2)$ is of the form

$$W_s(I_1,I_2)=\left[\frac{2}{P\pi}\right]^{1/2}\frac{\exp[(aI_1+aI_2-\frac{1}{2}I_1^2-\frac{1}{2}I_2^2-I_1I_2)/P]}{\{a\exp(a^2/2P)[1+\operatorname{erf}(a/\sqrt{2P})]+\sqrt{2P/\pi}\}}\ . \tag{14}$$

The mean light intensity is given by

$$\langle I_1\rangle=\langle I_2\rangle=\frac{a}{2}+\frac{P[1+\operatorname{erf}(a/\sqrt{2P})]}{2\{a[1+\operatorname{erf}(a/\sqrt{2P})]+\sqrt{2P/\pi}\exp(-a^2/2P)\}}\ . \tag{15}$$

The autocorrelation and cross correlation at equal time are given by

$$\lambda_{11}(0)=\lambda_{22}(0)=\frac{2P}{3\langle I_1\rangle^2}+\frac{2a}{3\langle I_1\rangle}-1 \tag{16}$$

and

$$\lambda_{12}(0)=\frac{P}{3\langle I_1\rangle^2}+\frac{a}{3\langle I_1\rangle}-1\ . \tag{17}$$

The steady-state distribution function $Q_s(I_1,I_2)$ is plotted in Fig. 1 as a function of one mode intensity I_1 while the other mode intensity I_2 is held fixed. Figure 1(a) shows the $Q_s(I_1,I_2)$ for different values of the pump parameter a with $P'=5$ and $I_2=0$. It is shown that the curve has a large peak value and drops very quickly for small values of a but has a small peak with a long tail when the value of a is quite large. The peak of the curve corresponding to the most probable distribution occurs when

$$I_{1\max}=\begin{cases}0, & a\le I_2+2P'\\ a-I_2-2P', & a>I_2+2P'\ .\end{cases} \tag{18}$$

Figure 1(b) is a plot of the distribution function $Q_s(I_1,I_2)$ versus I_1 for different values of the multiplicative noise P' with I_2 progressively increased and with a fixed value of a. It is shown that the height of the peak increased with increasing value of P' and the curve with a larger value of P' has a longer tail. Also, the curve with a larger value of P' has a more asymmetric shape. However, if there is no multiplicative noise included in the laser model, i.e., $P'=0$, the distribution function $Q_s(I_1,I_2)$ reduces to a Gaussian distribution expressed by $W_s(I_1,I_2)$ of Eq. (14) with a very high and sharp peak. The height of the peak of $W_s(I_1,I_2)$ is 3.6 times as large as that of $Q_s(I_1,I_2)$ for $P'=10$ when $a=25$. Due to the scale of the vertical axis, the height of the peak of $W_s(I_1,I_2)$ cannot be shown in Fig. 1(b).

Figure 2 is a plot of the mean light intensity $\langle I\rangle=\langle I_1\rangle+\langle I_2\rangle$ against the pump parameter a. It is clear that $\langle I\rangle$ increases with increasing P' when the laser is operated below threshold but approaches a when it is operated well above threshold.

The equal time autocorrelation and cross-correlation functions $\lambda_{11}(0)$ and $\lambda_{12}(0)$ are plotted in Figs. 3(a) and 3(b) for different values of P'. It is shown that if $P'=0$, $\lambda_{11}(0)$ and $\lambda_{12}(0)$ decrease monotonically as a function of $\langle I\rangle$ and also very fast near the threshold regime (from $\langle I\rangle=0.1$ to 10). However, there is a peak in the correlations $\lambda_{11}(0)$ and $\lambda_{12}(0)$ if P' is not equal to zero and the peak position shifts to small values of $\langle I\rangle$ as P' increases. Well below threshold, $\lambda_{11}(0)$ and $\lambda_{12}(0)$ are close to one and zero, respectively, while far above threshold, $\lambda_{11}(0)$ and $\lambda_{12}(0)$ approach $\frac{1}{3}$ and $-\frac{1}{3}$ respectively. In these two limiting cases, there is no big difference in $\lambda_{11}(0)$ and $\lambda_{12}(0)$ between the laser models with and without multiplicative noise. This can be directly shown by asymptotic expansions of Eqs. (11)–(17).

