Order-α corrections to the decay rate of orthopositronium in the Fried-Yennie gauge

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The order-α correction to the decay rate of orthopositronium is obtained using the Fried-Yennie gauge. The result $(m\alpha^7/\pi^2)$ $[-1.987\,84(11)]$ is consistent with but more accurate than the results of previous evaluations.

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I. INTRODUCTION

The orthopositronium decay rate is of current interest because recent high-precision measurements [1,2],

$$\Gamma^1_{\rm expt}=7.0514(14)\ \mu{\rm s}^{-1}\ , \tag{1}$$

$$\Gamma^2_{\rm expt}=7.0482(16)\ \mu{\rm s}^{-1}\ , \tag{2}$$

are in apparent disagreement with the theoretical prediction. Orthopositronium decays primarily to three photons. The lowest-order expression for the rate [3,4],

$$\Gamma_{\rm LO}=\frac{2}{9\pi}(\pi^2-9)m\alpha^6 =7.211\,17\ \mu{\rm s}^{-1}\ , \tag{3}$$

was first obtained by Ore and Powell [5]. Order-α and $\alpha^2\ln(1/\alpha)$ corrections were calculated by several authors, with the result [6–15]

$$\begin{aligned}\Gamma&=\Gamma_{\rm LO}\left[1-10.282(3)\frac{\alpha}{\pi}-\frac{1}{3}\alpha^2\ln\left[\frac{1}{\alpha}\right]+\cdots\right]\\&=\Gamma_{\rm LO}+\frac{m\alpha^7}{\pi^2}[-1.9869(6)]\\&\quad+\frac{m\alpha^8}{\pi}\ln\left[\frac{1}{\alpha}\right]\left[-\frac{4}{9}\zeta(2)+\frac{2}{3}\right]+\frac{m\alpha^8}{\pi^3}X+\cdots\\&=7.038\,31(5)\ \mu{\rm s}^{-1}+\frac{m\alpha^8}{\pi^3}X+\cdots\ .\end{aligned} \tag{4}$$

The discrepancy is $0.0099(16)\ \mu{\rm s}^{-1}$, which is 6.2 times the experimental uncertainty [using the experimental results of Eq. (2)]. This discrepancy corresponds to a value of 50 for X since $m\alpha^8/\pi^3=0.000\,20\ \mu{\rm s}^{-1}$. A calculation of the order-α^2 correction X is clearly required.

In this work we present a higher-precision calculation of the order-α correction to the orthopositronium decay rate using the Fried-Yennie gauge. The Fried-Yennie-gauge photon propagator is [16]

$$D^{\mu\nu}_{\rm FY}=\frac{-1}{k^2}\left[g^{\mu\nu}+2\frac{k^\mu k^\nu}{k^2}\right]\ . \tag{5}$$

This gauge is of interest because it has a relatively simple covariant structure (convenient for multiloop graphs), and it is well behaved in the infrared (convenient for the study of bound states). We believe that the Fried-Yennie gauge will be useful for the calculation of the order-α^2 corrections. Our present work helps lay the foundation for such a calculation.

II. METHOD AND LOWEST-ORDER RESULT

The expression for the rest-frame decay rate of orthopositronium into three photons is

$$\begin{aligned}\Gamma=\frac{1}{2M}\int\frac{(d^3k_1)'}{2\omega_1}\frac{(d^3k_2)'}{2\omega_2}\frac{(d^3k_3)'}{2\omega_3}&\\\times(2\pi)^4\delta(P-k_1-k_2-k_3)&\\\times\sum_{\epsilon_1,\epsilon_2,\epsilon_3}\frac{1}{3}\sum_{\epsilon_m}\frac{1}{3!}|\mathcal{M}|^2\ .&\end{aligned} \tag{6}$$

Here $M=2W$ is the orthopositronium mass, $P=(2W,\mathbf{0})$ is the (rest-frame) orthopositronium energy-momentum vector, k_i is the energy-momentum vector of the ith final-state photon with $k_i^0=\omega_i=|\mathbf{k}_i|$, ϵ_i is the polarization vector of the ith final-state photon, ϵ_m is the orthopositronium polarization vector, and $(d^3k)'=d^3k/(2\pi)^3$ is the photon-integration measure. The decay amplitude $\mathcal{M}$ has contributions from all the graphs in Fig. 1. The graph of Fig. 1(a) gives the lowest-order decay rate, and the rest contribute order-α corrections to the lowest-order rate.

