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Coherent states in a finite-dimensional basis: Their phase properties and relationship to coherent states of light

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The Pegg-Barnett phase-operator formalism utilizes a finite basis set to represent operators of the harmonic oscillator; this enables the phase to be represented by a Hermitian operator, but rests on taking the dimensionality of the basis set to infinity for observable quantities. Simultaneously, in their approach Pegg and Barnett consider quantum states of a harmonic oscillator which are normalized in the Fock space, i.e., the dimensionality of the basis set in which the states of the harmonic oscillator are defined is supposed to be infinite, while the phase operator is defined in the finite-dimensional basis set. In this paper we address the problem of a consistent definition of a coherent state within a finite state basis. We employ displacement operators to define such coherent states and numerically evaluate observables as a function of the size of the basis set. We investigate phase properties of these coherent states. We find that if the dimensionality of the state space is much larger than the mean occupation number of the coherent states, then the results obtained in the finite-dimensional basis are applicable in the case of an ordinary quantum-mechanical harmonic oscillator. These coherent states are minimum uncertainty states with respect to quadrature operators (i.e., the position and momentum operators) and do not exhibit quadrature squeezing. A weakly excited (compared with the dimensionality of the state space) coherent state in finite-dimensional basis is not strictly speaking a minimum uncertainty state with respect to the number and phase operators. We give definitions of amplitude and phase squeezing and show that weakly excited coherent states can be amplitude squeezed. In the high-intensity limit (again compared with the dimensionality of the state space) these states exhibit phase squeezing.

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I. INTRODUCTION

Even though the theory of nonrelativistic quantum mechanics was completed more than 60 years ago, there are fundamental problems in this theory which only recently have been clarified. One of these problems is related to the existence of a Hermitian phase operator of the harmonic oscillator (or a single mode of the electromagnetic field). The classical electromagnetic field can be described by its amplitude, i.e., the square root of the intensity of the field, and its phase. In the quantum theory, the amplitude of the field is proportional to the square root of the photon-number operator, but the question is how to define the phase operator.

From the complementarity principle [1,2] it follows that, for each degree of freedom, the dynamical variables are a pair of complementary observables [3]. This implies that there should be a Hermitian operator conjugate to the excitation (photon-) number operator such that a precise knowledge of one of them implies that all possible outcomes of measuring the other are equally probable. Dirac [4] was the first to introduce a Hermitian phase operator of the electromagnetic field. He utilized the Poisson-bracket–commutator correspondence principle [5] and suggested that the photon-number operator and the phase operator should obey the canonical commutation relation

$$[\hat{N},\hat{\Phi}]=-i \;, \tag{1}$$

and that the annihilation $\hat{a}$ and the creation $\hat{a}^{\dagger}$ operators of the single mode of the electromagnetic field (for which $[\hat{a},\hat{a}^{\dagger}]=1$) can be expressed in the polar form

$$\hat{a}=\exp(i\hat{\Phi})\sqrt{\hat{N}} \;, \quad \hat{a}^{\dagger}=\sqrt{\hat{N}}\exp(-i\hat{\Phi}) \;. \tag{2}$$

It was shown by Louisell [6] and Susskind and Glogower [7] that the number-phase commutator (1) is not consistent with the existence of a well-defined Hermitian phase operator (see also Refs. [8,9]). Lat there were several attempts to define Hermitian phase operators consistently by introducing periodic functions of the phase [6,10,11]. These attempts did not, however, solve the problem (for details see Refs. [7,9,12]). Susskind and Glogower [7] proposed exponential operators $\widehat{\exp}(i\Phi)$ and $\widehat{\exp}(-i\Phi)$, which are not functions of a common phase operator $\hat{\Phi}$ [13]. The Susskind-Glogower (SG) phase operators have been applied in a variety of problems in quantum optics. In particular, using this operator, Carruthers and Nieto [14] have studied the phase properties of coherent states.

Susskind and Glogower [7] realized that the main problem in the proper definition of a phase operator lies in the existence of a cutoff in the spectrum of the number operator, which excludes the negative integers. In fact, there are two possible ways to overcome this problem of the semiboundedness of the energy spectrum of the harmonic oscillator and hence to define the phase operator consistently. One possibility is to extend the normal harmonic-oscillator Hilbert space to include negative number states (i.e., the spectrum of the harmonic oscillator is unbounded, but simultaneously it is assumed that the negative-energy states are decoupled from the positive-energy ground state [15]). Recently it has been shown that this approach suffers from some inconsistencies [16] which are due to the unbounded state space. The second possibility to treat the problem of the phase operator is to suppose the spectrum of the harmonic oscillator to be bounded, that is, to consider a finite-dimensional Hilbert space of the harmonic oscillator [17].

Recently Pegg, and Barnett [18,19,12] defined the Hermitian phase operator in a finite-dimensional state space. They used the fact that, in this state space, one can define phase states rigorously. The phase operator is then defined as the projection operator on the particular phase state multiplied by the corresponding value of the phase (for details see Sec. II). The main idea of the Pegg-Barnett (PB) formalism consists in evaluation of all expectation values of physical variables in a finite-dimensional Hilbert space. This gives a real number which depends parametrically on the dimension of the Hilbert space. Because a complete description of a real harmonic oscillator involves an infinite set of number states, the infinite limit must be taken. This limit is taken only *after* the physical results (mean values of observables) are evaluated thereby leading to a proper limit which corresponds to the results obtainable in ordinary quantum mechanics (for further work concerning the relation between the SG and the PB formalisms see the recent paper by Lukš and Peřinová [20]). It can be used for investigation of the phase properties of quantum states of the single mode of the electromagnetic field. In the past two years the PB formalism has been applied to various problems in quantum optics. In particular, it has been shown that the uncertainty product of the number and the phase fluctuations of a highly excited coherent state is minimized with increasing intensity of the coherent field [12]; it has also been shown that the number states of the single mode of the electromagnetic field are the minimum uncertainty states [21] of light. Phase properties of a single-mode squeezed vacuum have been analyzed and the relation between the squeezing parameter and the form of the phase probability distribution has been found [22–24]. In addition, phase properties of the cotangent states were recently analyzed [25] and the PB formalism has been used for investigation of the phase correlations between two modes of the electromagnetic field [26]. In particular, the phase properties of the two-mode squeezed vacuum have been studied [27] and the interesting feature of phase locking has been revealed. The PB formalism has been adopted to describe optical phase diffusion [28], phase-difference fluctuations in a quantum-beat laser [29], phase fluctuations in a laser with an atomic memory[30], and phase properties of coherent light interacting with a two-level atom [31] or nonlinear medium [32]. In all references mentioned above the quantum states of light employed in the calculations are defined in the infinite-dimensional basis, that is, they are properly normalized only in the infinite-dimensional state basis, whereas the Pegg-Barnett phase operator is defined in the finite-dimensional state space.

The aim of this paper is to study in detail the phase properties of coherent states, which are consistently defined in the *same* finite-dimensional basis set in which the phase operator acts. We find that if the dimensionality of the state space is much larger than the mean excitation number of the coherent state, then results obtained in the finite-dimensional basis are applicable in the case of the ordinary quantum-mechanical harmonic oscillator within an infinite-dimensional state space. But, as we show below, our finite-dimensional state space results are sufficiently general to encompass *both* bosonic coherent states *and* atomic [SU(2)] coherent states within a common formalism. Such coherent states, of course, can be variously squeezed or subfluctuant.

Coherent states are always associated with the "most" classical states one can imagine in the framework of quantum theory, so one would expect to find some connection between the mean values of the quantum phase operator and the phase of the classical coherent field. We will base our analysis on the PB formalism and concentrate our attention on phase fluctuations, on amplitude and phase squeezing, and on determining whether the coherent state of light is a minimum uncertainty state (MUS) or an intelligent state with respect to the number and phase operators. In Sec. II we present a short review of the PB formalism. In Sec. III we analyze in detail the algebraic properties of the PB phase operator and we show that one can construct phase annihilation and creation operators acting in the finite-dimensional Hilbert space of the quantum system which decrease or increase the phase by increments which depend on the dimensionality of the state space. In Sec. IV we give a precise definition of amplitude (number) squeezing and phase squeezing. In Sec. V we define the coherent state in the finite-dimensional Hilbert space and study the relation between this coherent state and the ordinary coherent state of the usual infinite-dimensional Hilbert-space harmonic oscillator. In Sec. VI we discuss the phase properties of the coherent state of light. We concentrate our attention on weakly excited coherent fields where the SG formalism [14] and the PB formalism [18,19] most strongly differ (see also Ref. [33]). Finally, we analyze the relation between amplitude squeezing and sub-Poissonian photon statistics [34]. Of course we recognize that there are many cases of interest where the state space is genuinely finite for which coherent states are of value. In the Appendix we illustrate the ideas advanced in this paper with the simplest possible finite-dimensional Hilbert space, namely that of a two-state system.

II. SHORT REVIEW OF PEGG-BARNETT FORMALISM

The concept of the phase operators recently introduced by Pegg and Barnett [18,19] is based on the idea of a *Hermitian* operator which has properties usually associated with phase. This operator is properly defined on a linear space Ψ of *finite* dimension spanned by the $(s+1)$ number states $|0\rangle, |1\rangle, \ldots, |s\rangle$. The physical variables (expectation values of Hermitian operators) are evaluated in the finite-dimensional space Ψ. These mean values depend parametrically on s. At the final stage of the calculations for bosonic systems the limit $s \to \infty$ is taken. As we will show, we need not insist on taking the infinite s limit before considering physical quantities: there are many finite-dimensional systems (the simplest being the two-level system) with interesting phase-dependent quantities.

Let us consider the finite-dimensional state (Hilbert) space Ψ of a "harmonic oscillator." The number states $|n\rangle \in \Psi$ are orthonormal:

$$\langle n|m\rangle = \delta_{n,m}, \quad \sum_{n=0}^{s} |n\rangle\langle n| = 1 . \tag{3}$$

The annihilation operator $\hat{a}$ acts on Ψ as usual, that is,

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle, \quad \hat{a}|0\rangle = 0 , \quad |n\rangle \in \Psi , \tag{4}$$

but the action of the creation operator $\hat{a}^{\dagger}$ is modified when acting on the state $|s\rangle$:

$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \hat{a}^{\dagger}|s\rangle = 0 , \quad |n\rangle \in \Psi . \tag{5}$$

The operators $\hat{a}$ and $\hat{a}^{\dagger}$ can be rewritten in terms of projection operators $|m\rangle\langle n|$:

$$\hat{a} = \sum_{n=1}^{s} \sqrt{n}\,|n-1\rangle\langle n|, \quad \hat{a}^{\dagger} = \sum_{n=1}^{s} \sqrt{n}\,|n\rangle\langle n-1| . \tag{6}$$

The commutation relation for the creation and annihilation operators in Ψ is

$$[\hat{a},\hat{a}^{\dagger}] = 1-(s+1)|s\rangle\langle s| , \tag{7}$$

which means that once the dimension of the Hilbert space of the harmonic oscillator is taken to be finite, the creation and annihilation operators are no longer related to the Weyl-Heisenberg algebra. The number operator $\hat{N}$ can be defined in a natural way:

$$\hat{N} = \sum_{n=1}^{s} n|n\rangle\langle n| , \tag{8}$$

and using the definition (6) of the creation and annihilation operators we can rewrite $\hat{N}$ as

$$\hat{N} = \hat{a}^{\dagger}\hat{a} . \tag{9}$$

The operators $\hat{N}$, $\hat{a}^{\dagger}$, and $\hat{a}$ obey the following commutation relations:

$$[\hat{N},\hat{a}^{k}] = -k\hat{a}^{k}, \quad [\hat{N},(\hat{a}^{\dagger})^{k}] = k(\hat{a}^{\dagger})^{k} . \tag{10}$$

As shown by Pegg and Barnett [18,19], the finite-dimensional state space Ψ can also be spanned by $(s+1)$ *phase states* $|\theta_m\rangle$ (see also Ref. [35]):

$$|\theta_m\rangle \equiv (s+1)^{-1/2} \sum_{n=0}^{s} \exp(i\theta_m n)|n\rangle , \tag{11}$$

with the following properties:

$$\langle \theta_m|\theta_n\rangle = \delta_{m,n} , \quad \sum_{m=0}^{s} |\theta_m\rangle\langle\theta_m| = 1 . \tag{12}$$

The phase θ_m is defined as

$$\theta_m = \theta_0 + 2\pi\frac{m}{s+1}, \quad m = 0,1,\ldots,s \tag{13}$$

where the value of the phase θ_0 is arbitrary and once chosen it defines a particular basis set.