If the laser is operated well above threshold with $a/P'\gg 1$ and

$$\Gamma(\beta+1)\gg\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha P/P')^{\beta+2+n}}{n!(\beta+1+n)(\beta+2+n)}\ , \tag{19}$$

then the asymptotic expressions of Eqs. (11)–(13) can be written as

$$\langle I_1\rangle=\langle I_2\rangle\approx\frac{(a/P')(\beta+1)}{2\alpha(a/P'-1)}-\frac{P}{2P'}\to\frac{a}{2}\left[1+\frac{P}{a^2}+\frac{PP'}{a^3}\right]\ , \tag{20}$$

$$\lambda_{11}(0)=\lambda_{22}(0)$$
$$\approx\left[\frac{(a/P')(\beta+1)}{2\alpha(a/P'-1)}-\frac{P}{2P'}\right]^{-2}\left\{\frac{1}{(a/P'-1)}\left[\frac{(\beta+1)(\beta+2)(\beta+3)}{3\alpha^2}-P(\beta+1)(\beta+2)\right.\right.$$
$$\left.\left.+\frac{2\alpha}{3}\left[\frac{P}{P'}\right]^3\right]+\left[\frac{P}{P'}\right]^2\right\}-1\to\frac{1}{3}\left[1+\frac{4}{a}\left[P'+\frac{P}{a}\right]\right]\ , \tag{21}$$

$$\lambda_{12}(0) \approx \frac{1}{2}\left[\frac{(a/P')(\beta+1)}{2\alpha(a/P'-1)}-\frac{P}{2P'}\right]^{-2}\left[\frac{(a/P'+1)(\beta+1)(\beta+2)}{(a/P'-1)\alpha^2}-\left[\frac{P}{P'}\right]^2\right]-\lambda_{11}(0)$$
$$-\frac{2P}{P'}\left[\frac{(a/P')(\beta+1)}{2\alpha(a/P'-1)}-\frac{P}{2P'}\right]^{-1}-2 \to -\frac{1}{3}\left[1-\frac{2}{a}\left[P'+\frac{P}{a}\right]\right] . \qquad (22)$$

Similarly, for the laser operated well above threshold, Eqs. (15)–(17) give

$$\langle I_1\rangle = \langle I_2\rangle \approx \frac{a}{2}\left[1+\frac{P}{a^2}\right] , \qquad (23)$$