Energy-momentum conservation and symmetry considerations can be used to reduce the phase-space integral from nine to two dimensions. The energy-momentum-conserving δ function enforces the conditions

$$2W=\omega_1+\omega_2+\omega_3\ , \tag{7a}$$

$$\mathbf{0}=\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3\ , \tag{7b}$$

and immediately reduces the dimensionality from nine to five. Physically, the three-momenta of the final-state photons all lie in a plane. Three Euler angles describe the orientation of that plane in space. By rotational symmetry the spin-averaged differential decay rate is independent of these Euler angles, so they can be integrated out. Two degrees of freedom remain to describe the relative orientation of the photons in the decay plane. These two variables can be taken to be the energies of two of the photons since the angles between photons are given in

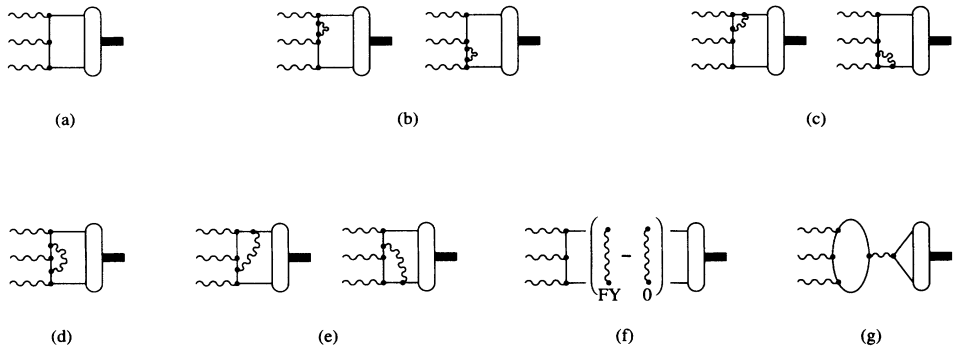

FIG. 1. Graphs contributing to the decay rate of orthopositronium. (a) is the lowest-order graph which leads to the Ore and Powell result [Eq. (3)] for the lowest-order decay rate. (b)–(g) represent order-α corrections: (b) self-energy, (c) outer vertex, (d) inner vertex, (e) double vertex, (f) subtracted binding diagram, and (g) annihilation. The subtracted binding diagram involves the difference between a Fried-Yennie gauge photon and a reference photon.

terms of the energies. The relation is

$$\cos(\theta_{ij})=\frac{x_k^2-x_i^2-x_j^2}{2x_ix_j}\ , \tag{8}$$

where θ_{ij} is the angle between photon i and photon j, photon k is the third photon, and $x_i=\omega_i/W$ is the normalized energy of photon i. On taking, for example, x_1 and x_3 as the remaining variables, one has that [10]

$$\Gamma=\frac{W}{2^7\pi^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_3\sum_{\epsilon_1,\epsilon_2,\epsilon_3}\frac{1}{3}\sum_{\epsilon_m}\frac{1}{3!}|\mathcal{M}|^2\ . \tag{9}$$

We use a bound-state formalism having an exactly soluble reference problem that contains the main physics of the Coulombic binding [17]. The positronium wave function that appears in Fig. 1 is this reference wave function. It satisfies the reference bound-state equation shown in Fig. 2. The order-α calculation is not sensitive to the details of the particular reference problem used. An approximation to the reference wave function that is sufficient for order-α work is

$$\Psi(p)=(2\pi)\delta(p_0)(2m)^{1/2}\begin{bmatrix}0 & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\epsilon}}_m\\ 0 & 0\end{bmatrix}\phi(\mathbf{p})\ . \tag{10}$$

Here

$$\phi(\mathbf{p})=\phi_0\frac{8\pi\gamma}{(|\mathbf{p}|^2+\gamma^2)^2} \tag{11}$$

is the nonrelativistic ground-state wave function with $\gamma=m\alpha/2$, and

$$\phi_0=\phi(\mathbf{x}=\mathbf{0})=(\gamma^3/\pi)^{1/2} \tag{12}$$

is the value of that wave function at contact. The mass of the reference bound state is $2W=2m[1-(\gamma/m)^2]^{1/2}$.