The Hermitian phase operator $\hat{\Phi}_{\theta}$ is defined through the projection operators $|\theta_m\rangle\langle\theta_k|$:

$$\hat{\Phi}_{\theta} \equiv \sum_{m=0}^{s} \theta_m|\theta_m\rangle\langle\theta_m| , \tag{14}$$

from which it follows that $|\theta_m\rangle$ are the eigenstates of the Hermitian phase operator $\hat{\Phi}_{\theta}$:

$$\hat{\Phi}_{\theta}|\theta_m\rangle = \theta_m|\theta_m\rangle , \tag{15}$$

that is, $|\theta_m\rangle$ are states with well-defined phases. The phase operator $\hat{\Phi}_{\theta}$ can be rewritten in the number-state basis:

$$\hat{\Phi}_0 = \theta_0 + 2\pi\frac{s}{s+1} + \frac{2\pi}{s+1}\sum_{\substack{k,n=0\\ k\neq n}}^{s} \frac{\exp[-i(k-n)\theta_0]}{\exp[-i(k-n)2\pi/(s+1)]-1}|n\rangle\langle k| . \tag{16}$$

Using the above definitions, one can find the commutation relation for the conjugate operators $\hat{N}$ and $\hat{\Phi}_{\theta}$ [36] in terms of the projection operators in the number-state basis:

$$[\hat{N},\hat{\Phi}_{\theta}] = \frac{2\pi}{(s+1)} \times \sum_{\substack{k,n=0\\ k\neq n}}^{s} \frac{(n-k)\exp[-i(k-n)\theta_0]}{\exp[-i(k-n)2\pi/(s+1)]-1}|n\rangle\langle k| , \tag{17}$$

which differs from the commutation relation (1) proposed by Dirac [4] (for further discussion of Refs. [9,18,19,3,6]).

Once the phase operator $\hat{\Phi}_{\theta}$ is defined consistently on the space Ψ, functions of this operator can also be introduced. For instance, the functions $\exp(i\hat{\Phi}_{\theta})$, $\cos\hat{\Phi}_{\theta}$, and $\sin\hat{\Phi}_{\theta}$ defined as

$$\exp(i\hat{\Phi}_{\theta}) = \sum_{m=0}^{s} \exp(i\theta_m)|\theta_m\rangle\langle\theta_m| , \tag{18}$$

$$\cos\hat{\Phi}_{\theta} = \sum_{m=0}^{s} \cos\theta_m|\theta_m\rangle\langle\theta_m| , \tag{19}$$

$$\sin\hat{\Phi}_{\theta} = \sum_{m=0}^{s} \sin\theta_m|\theta_m\rangle\langle\theta_m| \tag{20}$$

take the following form in the number-state basis:

$$\exp(i\hat{\Phi}_{\theta}) = \sum_{n=1}^{s} |n-1\rangle\langle n| + e^{i(s+1)\theta_0}|s\rangle\langle 0| , \tag{21}$$

$$\cos\hat{\Phi}_{\theta} = \frac{1}{2}\left\{ \sum_{n=1}^{s} |n-1\rangle\langle n| + e^{i(s+1)\theta_0}|s\rangle\langle 0| + \text{H.c.} \right\} , \tag{22}$$

$$\sin\hat{\Phi}_{\theta} = \frac{1}{2i}\left\{ \sum_{n=1}^{s} |n-1\rangle\langle n| + e^{i(s+1)\theta_0}|s\rangle\langle 0| - \text{H.c.} \right\} . \tag{23}$$

The commutation relations of these operators with $\hat{N}$ are

$$[\exp(i\hat{\Phi}_\theta),\hat{N}]=\exp(i\hat{\Phi}_\theta)-(s+1)e^{i(s+1)\theta_0}|s\rangle\langle 0| \ , \quad (24)$$

$$[\cos\hat{\Phi}_\theta,\hat{N}]=i\sin\hat{\Phi}_\theta-\tfrac{1}{2}\{(s-1)e^{i(s+1)\theta_0}|s\rangle\langle 0|+\text{H.c.}\} \ , \quad (25)$$

$$[\sin\hat{\Phi}_\theta,\hat{N}] = -i\cos\hat{\Phi}_\theta-\frac{1}{2i}\{(s+1)e^{i(s+1)\theta_0}|s\rangle\langle 0|-\text{H.c.}\} \ . \quad (26)$$

III. THE CREATION AND ANNIHILATION OPERATOR OF PHASE QUANTA

In the linear space Ψ, which is spanned by $(s+1)$ states (either the number states $|n\rangle$ or the phase states $|\theta_m\rangle$), there exists a close analogy between the number operator $\hat{N}$ and the phase operator $\hat{\Phi}_\theta$. For instance, the phase states $|\theta_m\rangle$ are the eigenstates of the phase operator $\hat{\Phi}_\theta$ [see Eq. (15)], while the number states are the eigenstates of the number operator $\hat{N}$. Furthermore, the operator $\exp(i\hat{\Phi}_\theta)$ plays the role of a *step* operator with respect to the number states, that is,

$$\exp(i\hat{\Phi}_\theta)|n\rangle=|n-1\rangle \ , \qquad \exp(-i\hat{\Phi}_\theta)|n\rangle=|n+1\rangle \ . \quad (27)$$

Analogously, the operator $\exp[-i\hat{N}2\pi/(s+1)]$ is a step operator with respect to the phase states:

$$\exp\left[-i\hat{N}\frac{2\pi}{s+1}\right]|\theta_m\rangle=|\theta_{m-1}\rangle \ , \qquad \exp\left[i\hat{N}\frac{2\pi}{s+1}\right]|\theta_m\rangle=|\theta_{m+1}\rangle \ . \quad (28)$$

The operator $\exp(i\hat{\Phi}_\theta)$ in the number-state basis can be written as

$$\exp(i\hat{\Phi}_\theta)=\sum_{n=1}^{s}|n-1\rangle\langle n|+e^{i(s+1)\theta_0}|s\rangle\langle 0| \ . \quad (29)$$

while the operator $\exp[-i\hat{N}2\pi/(s+1)]$ in the phase-state basis takes the form

$$\exp\left[-i\hat{N}\frac{2\pi}{s+1}\right]=\sum_{m=1}^{s}|\theta_{m-1}\rangle\langle\theta_m|+|\theta_s\rangle\langle\theta_0| \ . \quad (30)$$

Now we turn our attention to the fact that the number operator $\hat{N}$ is related to the photon creation ($\hat{a}^\dagger$) and annihilation ($\hat{a}$) operators as $\hat{N}=\hat{a}^\dagger\hat{a}$. Because $\hat{N}$ is the counterpart of the phase operator $\hat{\Phi}_\theta$, the natural question arises whether $\hat{\Phi}_\theta$ can be expressed as

$$\hat{\Phi}_\theta=\hat{\phi}^\dagger\hat{\phi} \ , \quad (31)$$

where $\hat{\phi}^\dagger$ and $\hat{\phi}$ play the role of the creation and annihilation operators of the *phase quanta*, respectively (that is, one can increase or decrease the phase by discrete increments which depend on the dimensionality of the state space). To answer this question we first note that the operators $\hat{a}^\dagger$ and $\hat{a}$ can be expressed through the photon number and the phase operators as [4,9,18,19]

$$\hat{a}=\exp(i\hat{\Phi}_\theta)(\hat{N})^{1/2}, \quad \hat{a}^\dagger=(\hat{N})^{1/2}\exp(-i\hat{\Phi}_\theta) \ . \quad (32)$$

From the analogy between $\hat{N}$ and $\hat{\Phi}_\theta$ it follows that the phase creation and annihilation operators should take the form

$$\hat{\phi}=\exp\left[-i\hat{N}\frac{2\pi}{s+1}\right](\hat{\Phi}_\theta)^{1/2} \ , \qquad \hat{\phi}^\dagger=(\hat{\Phi}_\theta)^{1/2}\exp\left[i\hat{N}\frac{2\pi}{s+1}\right] \ , \quad (33)$$

which implies that $\hat{\Phi}_\theta=\hat{\phi}^\dagger\hat{\phi}$. The operators $\hat{\phi}$ and $\hat{\phi}^\dagger$ act on the phase states in the same way as the operators $\hat{a}$ and $\hat{a}^\dagger$ act on the photon-number states, that is (see Fig. 1),

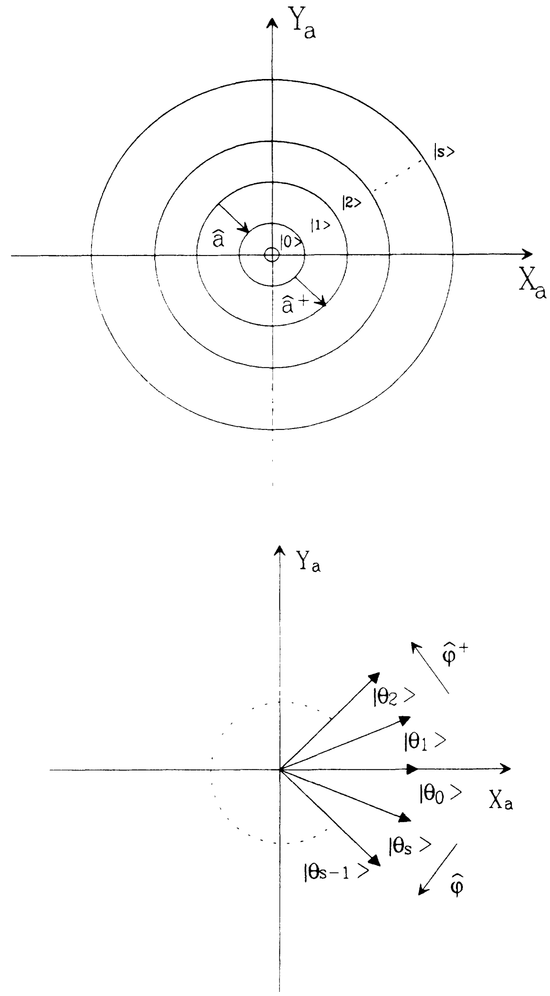

FIG. 1. We consider the phase space of the "harmonic oscillator" which is defined on the finite-dimensional state space. In (a) we show number states of this harmonic oscillator as annulae in phase space. The action of the photon creation and annihilation operators corresponds to the transition from one annulus to another. In (b) we show the phase states as vectors in the phase space and as seen from this picture the phase creation and annihilation operators are responsible for transitions between these vectors.