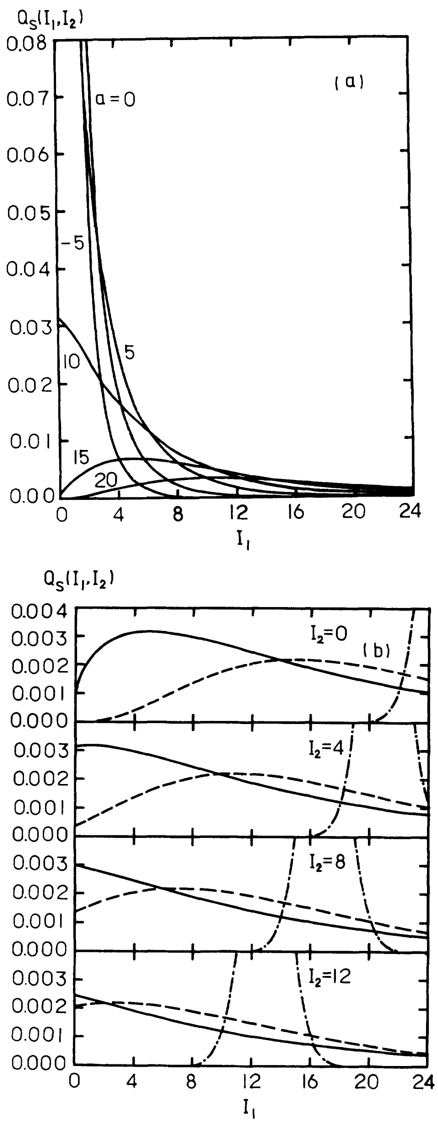

FIG. 1. The steady-state intensity distribution function $Q_s(I_1,I_2)$ as a function of one mode intensity I_1 with the other mode intensity I_2 held fixed. (a) $Q_s(I_1,I_2)$ vs I_1 for different pump parameters a with $P'=5$ and $I_2=0$. (b) $Q_s(I_1,I_2)$ vs I_1 for different multiplicative noise strength P' with $a=25$ and I_2 increased progressively shown in the figure. —·—·—·: $P'=0$; — — —: $P'=5$; ——: $P'=10$.

$$\lambda_{11}(0)=\lambda_{22}(0)\approx\frac{1}{3}\left[1+\frac{4P}{a^2}\right] , \qquad (24)$$

$$\lambda_{12}(0)\approx-\frac{1}{3}\left[1-\frac{2P}{a^2}\right] . \qquad (25)$$

It is clear that in the mean light intensity $\langle I\rangle$ the multiplicative noise P' only gives a correction term an order smaller than that from P. However, in the equal time correlations $\lambda_{11}(0)$ and $\lambda_{12}(0)$, the multiplicative noise P' gives a correction term an order larger than that from P. Thus Eqs. (21) and (22) approach $\frac{1}{3}$ and $-\frac{1}{3}$ slower than Eqs. (24) and (25), respectively.

V. DISCUSSION

The statistical fluctuations of the light emitted by an inhomogeneously broadened two-mode laser that includes both additive and multiplicative white noise are investigated through a Fokker-Planck equation. Analytic expressions of the mean, the equal time auto- and crosscorrelation of the steady-state two-mode intensities show the dependence of these quantities on the additive and multiplicative noise. The anomalously large fluctuations appearing in $\lambda_{11}(0)$ and $\lambda_{12}(0)$ are entirely due to the effects of multiplicative noise which plays an important role in the threshold regime even though the laser is operated at steady state. It is also seen from Eqs. (16) and (17) that for a conventional laser model with $P'=0$,

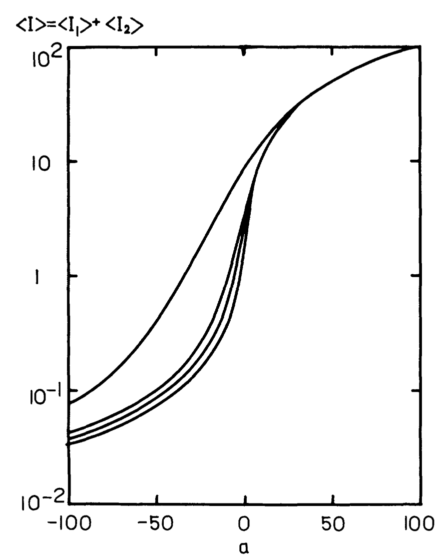

FIG. 2. The mean light intensity $\langle I\rangle=\langle I_1\rangle+\langle I_2\rangle$ vs the pump parameter a for different values of P' (from bottom to top): $P'=0,5,10,50$.

$\langle I_1^2 \rangle = \langle I_2^2 \rangle = \lambda_{11}(0)+1$ is always twice as large as $\langle I_1 I_2 \rangle = \lambda_{12}(0)+1$. However, in a laser model with $P' \neq 0$, there is no such relation for arbitrary value of the pump parameter a. Only when the laser is operated far above threshold, $\langle I_1^2 \rangle \approx 2\langle I_1 I_2 \rangle$, which is shown in Eqs. (21) and (22). It is also clear that when the laser operates high above threshold, the equal time intensity autocorrelation and cross-correlation functions $\lambda_{11}(0)$ and $\lambda_{12}(0)$ are determined by quantum noise for a conventional laser model with $P'=0$. However, for a laser model with $P' \neq 0$, the equal time intensity correlation functions $\lambda_{11}(0)$ and $\lambda_{12}(0)$ are determined by pump fluctuations. The first-order correction term in $\lambda_{11}(0)$ and $\lambda_{12}(0)$ from the multiplicative pump noise P' is an order larger than that from the additive quantum noise P. This is shown in Eqs. (20)–(25). For correlation functions which involve the phases, behavior similar to that in a single-mode laser [12] may be expected. Finally, these conclusions should lend themselves to experimental verification in a manner similar to those for a single-mode laser [9].