The lowest-order decay rate is obtained from the graph of Fig. 1(a). With the momentum and polarization labels as in Fig. 3, the decay amplitude is

$$\mathcal{M}_{\rm LO}=\int(dp)'\sum_{\sigma\in S_3}\mathrm{tr}\left[(-ie\gamma\epsilon_{\sigma(3)})\frac{i}{[\gamma(-\frac{1}{2}P+p+k_{\sigma(3)})-m]}(-ie\gamma\epsilon_{\sigma(2)})\right.$$
$$\left.\times\frac{i}{[\gamma(\frac{1}{2}P+p-k_{\sigma(1)})-m]}(-ie\gamma\epsilon_{\sigma(1)})\Psi(p)\right]\ , \tag{13}$$

where $(dp)'=d^4p/(2\pi)^4$. The sum is over the six permutations σ of the three final-state photons. The p_0 integral can be done immediately by use of the δ function in the wave function. The three momentum $\mathbf{p}$ is kept small (of order γ) by the wave function, and can be ignored (to lowest order) when compared to P and the photon momenta which have scales set by the electron mass. The $\mathbf{p}$ integral becomes simply

$$\int(d^3p)'\frac{8\pi\gamma}{(|\mathbf{p}|^2+\gamma^2)^2}=1\ . \tag{14}$$

Using the approximations

$$W\approx m\ , \tag{15a}$$

$$(\tfrac{1}{2}P-k_i)^2-m^2\approx-2m^2x_i\ , \tag{15b}$$

and the identity $e^2=4\pi\alpha$ one has

$$\mathcal{M}_{\rm LO}\approx -i\pi\alpha^3 \sum_{\sigma\in S_3} \frac{x_{\sigma(2)}}{x_1x_2x_3}\,\mathrm{tr}\left[\gamma\epsilon_{\sigma(3)}(-\gamma R_{\sigma(3)}+1)\gamma\epsilon_{\sigma(2)}(\gamma R_{\sigma(1)}+1)\gamma\epsilon_{\sigma(1)}\begin{bmatrix}0 & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\epsilon}}_m\\ 0 & 0\end{bmatrix}\right], \tag{16}$$

where $R_i=N-K_i$ with $N=(1,\mathbf{0})$ and $K_i=k_i/W$. Use of this decay amplitude in Eq. (9) gives the lowest-order decay rate. We performed the polarization sums by

$$\sum_{\epsilon}\epsilon_\mu^*\epsilon_\nu=-g_{\mu\nu}\ , \tag{17}$$

and the spin average by

$$\tfrac{1}{3}\sum_{\epsilon_m}\begin{bmatrix}0 & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\epsilon}}_m\\ 0 & 0\end{bmatrix}^*\otimes\begin{bmatrix}0 & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\epsilon}}_m\\ 0 & 0\end{bmatrix}=\tfrac{1}{3}[-\tfrac{1}{2}(1+\gamma_0)\gamma\epsilon_m]^*\otimes[-\tfrac{1}{2}(1+\gamma_0)\gamma\epsilon_m]$$

$$=\tfrac{1}{12}[(1+\gamma_0)\gamma^\alpha]^*\otimes[(1+\gamma_0)\gamma^\beta](-g_{\alpha\beta}+N_\alpha N_\beta)\ , \tag{18}$$

since ϵ_m has only spatial components. The resulting traces were done using REDUCE [18]. We found it convenient to delay doing one of the permutation-symmetrization sums until the end using the group property

$$\sum_\sigma F(\sigma)\sum_\tau G(\tau)=\sum_\sigma\left[F(\sigma)\sum_\tau G(\sigma\tau)\right]. \tag{19}$$