$$\hat{\phi}|\theta_m\rangle=\sqrt{\theta_m}|\theta_{m-1}\rangle,\quad \hat{\phi}^\dagger|\theta_m\rangle=\sqrt{\theta_{m+1}}|\theta_{m+1}\rangle\ , \tag{34}$$

and

$$\hat{\phi}|\theta_0\rangle=\sqrt{\theta_0}|\theta_s\rangle,\quad \hat{\phi}^\dagger|\theta_s\rangle=\sqrt{\theta_0}|\theta_0\rangle\ . \tag{35}$$

In the phase-state basis these operators take the form

$$\hat{\phi}=\sum_{m=1}^{s}\sqrt{\theta_m}|\theta_{m-1}\rangle\langle\theta_m|+\sqrt{\theta_0}|\theta_s\rangle\langle\theta_0|\ , \tag{36}$$

$$\hat{\phi}^\dagger=\sum_{m=1}^{s}\sqrt{\theta_m}|\theta_m\rangle\langle\theta_{m-1}|+\sqrt{\theta_0}|\theta_0\rangle\theta_s|\ . \tag{37}$$

The commutation relation for these operators is

$$[\hat{\phi},\hat{\phi}^\dagger]=\frac{2\pi}{s+1}-2\pi|\theta_s\rangle\langle\theta_s|\ . \tag{38}$$

If the operator $\hat{\phi}^\dagger$ acts k times, where $1\le k\le s$, on the phase "vacuum" state $|\theta_0\rangle$ we obtain

$$(\hat{\phi}^\dagger)^k|\theta_0\rangle=\left[\prod_{j=1}^{k}\theta_j\right]^{1/2}|\theta_k\rangle\ . \tag{39}$$

On the other hand, if the operator $\hat{\phi}^\dagger$ acts k times, where $k\ge s+1$ $[k=n(s+1)+m]$ on the phase "vacuum" state $|\theta_0\rangle$, then we obtain

$$(\hat{\phi}^\dagger)^k|\theta_0\rangle=\left[\prod_{j=0}^{s}\theta_j\right]^{n/2}\left[\prod_{j=1}^{m}\theta_j\right]^{1/2}|\theta_m\rangle\ .$$

From the above it follows that, in general,

$$(\hat{\phi}^\dagger)^k|\theta_0\rangle\neq(\hat{\phi}^\dagger)^m|\theta_0\rangle\ . \tag{40}$$

Here we note that the phase θ_m is defined as $\theta_m=\theta_0+2\pi m/(s+1)$. From Eq. (33) it follows that, if θ_0 is taken to be zero, then

$$(\hat{\phi}^\dagger)^k|\theta_0\rangle=\begin{cases}|\theta_0\rangle & \text{if } k=0\\ \left[\prod_{j=1}^{k}\theta_j\right]^{1/2}|\theta_k\rangle=\left[\frac{2\pi}{s+1}\right]^{k/2}\sqrt{k!}|\theta_k\rangle & \text{if } 1\le k\le s\\ 0 & \text{if } k\ge s+1\ ,\end{cases} \tag{41}$$

which means that, in this case, the complete analogy between the operators $\hat{\phi}$ and $\hat{a}$ acting on the states $|\theta_m\rangle$ and $|n\rangle$, respectively, is established. Obviously, the phase creation and annihilation operators are not Hermitian, so they are not observable (in the same way as $\hat{a}$ and $\hat{a}^\dagger$ are not observable operators). Nevertheless, they offer intuitive and calculation advantages in their use for description of the phase states of the harmonic oscillator [see Eq. (39)].

Once we have introduced the annihilation and creation operators of the phase, we can define new Hermitian (quadrature) operators $\hat{X}_\phi$ and $\hat{Y}_\phi$ as

$$\hat{X}_\phi=\frac{\hat{\phi}+\hat{\phi}^\dagger}{2}=\frac{\exp\{-i\hat{N}[2\pi/(s+1)]\}(\hat{\Phi}_\theta)^{1/2}+(\hat{\Phi}_\theta)^{1/2}\exp\{i\hat{N}[2\pi/(s+1)]\}}{2}\ , \tag{42}$$

$$\hat{Y}_\phi=\frac{\hat{\phi}-\hat{\phi}^\dagger}{2i}=\frac{\exp\{-i\hat{N}[2\pi/(s+1)]\}(\hat{\Phi}_\theta)^{1/2}-(\hat{\Phi}_\theta)^{1/2}\exp\{i\hat{N}[2\pi/(s+1)]\}}{2i}\ , \tag{43}$$

which are *not* equal to the operators $\cos\hat{\Phi}_\theta$ and $\sin\hat{\Phi}_\theta$ [see Eqs. (19) and (20)], but nevertheless are well-defined Hermitian quadrature phase operators analogous to the "position" ($\hat{X}_a$) and "momentum" ($\hat{Y}_a$) operators of the harmonic oscillator, which are defined as

$$\hat{X}_a=\frac{\hat{a}+\hat{a}^\dagger}{2},\quad \hat{Y}_a=\frac{\hat{a}-\hat{a}^\dagger}{2i}\ . \tag{44}$$

Finally we emphasize the fact that the phase creation and annihilation operators $\hat{\phi}^\dagger$ and $\hat{\phi}$ are defined consistently *only* when acting on the finite-dimensional state space Ψ, which means that the assumption concerning the finite dimension of the Hilbert space of the harmonic oscillator plays a central role in the proper definition of the phase creation and annihilation operators. In the Appendix we illustrate the ideas of the phase operator $\hat{\Phi}_\theta$ and the phase creation and annihilation operators with the simplest possible finite-dimensional Hilbert space, namely that of a two-state system (for instance, a two-level atom).

IV. $\hat{A}$-$\hat{B}$ SQUEEZING IN THE FINITE-DIMENSIONAL STATE SPACE

Let us consider two noncommuting Hermitian operators $\hat{A}$ and $\hat{B}$ acting on the finite-dimensional state space Ψ. The variances of these operators, $\langle(\Delta\hat{A})^2\rangle=\langle\hat{A}^2\rangle-\langle\hat{A}\rangle^2$ and $\langle(\Delta\hat{B})^2\rangle=\langle\hat{B}^2\rangle-\langle\hat{B}\rangle^2$, obey the uncertainty relation [1,2,37,38]

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle\ge\tfrac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2\ . \tag{45}$$

In the case where the commutator $[\hat{A},\hat{B}]$ is an operator the right-hand side of the uncertainty relation (45) is state dependent. Uncertainty relations with a state-dependent right-hand side are well known from earlier studies of the atomic coherent states (SU(2) coherent states) [38–40]. The states for which the left- and the right-hand sides of the relation (45) are equal, but where $\langle[\hat{A},\hat{B}]\rangle$ does not reach its local minimum, are called *intelligent* states [39]. If a local minimum is reached, then the state is called the *minimum uncertainty state*. Following Wódkiewicz and

Eberly [38] we shall say that the variances (fluctuations) of the operators $\hat{A}$ and $\hat{B}$ are squeezed if

$$\langle(\Delta\hat{A})^2\rangle<\tfrac{1}{2}|\langle[\hat{A},\hat{B}]\rangle \quad \text{or} \quad \langle(\Delta\hat{B})^2\rangle<\tfrac{1}{2}|\langle[\hat{A},\hat{B}]\rangle| \;. \tag{46}$$

To measure the degree of squeezing with respect to the pair of operators $\hat{A}$ and $\hat{B}$ we introduce two parameters S_A and S_B:

$$S_A=\frac{\langle(\Delta\hat{A})^2\rangle-\frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|}{\frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|}\;, \tag{47}$$

$$S_B=\frac{\langle(\Delta\hat{B})^2\rangle-\frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|}{\frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|}\;. \tag{48}$$

The squeezing condition now takes the simple form

$$S_A<0 \quad \text{or} \quad S_B<0\;, \tag{49}$$

and the maximum (100%) squeezing of variance $\langle(\Delta\hat{A})^2\rangle$ $[\langle(\Delta\hat{B})^2\rangle]$ corresponds to $S_A=1$ $[S_B=-1]$. Using the terminology introduced by Glauber and Lewenstein [41] we can say that the variable $\hat{A}$ ($\hat{B}$) is subfluctuant (superfluctuant) when $S_A<0$ ($S_B>0$). Obviously, the operator is 100% squeezed (subfluctuant) in its eigenstate.

V. COHERENT STATES IN A FINITE-DIMENSIONAL STATE SPACE

As we said earlier the phase operators are well defined when acting on the space Ψ spanned by $(s+1)$ state vectors. This is as it should be, for we know of many finite-dimensional systems for which excitation number and phase are relevant quantities. The coherent states of the two-level system are well known to exhibit squeezing in suitably chosen operator pairs. We elaborate on this point in the Appendix. Now the task is to define consistently the states of the harmonic oscillator in Ψ which in the limit $s\to\infty$ will approach the well-defined (physical) states of an infinite-dimensional (Fock) state space. This problem is generally not as simple as one expects at first sight.

For example, let us study the *analog* of Glauber's coherent state is a finite-dimensional space Ψ. Such a state *cannot* be defined as the eigenstate of the annihilation operator $\hat{a}$ (6) because the *only* eigenstate of this operator in Ψ is the vacuum state $|0\rangle$ with the eigenvalue equal to zero. The alternative possibility is to define the coherent state $|\alpha\rangle$ as the result of the action of the displacement operator $\hat{D}(\alpha)$,

$$\hat{D}(\alpha)=\exp(\alpha\hat{a}^{\dagger}-\alpha^*\hat{a})\;, \tag{50}$$

on the vacuum state $|0\rangle$, that is,

$$|\alpha\rangle=\hat{D}(\alpha)|0\rangle\;. \tag{51}$$

Generally, this is the most appropriate way [40] to define generalized coherent states. However, the commutator of the operators $\hat{a}$ and $\hat{a}^{\dagger}$(7) is not a c number and therefore the Baker-Hausdorff formula (the disentangling theorem for the Weyl-Heisenberg algebra [37]) cannot be used to evaluate the probability distribution of $|\alpha\rangle$ in the number-state basis of Ψ. To find the explicit expression for the coherent state $|\alpha\rangle$ in the finite-dimensional Hilbert space, that is to find the coefficients $C_n^{(s)}$ in the decomposition

$$|\alpha\rangle=\sum_{n=0}^{s}C_n^{(s)}|n\rangle\;, \tag{52}$$

we expand $\exp(\alpha\hat{a}^{\dagger}-\alpha^*\hat{a})$ in a formal infinite Taylor series and evaluate the action of the operators $(\alpha\hat{a}^{\dagger}-\alpha^*\hat{a})^k$ on the vacuum state $|0\rangle$ having in mind that $(\hat{a}^{\dagger})^k|0\rangle=0$ if $k>s$ (in the Appendix we illustrate the formalism presented above using the state space Ψ spanned by only two vectors $|0\rangle$ and $|1\rangle$). We are not able to express the coefficient obtained by this procedure in a closed analytical form. Nevertheless, it can be proved at least numerically (see below) that

$$\lim_{s\to\infty}C_n^{(s)}=\exp(-|\alpha|^2/2)\frac{\alpha^n}{\sqrt{n!}}\;, \tag{53}$$

which means that as s increases, the state $|\alpha\rangle$ given by Eq. (51) approaches the ordinary coherent state. We should note here that for finite s the mean excitation number $\langle\hat{N}\rangle$ is not equal to $|\alpha|^2$ (see the Appendix). Obviously, in the limit $s\to\infty$ we find that $\langle\hat{N}\rangle=|\alpha|^2$. The same problem as with the coherent state arises when the analog of the squeezed vacuum $|\xi\rangle$ in Ψ, that is, the state given by the relation [42]

$$|\xi\rangle=\exp[\xi(\hat{a}^{\dagger})^2-\xi^*\hat{a}^2]|0\rangle\;, \tag{54}$$

or any other generalized coherent state [40] in Ψ, is studied. Pegg and Barnett [18,19] overcome the problem of the precise definition of states of the harmonic oscillator in Ψ by assuming that s can always be taken large enough, such that

$$1-\sum_{n=0}^{s}|C_n|^2<\epsilon \tag{55}$$

for any arbitrarily small ϵ, where C_n are the coefficients evaluated in the infinite-dimensional state space, that is, the normalization condition is strictly fulfilled only in the limit $s\to\infty$.