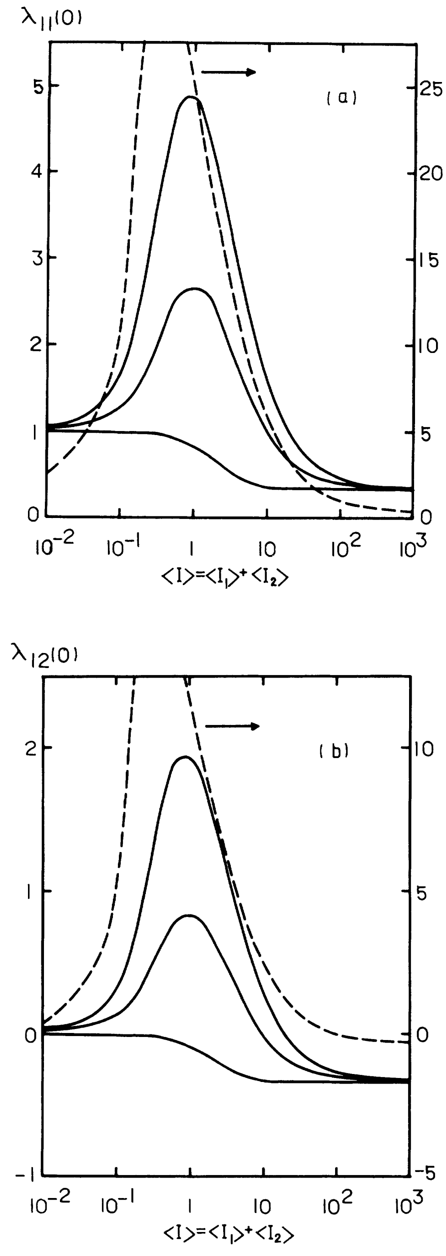

FIG. 3. The equal time autocorrelation and cross correlation for different values of P' (from bottom to top): $P'=0,5,10,50$. The dashed curve should be read from the vertical scale on the right-hand side axis which is five times as large as that on the left. (a) The autocorrelation $\lambda_{11}(0)$ vs $\langle I \rangle$. (b) The cross correlation $\lambda_{12}(0)$ vs $\langle I \rangle$.

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APPENDIX

If the two-mode laser fields E_1 and E_2 are written as $E_1 = x_1 + ix_2$ and $E_2 = x_3 + ix_4$, the Fokker-Planck equation corresponding to Eqs. (1)–(3) for the probability density function $\tilde{Q}(x_1, x_2; x_3, x_4; t)$ is found to be [6]

$$\frac{\partial \tilde{Q}}{\partial t} = -\sum_{i=1}^{4} \frac{\partial}{\partial x_i}(\tilde{D}_i \tilde{Q}) + \sum_{i=1}^{4}\sum_{j=1}^{4} \frac{\partial^2}{\partial x_i \partial x_j}(\tilde{D}_{ij}\tilde{Q}) , \quad \text{(A1)}$$

where

$$\begin{aligned} \tilde{D}_1 &= (a_1 - |E_1|^2 - \xi |E_2|^2)x_1 , \\ \tilde{D}_2 &= (a_1 - |E_1|^2 - \xi |E_2|^2)x_2 , \\ \tilde{D}_3 &= (a_2 - |E_2|^2 - \xi |E_1|^2)x_3 , \\ \tilde{D}_4 &= (a_2 - |E_2|^2 - \xi |E_1|^2)x_4 , \end{aligned} \quad \text{(A2)}$$