For the averaged and summed amplitude squared we obtained

$$\sum_{\epsilon_1,\epsilon_2,\epsilon_3}\frac{1}{3}\sum_{\epsilon_m}\frac{1}{3!}|\mathcal{M}_{\rm LO}|^2=\frac{64}{3}\frac{\pi^2\alpha^6}{(x_1x_2x_3)^2}\frac{1}{3!}\sum_{\sigma\in S_3}x_{\sigma(2)}(-x_{\sigma(1)}^3x_{\sigma(3)}-4x_{\sigma(1)}^3-4x_{\sigma(1)}^2x_{\sigma(3)}^2-7x_{\sigma(1)}^2x_{\sigma(3)}+16x_{\sigma(1)}^2$$
$$-x_{\sigma(1)}x_{\sigma(3)}^3-7x_{\sigma(1)}x_{\sigma(3)}^2+28x_{\sigma(1)}x_{\sigma(3)}-20x_{\sigma(1)}$$
$$-4x_{\sigma(3)}^3+16x_{\sigma(3)}^2-20x_{\sigma(3)}+8)\ . \tag{20}$$

The lowest-order decay rate was found by integrating one term of the permutation sum (since the phase space itself is symmetric in the final-state photons). The result is given in Eq. (3).

III. ORDER-α CORRECTIONS

The various order-α corrections to the decay amplitude give rise to order-α corrections to the decay rate. After taking into account the exact cancellation between the lowest-order process [Fig. 1(a)] and the subtraction term in the binding diagram (BD) [the second term in Fig. 1(f)], which follows from the reference bound-state equation of Fig. 2, one has

$$\mathcal{M}=\mathcal{M}_{\rm SE}+\mathcal{M}_{\rm OV}+\mathcal{M}_{\rm IV}+\mathcal{M}_{\rm DV}+\mathcal{M}_{\rm BD}+\mathcal{M}_A+\cdots\ , \tag{21}$$

where the terms are the self-energy (SE), outer-vertex (OV), inner-vertex (IV), double-vertex (DV), binding-diagram (BD), and the annihilation (A) contributions. Now the (unsubtracted) binding-diagram amplitude $\mathcal{M}_{\rm BD}$ has in it the lowest-order amplitude

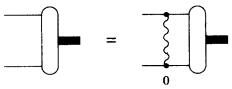

FIG. 2. The reference homogeneous bound-state equation.

$$\mathcal{M}_{\rm BD}=\mathcal{M}_{\rm LO}+\mathcal{M}_{\rm BD'} \tag{22}$$

in a way that will be made explicit shortly. So, one has

$$\mathcal{M}=\mathcal{M}_{\rm LO}+\mathcal{M}_{\rm SE}+\mathcal{M}_{\rm OV}+\mathcal{M}_{\rm IV}+\mathcal{M}_{\rm DV}+\mathcal{M}_{\rm BD'}+\mathcal{M}_A+\cdots\ , \tag{23}$$

and

$$|\mathcal{M}|^2=|\mathcal{M}_{\rm LO}|^2+2\ \mathrm{Re}[(\mathcal{M}_{\rm LO})^*(\mathcal{M}_{\rm SE}+\mathcal{M}_{\rm OV}+\mathcal{M}_{\rm IV}+\mathcal{M}_{\rm DV}+\mathcal{M}_{\rm BD'}+\mathcal{M}_A)]+\cdots\ . \tag{24}$$

The decay rate has corresponding contributions

$$\Gamma=\Gamma_{\rm LO}+\Gamma_{\rm SE}+\Gamma_{\rm OV}+\Gamma_{\rm IV}+\Gamma_{\rm DV}+\Gamma_{\rm BD'}+\Gamma_A+\cdots\ . \tag{25}$$

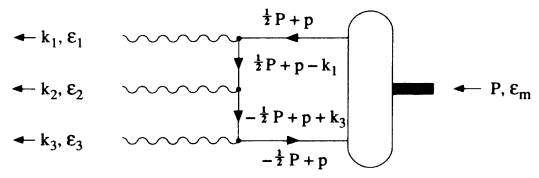

FIG. 3. The lowest-order orthopositronium decay diagram with momentum and polarization labels.