Nevertheless, we would like to emphasize that to make all the calculations self-consistent it is better to utilize the properly normalized states in Ψ (that is $\sum_{n=0}^{s}|C_n^{(s)}|^2=1$). To do so we use the definition (51) from which it is seen that the coherent state $|\alpha\rangle$ is normalized to unity in Ψ. In order to evaluate the coefficients $C_n^{(s)}$ in Eq. (52) we turn our attention to the fact that the operators $\hat{a}$ and $\hat{a}^{\dagger}$

given by Eq. (6) can be expressed in Ψ as $(s+1)\times(s+1)$ matrices:

$$\hat{a}=\begin{bmatrix} 0 & \sqrt{1} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{2} & & 0 \\ \cdots & & & & \cdots \\ 0 & & & 0 & \sqrt{s} \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \tag{56}$$

$$\hat{a}^{\dagger}=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \sqrt{1} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \sqrt{s-1} & 0 & 0 \\ 0 & \cdots & 0 & \sqrt{s} & 0 \end{bmatrix},$$

and state vectors can be described as $(s+1)$-dimensional column vectors in Ψ. The displacement operator $\hat{D}(\alpha)$ given by Eq. (50) can be rewritten as

$$\hat{D}(\alpha)=\exp(\hat{T})=\sum_{n=0}^{\infty}\frac{1}{n!}(\hat{T})^n , \tag{57}$$

where the operator $\hat{T}$ is an $(s+1)\times(s+1)$ matrix

$$\hat{T}=\begin{bmatrix} 0 & -\alpha^*\sqrt{1} & 0 & \cdots & 0 \\ \alpha\sqrt{1} & 0 & -\alpha^*\sqrt{2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \alpha\sqrt{s-1} & 0 & -\alpha^*\sqrt{s} \\ 0 & \cdots & 0 & \alpha\sqrt{s} & 0 \end{bmatrix}. \tag{58}$$

Using the fact that such tridiagonal matrices can easily be multiplied numerically, we can evaluate the coefficients C_n^s in Eq. (52) straightforwardly and then find all quantities of interest in the $(s+1)$-dimensional Hilbert space. Of course, all these quantities depend parametrically on the dimension of the state space. If we are describing a real finite-dimensional system (for instance, a spin system of the type discussed in the Appendix), this dependence is as it should be. Nevertheless, these results can also be applied to harmonic oscillators (and light fields) if suitable care is taken ensuring that the limiting process $s\to\infty$ is performed appropriately. We remind the reader that our states are properly normalized throughout.

In Fig. 2 we plot the "photon-number distribution" $P_n^{(s)}$ defined as

$$P_n^{(s)}=|\langle n|\hat{D}(\alpha)|0\rangle|^2=|C_n^{(s)}|^2 \tag{59}$$

for the coherent state $|\alpha\rangle$ with $\alpha=4$ in $(s+1)$-dimensional state space for various values of s. We see that for $s\le|\alpha|^2$, $P_n^{(s)}$ significantly deviates from the Poissonian distribution

$$P_n=\exp(-|\alpha|^2)\frac{|\alpha|^{2n}}{n!} , \tag{60}$$

corresponding to the ordinary coherent state. The shape of the distribution (59) becomes more Poissonian in the large s limit, that is, if $s\gg|\alpha|^2$. In Fig. 3 we plot the mean photon number $\bar{n}$

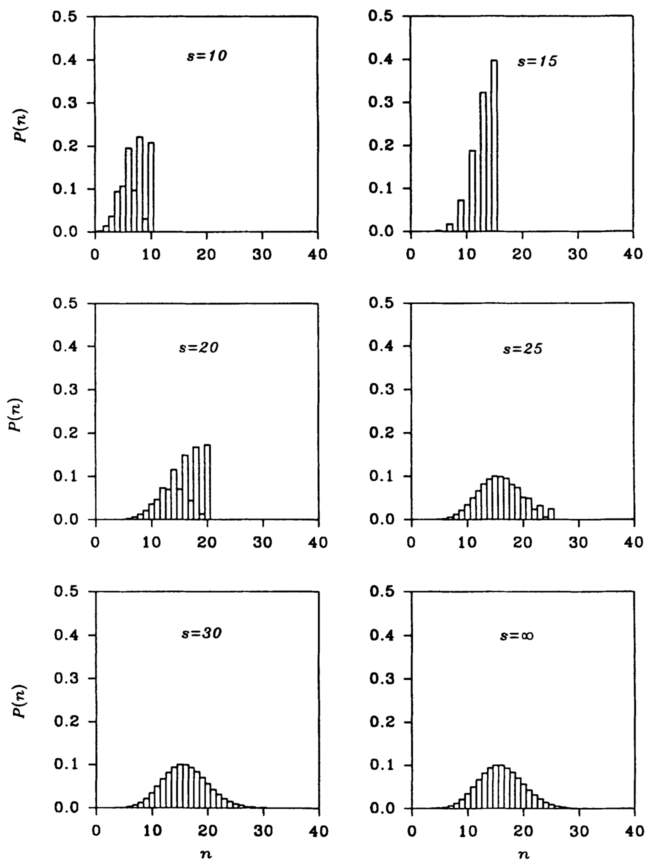

FIG. 2. Photon-number distribution $P_n^{(s)}$ of the coherent state as given by Eq. (59) for $\alpha=4$ (α is supposed to be real) and for various values of s. We see that for $s\gg\alpha^2=16$, $P_n^{(n)}$ very rapidly converges to the Poissonian distribution ($s\to\infty$).

$$\bar{n}=\sum_{n=0}^{s} n|C_n^{(s)}|^2 \tag{61}$$

as a function of s for the state given by Eq. (52). From this figure it is clearly seen that in the limit of $s\gg|\alpha|^2$ the mean photon number is equal to $|\alpha|^2$, as is expected for the Glauber coherent state.

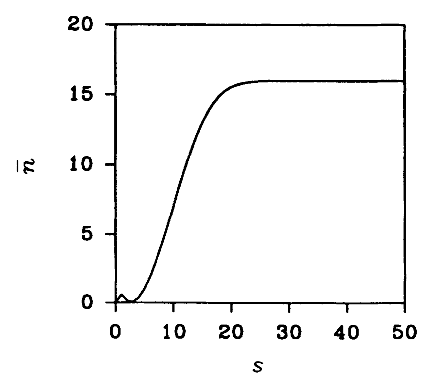

FIG. 3. Mean photon number $\bar{n}$ [see Eq. (61)] of the coherent state with $\alpha=4$ is plotted as a function of s. If $s\gg\alpha^2$, then $\bar{n}\simeq\alpha^2$, as one would expect for the ordinary coherent state.

VI. PHASE PROPERTIES OF COHERENT STATES

A. Phase probability distribution

We start our discussion on the phase properties of coherent states by exploring the phase-probability distribution of these states. The phase-probability distribution $P(\theta_m)$ is defined as [18,19]

$$P(\theta_m)=|\langle\theta_m|\alpha\rangle|^2 , \quad (62)$$

with the normalization condition

$$\sum_{m=0}^{s} P(\theta_m)=1 . \quad (63)$$

For the coherent state (52) we obtain

$$P(\theta_m)=\frac{1}{s+1}\left|\sum_{n=0}^{s} e^{-i\theta_m n}C_n^{(s)}\right|^2 . \quad (64)$$

This expression is plotted in Fig. 4 for $\alpha\leq4$ and $s=100$, that is, s is large enough so that the results obtained are physical. We have chosen the value of θ_0 such that the variance of the phase operator $\langle(\Delta\hat{\Phi}_\theta)^2\rangle$ is minimized (see discussion below). In particular, for real α we have chosen $\theta_0=-\pi$. From Fig. 4 it is seen that for $\alpha=0$, i.e., for the vacuum state, the phase probability distribution is uniform and equal to $1/(s+1)$ (i.e., is inversely proportional to the density of phase states). In the continuum limit the normalization condition (64) for the phase-probability limit should be rewritten in the form [18,19]

$$\frac{s+1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} P(\theta)d\theta=1 , \quad (65)$$

and the phase-probability distribution of the vacuum state is again uniform, but equal to $1/2\pi$. The mean value of the phase in the vacuum state is

$$\langle 0|\hat{\Phi}_\theta|0\rangle=\theta_0+\pi\frac{s}{s+1} , \quad (66)$$

and its variance $\langle(\Delta\hat{\Phi}_\theta)^2\rangle=\langle\hat{\Phi}_\theta^2\rangle-\langle\hat{\Phi}_\theta\rangle^2$ is

$$\langle(\Delta\hat{\Phi}_\theta)^2\rangle=\frac{\pi^2}{(s+1)^2}[\tfrac{1}{3}s^2+\tfrac{2}{3}s] . \quad (67)$$

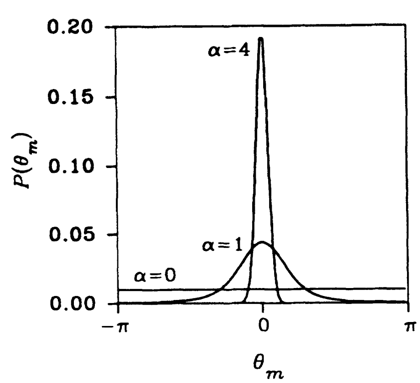

FIG. 4. Phase-probability distribution of the coherent field. The amplitude of the coherent field is supposed to be real; $\theta_0=-\pi$ and $s=100$. It is clearly seen that the higher the amplitude of the coherent field the smaller is the variance of the phase.