and

$$\begin{aligned} \tilde{D}_{11} &= \tilde{D}_{22} = \tfrac{1}{2}(P + P'|E_1|^2) , \\ \tilde{D}_{12} &= \tilde{D}_{21} = \tilde{D}_{34} = \tilde{D}_{43} = 0 , \\ \tilde{D}_{13} &= \tilde{D}_{31} = \tilde{D}_{24} = \tilde{D}_{42} = \tfrac{1}{2}(P + P'x_1x_3 + P'x_2x_4) , \\ \tilde{D}_{14} &= \tilde{D}_{41} = \tfrac{1}{2}P'(x_1x_4 - x_2x_3) , \\ \tilde{D}_{23} &= \tilde{D}_{32} = \tfrac{1}{2}P'(x_2x_3 - x_1x_4) , \\ \tilde{D}_{33} &= \tilde{D}_{44} = \tfrac{1}{2}(P + P'|E_2|^2) . \end{aligned} \quad \text{(A3)}$$

The use of polar coordinates

$$\begin{aligned} x_1 &= (I_1)^{1/2}\cos\theta_1 , \\ x_2 &= (I_1)^{1/2}\sin\theta_1 , \\ x_3 &= (I_2)^{1/2}\cos\theta_2 , \\ x_4 &= (I_2)^{1/2}\sin\theta_2 , \end{aligned} \quad \text{(A4)}$$

with

$$\bar{Q}(I_1, I_2; \theta_1, \theta_2; t) = \frac{1}{(I_1 I_2)^{1/2}} \tilde{Q}(x_1, x_2; x_3, x_4; t) \quad \text{(A5)}$$

leads to the Fokker-Planck equation

$$\frac{\partial \bar{Q}}{\partial t} = -\frac{\partial}{\partial I_1}\left\{2(a_1 I_1 - I_1^2 - \xi I_1 I_2 + P + P' I_1)\bar{Q} - 2\frac{\partial}{\partial I_2}\{[(I_1 I_2)^{1/2} P \cos(\theta_1-\theta_2) + P' I_1 I_2]\}\bar{Q} - 2\frac{\partial}{\partial I_1}[(P+P'I_1)I_1\bar{Q}]\right\}$$
$$-\frac{\partial}{\partial I_2}\left\{2(a_2 I_2 - I_2^2 - \xi I_1 I_2 + P + P' I_2)\bar{Q} - 2\frac{\partial}{\partial I_1}\{[(I_1 I_2)^{1/2} P \cos(\theta_1-\theta_2) + P' I_1 I_2]\bar{Q}\} - 2\frac{\partial}{\partial I_2}[(P+P'I_2)I_2\bar{Q}]\right\} + \frac{\partial^2}{\partial I_1 \partial \theta_2}[2(I_1)^{1/2}\sin(\theta_1-\theta_2)\bar{Q}]$$
$$-\frac{\partial^2}{\partial I_2 \partial \theta_1}[2(I_2)^{1/2} P \sin(\theta_1-\theta_2)\bar{Q}] + \frac{\partial^2}{\partial \theta_1 \partial \theta_2}\left[\left\{\frac{P}{(I_1 I_2)^{1/2}}\cos(\theta_1-\theta_2) + P'\right\}\bar{Q}\right]$$
$$+\frac{\partial^2}{\partial \theta_1^2}\left\{\frac{1}{2I_1}(P+P'I_1)\bar{Q}\right\} + \frac{\partial^2}{\partial \theta_2^2}\left\{\frac{1}{2I_2}(P+P'I_2)\bar{Q}\right\} . \qquad \text{(A6)}$$

If the phase variables θ_1 and θ_2 are integrated over in Eq. (A6) with

$$Q(I_1,I_2;t) = \int_0^{2\pi} d\theta_2 \int_0^{2\pi} d\theta_1 \bar{Q}(I_1,I_2;\theta_1,\theta_2;t) , \qquad \text{(A7)}$$

Eq. (A6) reduces exactly to Eq. (4) [1,2,11].

*Address for correspondence.

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