If we write the amplitude for one of the contributions as

$$\mathcal{M}_X=-i\pi\alpha^3\left[\frac{\alpha}{\pi}\right]\sum_{\tau\in S_3}\mathrm{tr}\left[X^{\tau}_{\mu_3,\mu_2,\mu_1}\begin{bmatrix}0 & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\epsilon}}_m\\ 0 & 0\end{bmatrix}\right]\times\epsilon^{\mu_3}_{\tau(3)}\epsilon^{\mu_2}_{\tau(2)}\epsilon^{\mu_1}_{\tau(1)}\ , \tag{26}$$

then the corresponding decay-rate correction is

$$\Gamma_X=\frac{m\alpha^7}{\pi^2}\frac{1}{48}\int_0^1 dx_1\int_{1-x_1}^1 dx_3\frac{1}{x_1x_2x_3}T\ , \tag{27}$$

where

$$T=\sum_{\sigma\in S_3}x_{\sigma(2)}\tfrac{1}{4}\mathrm{tr}[\,\gamma^{\mu_{\sigma(3)}}(-\gamma R_{\sigma(3)}+1)\times\gamma^{\mu_{\sigma(2)}}(\gamma R_{\sigma(1)}+1)\gamma^{\mu_{\sigma(1)}}(1+\gamma^0)\gamma^{\alpha}]\times\tfrac{1}{4}\mathrm{tr}[X^I_{\mu_3,\mu_2,\mu_1}(1+\gamma^0)\gamma^{\beta}](g_{\alpha\beta}-N_{\alpha}N_{\beta})\ . \tag{28}$$

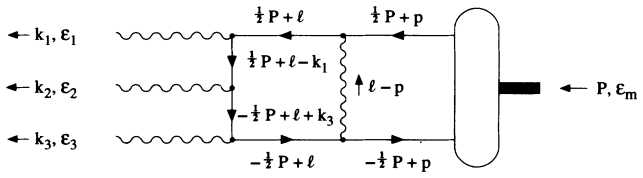

FIG. 4. The binding diagram with momentum and polarization labels.

Since the phase space is symmetric under permutations of the final-state photons, it is not necessary to symmetrize in both $\mathcal{M}^*_{\mathrm{LO}}$ and $\mathcal{M}_X$, so for $\mathcal{M}_X$ we used 3! times the term specified by the identity permutation ($\tau=I$).

The binding diagram requires the greatest care in evaluation. The (unsubtracted) binding diagram with momentum assignments is pictured in Fig. 4. The corresponding amplitude is

$$\mathcal{M}_{\mathrm{BD}}=(3!)\int(dl)'(dp)'\,\mathrm{tr}\left[(-ie\gamma^{\lambda})\frac{i}{\gamma(-\frac{1}{2}P+l)-m}(-ie\gamma\epsilon_3)\frac{i}{\gamma(-\frac{1}{2}P+k_3+l)-m}\times(-ie\gamma\epsilon_2)\frac{i}{\gamma(\frac{1}{2}P-k_1+l)-m}(-ie\gamma\epsilon_1)\frac{i}{\gamma(\frac{1}{2}P+l)-m}(-ie\gamma^{\kappa})\Psi(p)\right]\times\frac{-i}{(l-p)^2}\left[g_{\lambda\kappa}+2\frac{(l-p)_{\lambda}(l-p)_{\kappa}}{(l-p)^2}\right]. \tag{29}$$

After doing the p^0 integral this can be written as

$$\mathcal{M}_{\mathrm{BD}}=-i\alpha^4m^2(3!)\int(d^3p)'\frac{8\pi\gamma}{(|\mathbf{p}|^2+\gamma^2)^2}\int(dl)''\frac{1}{(l-p)^2[(l-\frac{1}{2}P)^2-m^2][(l+\frac{1}{2}P)^2-m^2]}\times\frac{\mathrm{tr}^{\lambda\kappa}(l)}{Z(l)}\left[g_{\lambda\kappa}+2\frac{(l-p)_{\lambda}(l-p)_{\kappa}}{(l-p)^2}\right], \tag{30}$$

where $p=(0,\mathbf{p})$ and $(dl)''=d^4l/(i\pi^2)=(4\pi)^2(dl)'/i$. The trace and denominator factors are