We see from Eq. (67) that the variance of the phase operator in the vacuum state does not depend on the particular value of θ_0 and in the limit $s\rightarrow\infty$ is equal to $\pi^2/3$. Generally this is the value of the variance for any state of random phase [18,19], that is, for any number state. On the other hand, in the limit $s\rightarrow\infty$ the mean value of the phase operator is equal to $\theta_0+\pi$, that is, it depends on the "reference" phase θ_0. This is also the case in classical optics, where the mean value of the phase $\bar{\phi}$ can be evaluated from the formula

$$\bar{\phi}=\int_{\theta_0}^{\theta_0+2\pi}\phi P(\phi)d\phi ,$$

with the phase distribution $P(\phi)=1/2\pi$.

The phase-probability distribution (64) for the nonzero value of α of the coherent state $|\alpha\rangle$ is localized around the phase of the coherent amplitude α. In our case α is supposed to be real and therefore $P(\theta_m)$ is localized around zero phase. This is clearly seen from Fig. 4, where we have chosen $\theta_0=-\pi$. As seen from this figure, the variance $\langle(\Delta\hat{\Phi}_\theta)^2\rangle$ depends on the intensity of the coherent field. Generally, the higher the intensity, the smaller the variance. It can be shown [9] that for high enough intensities $\bar{n}$ of the coherent field the variance of the phase is approximately equal to $1/4\bar{n}$. It should be emphasized here that the value of the variance of the phase depends not only on the intensity of the coherent field but also on the actual value of the reference phase θ_0. If $\theta_0=\phi-\pi$, where ϕ is the phase of the amplitude of the coherent field, that is, $\alpha=|\alpha|\exp(i\phi)$, the variance of the phase operator is minimized. On the other hand, if $\theta_0=\phi$, the variance reaches its maximal value. This can be clearly seen in Fig. 5 where the variance of the phase operator is plotted as a function of θ_0 for various values of α (the phase ϕ is supposed to be zero). We return later

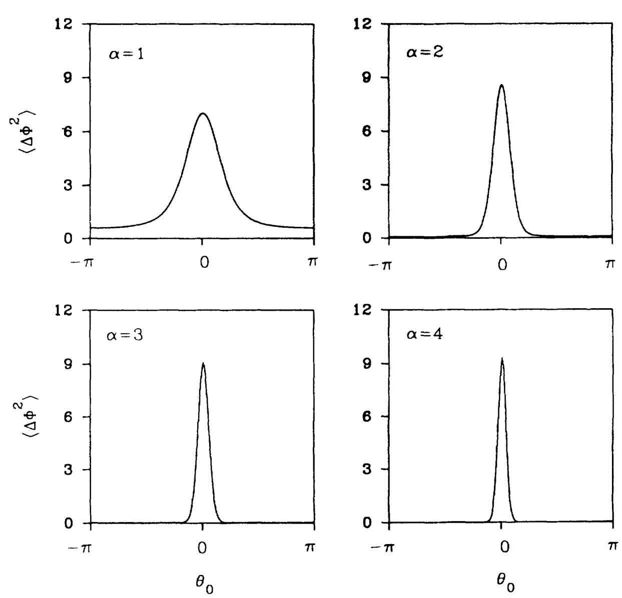

FIG. 5. Variance of the phase operator is plotted as a function of θ_0 for various values of α (the phase ϕ is supposed to be zero). The variance reaches its maximum for $\theta_0=\phi$ ($s=100$).

to the problem of the reference phase θ_0, but now we proceed to a discussion of the amplitude and phase squeezing.

B. Amplitude and phase squeezing

To find the degree of squeezing in $\hat{N}$ and $\hat{\Phi}_\theta$ we have to evaluate not only the variances of these operators but also the mean value of their commutator (see Sec. IV). As follows from Eq. (17), this commutator is an operator rather than a c number, which means that the mean value of $[\hat{N},\hat{\Phi}_\theta]$ is state dependent. Moreover, the mean value $\langle[\hat{N},\hat{\Phi}_\theta]\rangle$ does not depend only on the intensity of the coherent field but also on the value of the reference phase θ_0. In Fig. 6 we have plotted the mean value $|\langle[\hat{N},\hat{\Phi}_\theta]\rangle|$ as a function of θ_0 for various values of α. From this figure it is clearly seen that for $\alpha>0$ and $\theta_0=\pm\pi$ the mean value $|\langle[\hat{N},\hat{\Phi}_\theta]\rangle|$ approaches unity as was anticipated by Dirac [4]. Nevertheless, in the vicinity of θ_0 this mean value deviates significantly from unity. It should be stressed here that $|\langle[\hat{N},\hat{\Phi}_\theta]\rangle|$ is equal to unity for the same value for which the variance of the phase operator is minimized. Our numerical results are in agreement with the observation of Pegg and Barnett [18,19], who investigated the value of $\langle[\hat{N},\hat{\Phi}_\theta]\rangle$ in the large α limit and showed that [for $\alpha=|\alpha|\exp(i\phi)$]

$$\lim_{|\alpha|\to\infty}\langle[\hat{N},\hat{\Phi}_\theta]\rangle=i-2\pi i\delta(\phi-\theta_0)\ .$$

Once we know the value of the variance of the phase operator as well as the mean value of the commutator we are ready to evaluate the parameter S_Φ corresponding to the degree of squeezing in $\hat{\Phi}_\theta$. Analogously we can evaluate the parameter S_N, which measures the degree of squeezing in $\hat{N}$. The parameters S_N and S_Φ [see Eqs. (47) and (48)] are plotted in Figs. 7(a) and 7(b) as a function of α for $\theta_0=-\pi$. First we turn our attention to the parameter S_N [Fig. 7(a)], which provides us with information about the degree of squeezing in $\hat{N}$. In other words, this parameter can serve as a measure of the amplitude squeezing. From Fig. 7(a) it follows that in the limit $\alpha\to 0$ the parameter S_N tends to -1, which means that the vacuum state is 100% squeezed with respect to the operator $\hat{N}$ (see also the discussion in the paper by Vaccaro and Pegg [21]). This is a consequence of the fact that the vacuum state is the eigenstate of the number operator. As α increases the degree of squeezing in $\hat{N}$ decreases and for α larger than unity the parameter S_N is

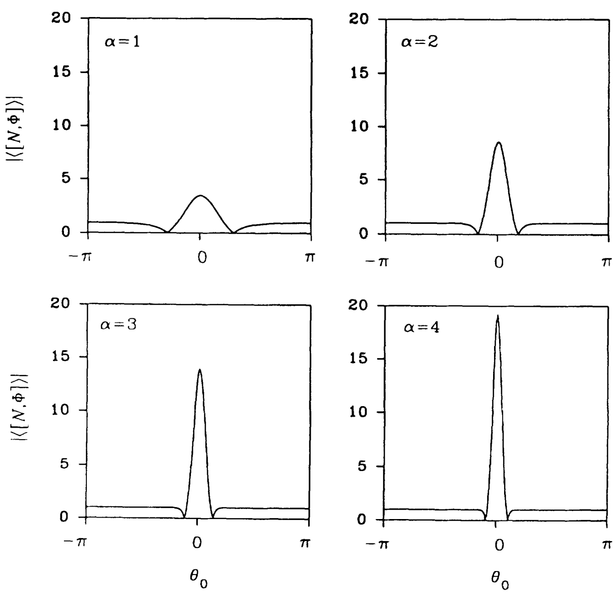

FIG. 6. Mean value $|\langle[\hat{N},\hat{\Phi}_\theta]\rangle|$ as a function of θ_0 for various values of α ($s=100$).

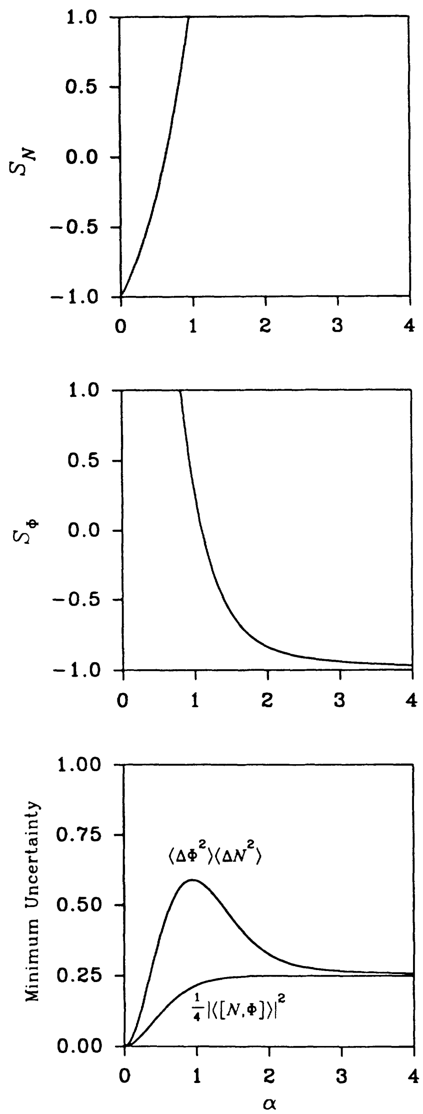

FIG. 7. Parameters (a) S_N and (b) S_Φ are plotted vs the coherent amplitude α. The reference phase $\theta_0=-\pi$ and $s=100$. We see that for small enough values of α the operator $\hat{N}$ is subfluctuant in the coherent field (that is, one can observe amplitude squeezing), while for $\alpha>1$ this operator is superfluctuant. Simultaneously, the photon statistics is Poissonian for any value of α. In (c) the product of the variances of the phase and the number operators as well as the mean value of the commutator of these operators are plotted vs the intensity of the coherent field. It is clear that the coherent field is not the MUS for small values of α.

greater than zero, which means that there is no amplitude squeezing for these values of α. It should be emphasized here that the photon-number distribution of the coherent field under consideration is Poissonian for any value of α for large enough values of s. In other words, the Mandel Q parameter [43]

$$Q=\frac{\langle(\Delta\hat{N})^2\rangle-\langle\hat{N}\rangle}{\langle\hat{N}\rangle} \tag{68}$$

is equal to zero for any value of α if s is large enough. Nevertheless, the number operator can be either subfluctuant or superfluctuant in this state, depending also on the value of the reference phase θ_0 (see below). From this we can conclude that one should carefully distinguish between the notion of sub-Poissonian statistics ($Q<0$) and that of amplitude squeezing ($S_N<0$). These two concepts are not identical.

From the parameter S_Φ we obtain information about the degree of squeezing in $\hat{\Phi}_\theta$. In particular, from Fig. 7(b) it follows that for small values of α the operator $\hat{\Phi}_\theta$ is superfluctuant in the coherent field, but as α increases this operator becomes subfluctuant. In the limit of high intensities of coherent fields the parameter S_Φ leads to -1, which means that these states describe a precisely defined phase as discussed in Ref. [9].

As we said earlier the degree of squeezing in $\hat{N}$ and $\hat{\Phi}_\theta$ depends not only on the intensity of the coherent field but also on the particular value of θ_0. To illustrate this effect we plot in Fig. 8 the parameters S_N and S_Φ as functions of θ_0 for various values of α. We see that in the vicinity of $\theta_0=\phi$ these parameters are very sensitive to the value of the reference phase θ_0. This is mainly due to the fact that the mean value of the commutator $|\langle[\hat{N},\hat{\Phi}_\theta]\rangle|$ around $\theta_0=\phi$ attains very large values (see Fig. 6). It is interesting to note that for small values of α (in particular, for $\alpha=1$) the variation of the phase θ_0 can lead to the appearance of squeezing in $\hat{N}$.