$$\mathrm{tr}^{\lambda\kappa}(l)=\mathrm{tr}\left[\gamma^{\lambda}[\gamma(l-\tfrac{1}{2}P)+m]\gamma\epsilon_3[\gamma(l-\tfrac{1}{2}P+k_3)+m]\gamma\epsilon_2[\gamma(l+\tfrac{1}{2}P-k_1)+m]\gamma\epsilon_1\times[\gamma(l+\tfrac{1}{2}P)+m]\gamma^{\kappa}\begin{bmatrix}0 & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\epsilon}}_m\\ 0 & 0\end{bmatrix}\right], \tag{31}$$

$$Z(l)=[(l-\tfrac{1}{2}P+k_3)^2-m^2][(l+\tfrac{1}{2}P-k_1)^2-m^2]\ . \tag{32}$$

The l integral in Eq. (30) is sensitive in the infrared. If the usual approximations $|\mathbf{p}|\to 0$, $W^2-m^2\to 0$ were made in the l integral it would diverge. The separation of the lowest-order term comes from writing

$$\frac{\mathrm{tr}^{\lambda\kappa}(l)}{Z(l)}=\frac{\mathrm{tr}^{\lambda\kappa}(0)}{Z(0)}+\frac{1}{Z(l)}\left[[\mathrm{tr}^{\lambda\kappa}(l)-\mathrm{tr}^{\lambda\kappa}(0)]+\frac{\mathrm{tr}^{\lambda\kappa}(0)}{Z(0)}[Z(0)-Z(l)]\right]. \tag{33}$$

The second term vanishes as $l\to 0$, and so the l integral there can be carried through after making the usual approximations. This contributes a part of the $\mathcal{M}_{\mathrm{BD}}{}'$ term in Eq. (22). The first term in Eq. (33) is proportional to the lowest-order amplitude

$$\frac{\mathrm{tr}^{\lambda\kappa}(0)}{Z(0)} \approx \frac{-N^{\lambda}N^{\kappa}}{x_1 x_3}\mathrm{tr}\left[\gamma\epsilon_3(-\gamma R_3+1)\gamma\epsilon_2(\gamma R_1+1)\gamma\epsilon_1\begin{bmatrix}0 & \sigma\cdot\hat{\epsilon}_m\\ 0 & 0\end{bmatrix}\right] . \tag{34}$$

It makes a contribution to the decay amplitude of

$$\left[\frac{\alpha}{\pi}\right]\mathcal{M}_{\mathrm{LO}}\int(d^3p)'\frac{8\pi\gamma}{(|\mathbf{p}|^2+\gamma^2)^2}\int(dl)''\frac{-m^2}{(l-p)^2[(l-\frac{1}{2}P)^2-m^2][(l+\frac{1}{2}P)^2-m^2]}\left[1+2\frac{l_0 l_0}{(l-p)^2}\right]$$
$$\approx\left[\frac{\alpha}{\pi}\right]\mathcal{M}_{\mathrm{LO}}\int(d^3p)'\frac{8\pi\gamma}{(|\mathbf{p}|^2+\gamma^2)^2}\left[\frac{\pi m}{|\mathbf{p}|}\arctan\left[\frac{|\mathbf{p}|}{\gamma}\right]-3\right]=\left[\frac{\alpha}{\pi}\right]\mathcal{M}_{\mathrm{LO}}\left[\frac{\pi}{\alpha}-3\right]=\mathcal{M}_{\mathrm{LO}}\left[1-3\frac{\alpha}{\pi}\right] . \tag{35}$$

This term, naively of order α, is in fact of order one and contains the lowest-order amplitude. The approach to the binding diagram used here was worked out by Tomozawa [19]. More details are given in the Appendix. The $-3(\alpha/\pi)\mathcal{M}_{\mathrm{LO}}$ in Eq. (35) is the other part of the $\mathcal{M}_{\mathrm{BD}'}$ term in Eq. (22). It is interesting to note that in covariant gauges other than the Fried-Yennie gauge an anomalously large contribution of order $\alpha\ln(\alpha)\mathcal{M}_{\mathrm{LO}}$ appears here.

The correction amplitude $\mathcal{M}_{\mathrm{BD}'}$ is the sum of two parts: $-3(\alpha/\pi)\mathcal{M}_{\mathrm{LO}}$ from Eq. (35), and the contribution of the second term of Eq. (33). This last mentioned contribution was evaluated numerically [18,20]. It is the sum of two parts associated with the $g_{\mu\nu}$ and the $2k_\mu k_\nu/k^2$ terms in the Fried-Yennie gauge propagator, called respectively the Feynman part and the gauge part. The Feynman part has been evaluated previously [9–11]. This integral is finite in the infrared, yet difficult to evaluate. When done with Feynman parameters, the six-dimensional integral is unstable as discussed by Caswell, Lepage, and Sapirstein [9]. They had to impose cutoffs and perform an extrapolation procedure in order to obtain a result. When done by poles, there is a five-dimensional integral to do: over the magnitude and angles of l and the two-dimensional phase space. Terms with a numerator of the form $l\cdot k_i$ diverge as $|l|\to 0$ unless the angular integrals are done first. However, in a Monte Carlo integration routine, the integrals are all done together, and points with small $|l|$ are hit that are not cancelled by corresponding points with the same small $|l|$ but different angles. The problem terms are eliminated by averaging over the directions specified by l and $-l$ [11], or by symmetrizing over final-state photons. We used the latter procedure. Our result for the Feynman part of $\mathcal{M}_{\mathrm{BD}'}$ is

$$\Gamma^F_{\mathrm{BD}'}=\frac{m\alpha^7}{\pi^2}[-1.125\,03(10)] . \tag{36}$$

The gauge part was done using Feynman parameters. It is

$$\Gamma^G_{\mathrm{BD}'}=\frac{m\alpha^7}{\pi^2}[0.364\,460(16)] . \tag{37}$$

The net result is

$$\Gamma_{\mathrm{BD}'}=-6\frac{\alpha}{\pi}\Gamma_{\mathrm{LO}}+\frac{m\alpha^7}{\pi^2}[-0.760\,57(10)]$$
$$=\frac{m\alpha^7}{\pi^2}[-1.92004(10)] . \tag{38}$$

The rest of the order-α corrections were easier to obtain since they do not contain pieces having anomalously low order. They were all done using Feynman parameters. The results are shown in Table I. The self-energy and vertex corrections required knowledge of the renormalized one-loop self-energy and vertex functions in the Fried-Yennie gauge [21]. The self-energy and outer-vertex contributions to the decay rate were obtained analytically [22]. The inner-vertex contribution was obtained through a four-dimensional numeral integration. The double-vertex contribution has a Feynman piece and a gauge piece. The result of the five-dimensional numerical integration for the Feynman piece was

$$\Gamma^F_{\mathrm{DV}}=\frac{m\alpha^7}{\pi^2}[-0.689\,419(14)] . \tag{39}$$

TABLE I. These numbers, multiplied by $m\alpha^7/\pi^2$ or by $(\alpha/\pi)\Gamma_{\mathrm{LO}}$, give the order-$\alpha$ corrections to the orthopositronium decay rate.

Graph	Contribution (units of $m\alpha^7/\pi^2$)	Contribution [units of $(\alpha/\pi)\Gamma_{\mathrm{LO}}$]
SE	−0.007 132 9	−0.036 911 1
OV	0.732 986 4	3.793 033 6
IV	0.167 881 3(10)	0.868 747(6)
DV	−0.804 253(15)	−4.161 82(8)
BD′	−1.920 04(10)	−9.935 8(6)
A	−0.157 280(12)	−0.813 89(7)
Total	−1.987 84(11)	−10.286 6(6)

Dimensional regularization was used in the annihilation contribution to regulate the would-be ultraviolet divergence, which disappeared after symmetrization over permutations of the final-state photons. Of course, the annihilation contribution is independent of gauge. Our result is consistent with the trend of previous evaluations [6,7,9,10,23].

We have chosen to write the order-α correction as $m\alpha^7/\pi^2$ times a number instead of $\Gamma_{\rm LO}(\alpha/\pi)$ times a number since the first seems more natural. The two contributions that are known analytically, $\Gamma_{\rm SE}$ and $\Gamma_{\rm OV}$, [22], have the form $m\alpha^7/\pi^2$ times a term of polylogarithmic degree three or four [24,25]. It is artificial to separate out a factor of π^2-9 from these terms.