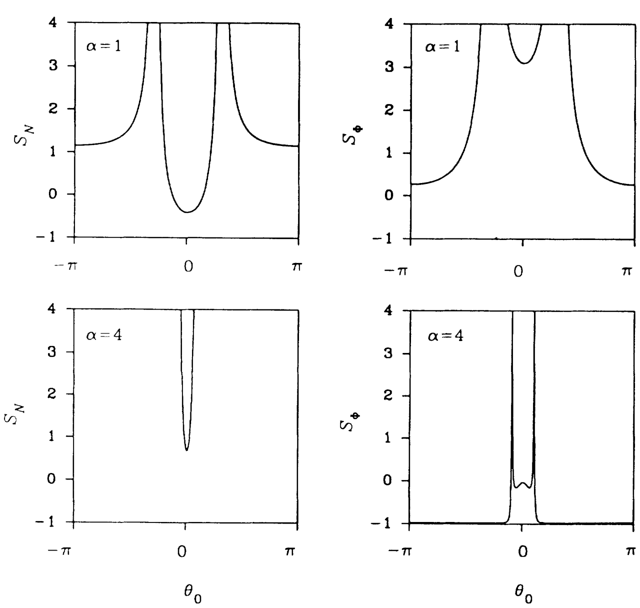

FIG. 8. Parameters S_N and S_Φ as functions of θ_0 for various values of α.

C. Number-phase uncertainty relation

We conclude our discussion on the phase properties of coherent fields with some remarks on the number-phase uncertainty relation

$$\langle(\Delta\hat{N})^2\rangle\langle(\Delta\hat{\Phi}_\theta)^2\geq\tfrac{1}{4}|\langle[\hat{N},\hat{\Phi}_\theta]\rangle| \tag{69}$$

for the coherent state. It is well known that coherent states are minimum uncertainty states with respect to the position and the momentum operators $\hat{X}_a$ and $\hat{Y}_a$ given by Eq. (44). This is true irrespective of the value of the intensity of the coherent field. The question is whether coherent states are minimum uncertainty states with respect to operators $\hat{N}$ and $\hat{\Phi}_\theta$. In Fig. 7(c) we have plotted both the left- and the right-hand sides of the uncertainty relation (69) as a function of α for $\theta_0=-\pi$. From this figure it follows that the coherent states under consideration are, strictly speaking, neither minimum uncertainty states nor intelligent states for any value of α except $\alpha=0$, that is, only the vacuum state is the MUS with respect to $\hat{N}$ and $\hat{\Phi}_\theta$ (this is in agreement with the discussion by Vaccaro and Pegg in Ref. [21]). For small values of α coherent states are far from being the MUS, but with increasing intensity they approach the MUS. In the limit of large intensities one can consider the coherent states to be the MUS with respect to $\hat{N}$ and $\hat{\Phi}_\theta$ (see also Ref. [9]). Similar results have been obtained also by Carruthers and Nieto [14], who have discussed the above problem in the framework of the SG formalism. The main different between the PB approach and the SG approach consists in dealing with the vacuum state. In particular, from the SG formalism it does not follow that in the limit $\alpha\to 0$ the coherent state under consideration becomes the MUS with respect to operators $\hat{N}$ and $\hat{\Phi}_\theta$.

VII. CONCLUSIONS

In this paper we have investigated in detail the phase properties of coherent states in a finite-dimensional Hilbert space. We have utilized the Pegg-Barnett formalism to define the phase operator. Because the phase operator is defined only in the finite-dimensional Hilbert space of a harmonic oscillator we have defined normalized coherent states using the analog of the Glauber displacement operator with the modified creation and annihilation operators given by Eq. (6). Our results can be applied for investigations of phase properties of physical systems having a finite number of states (such as a two-level atom, which is discussed in detail in the Appendix). We have shown that our results are valid also in the limit of large s. Namely, we have found that if the intensity of the coherent state is much smaller than the dimension of the state space, then the ordinary coherent state and the coherent state defined in the finite-dimensional state space have equal statistical and phase properties. We have introduced consistently the concept of amplitude and phase squeezing. We have shown that the effect of amplitude squeezing is not identical with sub-Poissonian photon statistics. In particular, we have shown that the weakly excited coherent state, which obviously has Poissonian photon statistics, can exhibit amplitude squeezing. We have analyzed in detail the dependence of the phase fluctuations and the amplitude squeezing on the value of the reference phase θ_0. We have shown that coherent states of weakly excited light fields are not minimum uncertainty states with respect to the phase and the number operators. Obviously, these states are minimum uncertainty states with respect to the quadrature operators.

ACKNOWLEDGMENTS

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APPENDIX: SIMPLE EXAMPLE— $\Psi=\{|0\rangle,|1\rangle\}$

It is instructive to see how the ideas advanced in this paper can be illustrated with the simplest finite-dimensional Hilbert space, namely that of a two-state system. Let us suppose that the state space Ψ is spanned just by two number-state vectors $|0\rangle$ and $|1\rangle$. Alternatively, Ψ can be spanned by phase states $|\theta_0\rangle$ and $|\theta_1\rangle$, which are related to $|0\rangle$ and $|1\rangle$ by [see Eq. (11)]

$$|\theta_0\rangle=\frac{1}{\sqrt{2}}[|0\rangle+e^{i\theta_0}|1\rangle] , \quad |\theta_1\rangle=\frac{1}{\sqrt{2}}[|0\rangle-e^{i\theta_0}|1\rangle] \tag{A1}$$

or

$$|0\rangle=\frac{1}{\sqrt{2}}[|\theta_0\rangle+|\theta_1\rangle], \quad |1\rangle=\frac{e^{-i\theta_0}}{\sqrt{2}}[|\theta_0\rangle-|\theta_1\rangle] . \tag{A2}$$

The states $|0\rangle$ and $|1\rangle$ can be associated with the lower and upper states of a two-level atom. In this case the phase states $|\theta_0\rangle$ and $|\theta_1\rangle$ describe coherent superpositions of the lower and upper states of the two-level atom. Recently it has been shown that these superposition states play a very important role in the dynamics of the two-level atom interacting with a coherent field [44] described in the framework of the Jaynes-Cummings model [45]. Physically the states $|\theta_{0,1}\rangle$ corresponds to the states with well defined values of the atomic-dipole amplitude which are either in phase or 180° out of phase with the applied field. As shown in [44], if the atom is initially prepared in the superposition state $|\theta_0\rangle$ or $|\theta_1\rangle$ and the cavity field is prepared in the coherent state, then in spite of complex quantum dynamics of the atom-field system governed by the Jaynes-Cummings Hamiltonian [45], the field as well as the atom remain in a pure state at $t>0$.

The operators of interest for the two-level system take, in the number-state basis $\{|0\rangle,|1\rangle\}$, the following form:

$$\begin{aligned}
&\hat{a}=|0\rangle\langle 1|, \quad \hat{a}^2=0 , \\
&\hat{N}=|1\rangle\langle 1|=\hat{N}^k , \\
&\hat{\Phi}_\theta=(\theta_0+\pi/2)-\frac{\pi}{2}(e^{-i\theta_0}|0\rangle\langle 1|+e^{i\theta_0}|1\rangle\langle 0|) , \\
&\hat{\Phi}_\theta^2=(\theta_0+\pi/2)^2+\frac{\pi^2}{4} \\
&\qquad -\pi(\theta_0+\pi/2)(e^{-i\theta_0}|0\rangle\langle 1|+e^{i\theta_0}|1\rangle\langle 0|) , \\
&[\hat{N},\Phi_\theta]=\frac{\pi}{2}(e^{-i\theta_0}|0\rangle\langle 1|-e^{i\theta_0}|1\rangle\langle 0|) , \\
&\cos\hat{\Phi}_\theta=\cos\theta_0(e^{-i\theta_0}|0\rangle\langle -e^{i\theta_0}|1\rangle\langle 0|) , \\
&\sin\hat{\Phi}_\theta=\sin\theta_0(e^{-i\theta_0}|0\rangle\langle 1|+e^{i\theta_0}|1\rangle\langle 0|) , \\
&\cos^2\hat{\Phi}_\theta=\cos^2\theta_0 , \\
&\sin^2\hat{\Phi}_\theta=\sin^2\theta_0 , \\
&[\cos\hat{\Phi}_\theta,\hat{N}]=\cos\theta_0(e^{-i\theta_0}|0\rangle\langle 1|-e^{i\theta_0}|1\rangle\langle 0|) , \\
&[\sin\hat{\Phi}_\theta,\hat{N}]=\sin\theta_0(e^{-i\theta_0}|0\rangle\langle 1|-e^{i\theta_0}|1\rangle\langle 0|) , \\
&\hat{\phi}=\frac{\sqrt{\theta_0}+\sqrt{\theta_1}}{2}(|0\rangle\langle 0|-1\rangle\langle 1|) \\
&\qquad +\frac{\sqrt{\theta_0}-\sqrt{\theta_1}}{2}(e^{-i\theta_0}|0\rangle\langle 1|-e^{i\theta_0}|1\rangle\langle 0|) , \\
&\hat{\phi}^2=\sqrt{\theta_0\theta_1} , \\
&[\hat{\phi},\hat{\phi}^\dagger]=\pi(e^{-i\theta_0}|0\rangle\langle 1+e^{i\theta_0}|1\rangle\langle 0|) ,
\end{aligned} \tag{A3a}$$

where $\theta_1=\theta_0+\pi$. These operators can be rewritten in terms of the energy (σ_3) and electric-dipole (σ_x,σ_y) operators, which can be expressed using the "spin-flip" operators σ_- and σ_+:

$$\sigma_-=|0\rangle\langle 1|, \quad \sigma_+=|1\rangle\langle 0| ,$$

$$\sigma_3=\tfrac{1}{2}(|1\rangle\langle 1|-|0\rangle\langle 0|)=\tfrac{1}{2}[\sigma_+,\sigma_-] .$$

Namely, we can define the electric dipole operators σ_x and σ_y as

$$\sigma_x=\frac{e^{i\theta_0}\sigma_++e^{-i\theta_0}\sigma_-}{2}, \quad \sigma_y=\frac{e^{i\theta_0}\sigma_+-e^{-i\theta_0}\sigma_-}{2i} ,$$

and then rewrite Eq. (A3a) as

$$\begin{aligned}
&\hat{a}=\sigma_-, \quad \hat{a}^2=0 , \\
&\hat{N}=\sigma_+\sigma_-=\sigma_3+1/2=\hat{N}^k , \\
&\hat{\Phi}_\theta=(\theta_0+\pi/2)-\pi\sigma_x , \\
&[\hat{N},\hat{\Phi}_\theta]=-i\pi\sigma_y , \\
&\cos\Phi_\theta=2\cos\theta_0\sigma_x , \\
&\sin\hat{\Phi}_\theta=2\sin\theta_0\sigma_x , \\
&[\cos\hat{\Phi}_0,\hat{N}]=-2i\cos\theta_0\sigma_y , \\
&[\sin\hat{\Phi}_\theta,\hat{N}]=-2i\sin\theta_0\sigma_y , \\
&\hat{\phi}=-(\sqrt{\theta_0}+\sqrt{\theta_1})\sigma_3-i(\sqrt{\theta_0}-\sqrt{\theta_1})\sigma_y , \\
&[\hat{\phi},\hat{\phi}^\dagger]=2\pi\sigma_x ,
\end{aligned} \tag{A3b}$$

which allows us to describe the phase properties of a two-level atom in terms of the dipole operators.

1. Coherent state in Ψ

If we adopt the definition of the coherent state given in Sec. V, then we have to define the state $|\alpha\rangle$ generated from the vacuum $|0\rangle$ by the action of the unitary displacement operator $\hat{D}(\alpha)$. This operator transforms $\hat{a}$ and $\hat{N}$ in Ψ as follows:

$$\hat{D}^{\dagger}(\alpha)\hat{a}\hat{D}(\alpha)=\hat{a}-e^{i\phi}\sin|\alpha|[\sin|\alpha|(e^{-i\phi}\hat{a}+e^{i\phi}\hat{a}^{\dagger})+(2\hat{N}-1)\cos|\alpha|]\ , \tag{A4}$$

$$\hat{D}^{\dagger}(\alpha)\hat{N}\hat{D}(\alpha)=\sin^2|\alpha|+[\sin|\alpha|\cos|\alpha|(e^{-i\phi}\hat{a}+e^{i\phi}\hat{a}^{\dagger})+\hat{N}\cos 2|\alpha|]\ , \tag{A5}$$

where $\alpha=|\alpha|e^{i\phi}$. The coherent state $|\alpha\rangle$ in the number-state basis reads

$$|\alpha\rangle=\cos|\alpha||0\rangle+e^{i\phi}\sin|\alpha||1\rangle\ . \tag{A6}$$

This coherent state can in fact be identified with the SU(2) coherent state for the two-level atomic system [38–40]. Obviously it is not an eigenstate of the annihilation operator $\hat{a}$. Coherent states (A6) are not orthogonal:

$$\langle\beta|\alpha\rangle=\cos|\alpha|\cos|\beta|+e^{i(\phi_\alpha-\phi_\beta)}\sin|\alpha|\sin|\beta|\ , \tag{A7}$$

and they are overcomplete, that is,

$$\int d\mu(\alpha)|\alpha\rangle\langle\alpha|=1\ , \tag{A8}$$

with properly chosen measured $d\mu(\alpha)$ (see, for instance, Ref. [38]). Finally we note that the mean value of the excitation operator $\hat{N}$ in the state (A6) is

$$\langle\alpha|\hat{N}|\alpha\rangle=|\langle 1|\alpha\rangle|^2=\sin^2|\alpha|\ .$$

2. $\hat{\Phi}_\theta-\hat{N}$ squeezing

The variances of the phase and the number operator and the mean value of their commutator in the coherent state (A6) are

$$\langle(\Delta\hat{\Phi}_\theta)^2\rangle=\frac{\pi^2}{4}[1-\sin^2 2|\alpha|\cos^2(\theta_0-\phi)]\ , \tag{A9}$$

$$\langle(\Delta\hat{N})^2\rangle=\tfrac{1}{4}\sin^2 2|\alpha|\ , \tag{A10}$$

$$\langle[\hat{N},\hat{\Phi}_\theta]\rangle=\frac{\pi}{2i}\sin(\theta_0-\phi)\sin 2|\alpha|\ . \tag{A11}$$

In what follows we will study the coherent states (A6) in which the operators $\hat{N}$ and $\hat{\Phi}_\theta$ are subfluctuant, that is, in which squeezing of the variances of these operators can be observed.

a. Squeezing in $\hat{N}$

Let suppose $\theta_0-\phi=\pi n+\pi/2$. In this case the coherent state (A6) takes the form

$$|\alpha\rangle=\cos|\alpha||0\rangle\pm\sin|\alpha|e^{i\theta_0}|1\rangle\ . \tag{A12}$$

It is easy to check that this is the *intelligent state* with respect to the operators $\hat{\Phi}_\theta$ and $\hat{N}$. The squeezing parameters given by Eqs. (47) and (48) describing the degree of squeezing of the variances of the operators $\hat{\Phi}_\theta$ and $\hat{N}$ in the state (A12) are, respectively,

$$S_\Phi=\frac{\pi-|\sin 2|\alpha||}{|\sin 2|\alpha||} \tag{A13}$$

and

$$S_N=\frac{\sin^2 2|\alpha|-\pi|\sin 2|\alpha||}{\pi|\sin 2|\alpha||}\ . \tag{A14}$$

From the above it follows that $\hat{\Phi}_\theta$ is superfluctuant in the coherent state (A12) (i.e., $S_\Phi>0$). On the other hand, the operator $\hat{N}$ is subfluctuant in (A12) ($S_N<0$) for *any* value of $|\alpha|$. Moreover, if $\sin 2|\alpha|\to 0$ (i.e., $|\alpha|=\pi n$ or $\pi n+\pi/2$), then $S_N\to -1$, while $S_\Phi\to\infty$. Simultaneously the uncertainty relation reaches its minimum, which means that the coherent state under consideration is the minimum uncertainty state. From (A12) we find that the minimum uncertainty states under consideration are

$$\lim_{|\alpha|\to\pi n}|\alpha\rangle=\pm|0\rangle \tag{A15}$$

and

$$\lim_{|\alpha|\to\pi n+\pi/2}|\alpha\rangle=\pm ie^{i\theta_0}|1\rangle\ . \tag{A16}$$

Obviously, these are the number states, in which the operator $\hat{N}$ is ultimately subfluctuant (100% squeezed).

We emphasize here that, in spite of the fact that the mean value of the operator $\hat{\Phi}_\theta$ as well as its variance are constant when $\theta_0-\phi=\pi/2$ or $3\pi/2$, that is [see also Eqs. (66) and (67) with $s=1$]

$$\langle\hat{\Phi}_\theta\rangle=\theta_0+\pi/2,\quad \langle(\Delta\hat{\Phi}_\theta)^2\rangle=\pi^2/4\ , \tag{A17}$$

the degree of squeezing S_Φ is sensitive to the value of $|\alpha|$, which means that the degree of fluctuations in $\hat{\Phi}_\theta$ does not depend *only* on the value of the variance $\langle(\Delta\hat{\Phi}_\theta)^2\rangle$.

b. Squeezing in $\hat{\Phi}_\theta$

If $|\sin 2|\alpha||=1$ (i.e., $|\alpha|=n\pi+\pi/4$ or $n\pi+3\pi/4$), then the coherent state (A6) is the intelligent state with respect to the operators $\hat{\Phi}_\theta$ and $\hat{N}$. The variances of these operators are

$$\langle(\Delta\hat{\Phi}_\theta)^2\rangle=\frac{\pi^2}{4}\sin^2(\theta_0-\phi)\ ,\quad \langle(\Delta\hat{N})^2\rangle=\tfrac{1}{4}\ , \tag{A18}$$

and the corresponding squeezing parameters take the form

$$S_\Phi=\frac{\pi\sin^2(\theta_0-\phi)-|\sin(\theta_0-\phi)|}{|\sin(\theta_0-\phi)|}\ , \tag{A19}$$

$$S_N=\frac{1-\pi|\sin(\theta_0-\phi)|}{\pi|\sin(\theta_0-\phi)|}\ . \tag{A20}$$

From above it follows that the coherent state under consideration becomes the minimum uncertainty state when $\theta_0-\phi=n\pi$. In this case $S_\Phi=-1$, while $S_N\to\infty$. Moreover, it can be found that

$$|\alpha\rangle = \begin{cases} +|\theta_0\rangle & \text{if } |\alpha|=\pi/4,\ \theta_0-\phi=0 \\ -|\theta_0\rangle & \text{if } |\alpha|=3\pi/4,\ \theta_0-\phi=\pi \\ +|\theta_1\rangle & \text{if } |\alpha|=\pi/4,\ \theta_0-\phi=\pi \\ -|\theta_1\rangle & \text{if } |\alpha|=3\pi/4,\ \theta_0-\phi=0\ . \end{cases} \tag{A21}$$

We can conclude that the phase and number states are the minimum uncertainty states with respect to operators $\hat{\Phi}_\theta$ and $\hat{N}$, which exhibit 100% squeezing in the phase and number states, respectively (see also Sec. VI and Ref. [21]).

3. $\cos\Phi_\theta - \hat{N}$ squeezing

The mean values of the operator $\cos\hat{\Phi}_\theta$ and its variance $\langle(\Delta\cos\hat{\Phi}_\theta)^2\rangle$ in the state $|\alpha\rangle$ (A6) are

$$\langle\cos\hat{\Phi}_\theta\rangle = \cos\theta_0 \sin 2|\alpha| \cos(\theta_0-\phi)\ , \tag{A22}$$

$$\langle(\Delta\cos\hat{\Phi}_\theta)^2\rangle = \cos^2\theta_0[1-\sin^2 2|\alpha|\cos^2(\theta_0-\phi)]\ , \tag{A23}$$

while the mean value of the commutator $[\cos\hat{\Phi}_\theta,\hat{N}]$ in this state is

$$\langle[\cos\hat{\Phi}_\theta,\hat{N}]\rangle = -i\cos\theta_0\sin 2|\alpha|\sin(\theta_0-\phi)\ . \tag{A24}$$

The mean value of the variance of the operator $\hat{N}$ in the state under consideration is given by Eq. (A10).

a. Squeezing in $\hat{N}$

If $\theta_0-\phi=(2n+1)\pi/2$, then the coherent state $|\alpha\rangle$

$$|\alpha\rangle = \cos|\alpha|\,|0\rangle \pm i e^{i\theta_0}\sin|\alpha|\,|1\rangle \tag{A25}$$

is the *intelligent state* with respect to the operators $\cos\hat{\Phi}_\theta$ and $\hat{N}$. In this case the squeezing parameters $S_{\cos\Phi}$ and S_N are

$$S_{\cos\Phi} = \frac{\cos^2\theta_0 - \frac{1}{2}|\cos\theta_0\sin 2|\alpha||}{\frac{1}{2}|\cos\theta_0 \sin 2|\alpha||}\ , \tag{A26}$$

$$S_N = \frac{\frac{1}{4}\sin^2 2|\alpha| - \frac{1}{2}|\cos\theta_0\sin 2|\alpha||}{\frac{1}{2}|\cos\theta_0\sin 2|\alpha||}\ . \tag{A27}$$

From these expressions it follows that if $\sin 2|\alpha|\to 0$ (i.e., if $|\alpha|=\pi n/2$), while $\theta_0\neq(2n+1)\pi/2$, then $S_{\cos\Phi}\to\infty$ and $S_N\to -1$, that is, the operator $\hat{N}$ is subfluctuant (with 100% degree of squeezing) in the number state $|0\rangle$ ($|\alpha|=\pi n$) and in the number state $|1\rangle$ [$|\alpha|=(2n+1)\pi/2$]. These number states are simultaneously minimum uncertainty states with respect to the pair of operators $\cos\hat{\Phi}_\theta$ and $\hat{N}$.

b. Squeezing in $\cos\hat{\Phi}_\theta$

Let us suppose $|\alpha|\to(2n+1)\pi/4$, while $\theta_0-\phi\neq(2n+1)\pi/2$. In this case the squeezing parameters $S_{\cos\Phi}$ and S_N describing the squeezing in the state (A6) are

$$S_{\cos\Phi} = \frac{\cos^2\theta_0\sin^2(\theta_0-\phi) - \frac{1}{2}|\cos\theta_0\sin(\theta_0-\phi)|}{\frac{1}{2}|\cos\theta_0\sin(\theta_0-\phi)|}\ , \tag{A28}$$

$$S_N = \frac{\frac{1}{4} - \frac{1}{2}|\cos\theta_0\sin(\theta_0-\phi)|}{\frac{1}{2}|\cos\theta_0\sin(\theta_0-\phi)|}\ . \tag{A29}$$

From above it follows that if $\theta_0-\phi=n\pi$ and $\theta_0\neq\pi/2$, then $S_\Phi=-1$, while $S_N\to\infty$ and the corresponding coherent state becomes the minimum uncertainty state. Moreover, it can be found that

$$|\alpha\rangle = \begin{cases} +|\theta_0\rangle & \text{if } |\alpha|=\pi/4,\ \theta_0-\phi=0 \\ -|\theta_0\rangle & \text{if } |\alpha|=3\pi/4,\ \theta_0-\phi=\pi \\ +|\theta_1\rangle & \text{if } |\alpha|=\pi/4,\ \theta_0-\phi=\pi \\ -|\theta_1\rangle & \text{if } |\alpha|=3\pi/4,\ \theta_0-\phi=0\ , \end{cases} \tag{A30}$$

i.e., the phase and number states are the minimum uncertainty states with respect to operators $\cos\hat{\Phi}_\theta$ and $\hat{N}$, which exhibit 100% squeezing in the phase and number states, respectively.

Finally we mention that both operators $\hat{\Phi}_\theta$ and $\cos\hat{\Phi}_\theta$ (as well as $\sin\hat{\Phi}_\theta$) can be used for measuring the phase of a state of a harmonic oscillator, but for particular values of α and θ the variances of these operators can exhibit different degree of squeezing. In the phase states $|\theta_m\rangle$ they exhibit 100% squeezing.

4. $\hat{X}_a$-$\hat{Y}_a$ squeezing

Using the definition of the quadrature operators given by Eq. (44) we find for the mean values of variances $\langle(\Delta\hat{X}_a)^2\rangle$ and $\langle(\Delta\hat{Y}_a)^2\rangle$ and for the mean value of the commutator $\langle[\hat{X}_a,\hat{Y}_a]\rangle$ in the state $|\alpha\rangle$ (A6) the following expressions:

$$\langle(\Delta\hat{X}_a)^2\rangle = \tfrac{1}{4}(1-\sin^2 2|\alpha|\cos^2\phi)\ , \tag{A31}$$

$$\langle(\Delta\hat{Y})^2\rangle = \tfrac{1}{4}(1-\sin^2 2|\alpha|\sin^2\phi)\ , \tag{A32}$$

and

$$\langle[\hat{X}_a,\hat{Y}_{a]}\rangle = \frac{i}{2}\cos 2|\alpha|\ . \tag{A33}$$

From here it follows that the coherent state $|\alpha\rangle$ is the intelligent state with respect to the operators $\hat{X}_a$ and $\hat{Y}_a$ if $\phi=n\pi$ or $(n+1/2)\pi$.

a. Squeezing in $\hat{X}_a$

Let $\phi=n\pi$. In this case the squeezing parameters S_{X_a} and S_{Y_a} are

$$S_{X_a} = \frac{\cos^2 2|\alpha| - |\cos 2|\alpha||}{|\cos 2|\alpha||} \tag{A34}$$

and

$$S_{Y_a} = \frac{1-|\cos 2|\alpha||}{|\cos 2|\alpha||}\ , \tag{A35}$$

which means that the operator $\hat{X}_a$ ($\hat{Y}_a$) is subfluctuant (superfluctuant) in the state

$$|\alpha\rangle=\cos|\alpha||0\rangle\pm\sin|\alpha|\,|1\rangle\ . \quad (A36)$$

In the limit $|\alpha|\to(\pi/2+n\pi)/2$ one can observe 100% squeezing in $\hat{X}_a$ quadrature, that is,

$$S_{X_a}\to -1,\quad S_{Y_a}\to\infty\ . \quad (A37)$$

In this case the coherent state (A36) takes the form

$$|\alpha\rangle=\frac{1}{\sqrt{2}}(|0\rangle\pm|1\rangle) \quad (A38)$$

and plays the role of the minimum uncertainty state with respect to the quadrature operators $\hat{X}_a$ and $\hat{Y}_a$. Moreover, this state is equal to the phase state $|\theta_0\rangle$ or $|\theta_1\rangle$ depending on the choice of the phase θ_0. This is in agreement with the observation by Schleich and co-workers [24,5], who related the squeezed vacuum exhibiting 100% squeezing (i.e., a *line* state, which is a quadrature eigenstate) to states with precisely defined phase.

b. Squeezing in $\hat{Y}_a$

Analogously, if $\phi=(2n+1)\pi/2$, then

$$S_{X_a}=\frac{1-|\cos 2|\alpha||}{|\cos 2|\alpha||} \quad (A39)$$

and

$$S_{Y_a}=\frac{\cos^2 2|\alpha|-|\cos 2|\alpha||}{|\cos 2|\alpha||}\ , \quad (A40)$$

which means that in this case the operator $\hat{Y}_a$ is subfluctuant and 100% squeezing in this quadrature can be observed in the limit $|\alpha|\to(\pi/2+n\pi)/2$. In this case the coherent state (A6) takes the form

$$|\alpha\rangle=\frac{1}{\sqrt{2}}(|0\rangle\pm i|1\rangle) \quad (A41)$$

and plays the role of the minimum uncertainty state with respect to the quadrature operators $\hat{X}_a$ and $\hat{Y}_a$.

Here we turn our attention to the fact that the coherent states (A38) [and analogously (A41)] for which 100% squeezing in $\hat{X}_a$ [$\hat{Y}_a$] can be observed are *equal* to the phase states $|\theta_{0,1}\rangle$ with $\theta_0=n\pi$ [$\theta_0=(n+1)\pi$].

5. $\hat{X}_\phi$-$\hat{Y}_\phi$ squeezing

In this subsection we turn our attention to the squeezing properties of the phase quadrature operators $\hat{X}_\phi$ and $\hat{Y}_\phi$ given by Eqs. (42) and (43). These operators in the phase-state basis in Ψ can be written as

$$\hat{X}_\phi=\frac{\sqrt{\theta_1}+\sqrt{\theta_0}}{2}[|\theta_0\rangle\langle\theta_1|+|\theta_1\rangle\langle\theta_0|]\ , \quad (A42)$$

$$\hat{Y}_\phi=\frac{\sqrt{\theta_1}-\sqrt{\theta_0}}{2i}[|\theta_0\rangle\langle\theta_1|-|\theta_1\rangle\langle\theta_0|]\ , \quad (A43)$$

and their commutator reads

$$[\hat{X}_\phi,\hat{Y}_\phi]=\frac{\theta_1-\theta_0}{2i}[|\theta_1\rangle\langle\theta_1|-|\theta_0\rangle\langle\theta_0|]\ . \quad (A44)$$

The coherent state $|\alpha\rangle$ (A6) can also be rewritten in the phase-state basis:

$$|\alpha\rangle=\beta_0|\theta_0\rangle+\beta_1|\theta_1\rangle\ , \quad (A45)$$

where

$$\beta_m=\frac{\cos|\alpha|+e^{i(\phi-\theta_m)}\sin|\alpha|}{\sqrt{2}}\ . \quad (A46)$$

Now one can straightforwardly evaluate the mean values of the variances of the quadrature operators $\hat{X}_\phi$ and $\hat{Y}_\phi$ as well as the mean value of their commutator:

$$\langle[\Delta\hat{X}_\phi(t)]^2\rangle=\left[\frac{\sqrt{\theta_1}+\sqrt{\theta_0}}{2}\right]^2\sin^2 2|\alpha|\ , \quad (A47)$$

$$\langle[\Delta\hat{Y}_\phi(t)]^2=\left[\frac{\sqrt{\theta_1}-\sqrt{\theta_0}}{2}\right]^2 \times[1-\sin^2 2|\alpha|\sin^2(\theta_0-\phi)]\ , \quad (A48)$$

$$\langle[\hat{X}_\phi,\hat{Y}_\phi]\rangle=i\frac{\theta_1-\theta_0}{2}\sin 2|\alpha|\cos(\theta_0-\phi)\ . \quad (A49)$$

a. Squeezing in $\hat{X}_\phi$

Let us suppose $\theta_0-\phi=0$. In this case the coherent state $|\alpha\rangle$ (A6) is the intelligent state with respect to the pair of operators $\hat{X}_\phi$ and $\hat{Y}_\phi$, i.e.,

$$\langle[\Delta\hat{X}_\phi(t)]^2\rangle\langle[\Delta\hat{Y}_\phi(t)]^2\rangle =\tfrac{1}{4}\langle[\hat{X}_\phi,\hat{Y}_\phi]\rangle|=\tfrac{1}{16}(\theta_1-\theta_0)^2\sin^2 2|\alpha|\ , \quad (A50)$$

and the squeezing parameter S_{X_ϕ} and S_{Y_ϕ} are

$$S_{X_\phi}=\frac{(\sqrt{\theta_1}+\sqrt{\theta_0})^2\sin^2 2|\alpha|-(\theta_1-\theta_0)|\sin 2|\alpha||}{(\theta_1-\theta_0)|\sin 2|\alpha||}\ , \quad (A51)$$

$$S_{Y_\phi}=\frac{(\sqrt{\theta_1}-\sqrt{\theta_0})^2-(\theta_1-\theta_0)|\sin 2|\alpha||}{(\theta_1-\theta_0)|\sin 2|\alpha||}\ . \quad (A52)$$

From the above it follows that

$$\lim_{|\alpha|\to\pi n/2} S_{X_\phi}=-1,\quad \lim_{|\alpha|\to\pi n/2} S_{Y_\phi}=\infty\ , \quad (A53)$$

which means that in the number state $|0\rangle$ the quadrature operator $\hat{X}_\phi$ is 100% squeezed, while the operator $\hat{Y}_\phi$ is superfluctuant.

b. Squeezing in $\hat{Y}_\phi$

Let us suppose $|\alpha|=(\pi n+\phi/2)/2$. In this case we find for the squeezing parameters S_{X_ϕ} and S_{Y_ϕ} the following expressions:

$$S_{X_\phi}=\frac{(\sqrt{\theta_1}+\sqrt{\theta_0})^2-(\theta_1-\theta_0)|\cos(\theta_0-\phi)|}{(\theta_1-\theta_0)|\cos(\theta_0-\phi)|}\,, \tag{A54}$$

$$S_{Y_\phi}=\frac{(\sqrt{\theta_1}-\sqrt{\theta_0})^2\cos^2(\theta_0-\phi)-(\theta_1-\theta_0)|\cos(\theta_0-\phi)|}{(\theta_1-\theta_0)|\cos(\theta_0-\phi)|}\,, \tag{A55}$$

from which it follows that in the limit $\theta_0-\phi\to\pi/2+\phi n$ the quadrature operator $\hat{Y}_\phi$ is 100% squeezed. The coherent state in which the quadrature operator $\hat{Y}_\phi$ is 100% squeezed is equal to the number state $|1\rangle$.

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