IV. CONCLUSION

Our result for the order-α correction to the orthopositronium decay rate is given in the bottom row of Table I. It is

$$\frac{m\alpha^7}{\pi^2}[-1.987\,84(11)]=\frac{\alpha}{\pi}\Gamma_{\rm LO}[-10.2866(6)]\ . \tag{40}$$

Our result is more precise that the earlier result in Eq. (4) mostly because of an improved value for the Feynman part of the subtracted-binding diagram. The present prediction for the decay rate is

$$\Gamma=7.038\,236(10)\ \mu{\rm s}^{-1}+\frac{m\alpha^8}{\pi^3}X+\cdots\ . \tag{41}$$

The increased precision of the order-α correction does nothing to resolve the apparent discrepancy with experiment.

The binding diagram makes the largest contribution to the order-α correction. There are two gauge-invariant subsets of graphs that contribute to this decay: those without an internal annihilation to a single photon [Figs. 1(a)–1(f)], and the one with [Fig. 1(g)]. The total order-α contribution of the first set (the nonannihilation "NA" set) is

$$\Gamma_{\rm NA}=\frac{m\alpha^7}{\pi^2}[-1.830\,56(10)]\ , \tag{42}$$

of which the binding diagram part is

$$\Gamma_{\rm BD}{}'=\frac{m\alpha^7}{\pi^2}[-1.920\,04(10)]\ . \tag{43}$$

The binding diagram contributes 105% of the total. The OV and DV contributions are also sizable, 38% and 42% of the BD′ contribution in magnitude. The annihilation diagram contributes

$$\Gamma_A=\frac{m\alpha^7}{\pi^2}[-0.157\,280(12)]\ , \tag{44}$$

only 8% of the binding diagram part. These results suggest that one place to look for large contributions to the order-α^2 corrections is in graphs containing binding photons.

ACKNOWLEDGMENT

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APPENDIX

In this appendix we give some of the details of the integration leading up to Eq. (35). The loop integral

$$B(\mathbf{p})=\int (dl)''\frac{-W^2}{(l-p)^2[(l-\frac{1}{2}P)^2-m^2][(l+\frac{1}{2}P)^2-m^2]}\left[1+2\frac{l_0 l_0}{(l-p)^2}\right]\ , \tag{A1}$$

where $p^0=0$ can be done using Feynman parameters. In parametric form, with

$$h=(1-x)|\mathbf{p}|^2+\gamma^2\ , \tag{A2a}$$

$$H=x(1-2u)^2W^2+h\ , \tag{A2b}$$

one has

$$B(\mathbf{p})=W^2\int_0^1 dx\int_0^1 du\left[\frac{x}{H}+2(1-x)\frac{h}{H^2}\right]$$
$$=\int_0^1 dx\left[\frac{W}{\sqrt{hx}}\arctan(W\sqrt{x/h}\,)+\frac{(1-x)W^2}{h+xW^2}\right]\ . \tag{A3}$$

Now h tends to be small in bound states, so we can use the identity

$$\arctan(W\sqrt{x/h}\,)=\frac{\pi}{2}-\arctan\left[\frac{1}{W}\sqrt{h/x}\right] \tag{A4}$$

to pick out the leading term of $B(\mathbf{p})$. One can express $B(\mathbf{p})$ as

$$B(\mathbf{p})=\int_0^1 dx\frac{W}{\sqrt{hx}}\left[\frac{\pi}{2}\right]+R(\mathbf{p})$$
$$=\frac{\pi W}{|\mathbf{p}|}\arctan\left[\frac{|\mathbf{p}|}{\gamma}\right]+R(\mathbf{p})\ , \tag{A5}$$

where the remainder is

$$R(\mathbf{p})=\int_0^1 dx\left[\frac{-W}{\sqrt{hx}}\arctan\left[\frac{1}{W}\sqrt{h/x}\right]+\frac{(1-x)W^2}{h+xW^2}\right]. \tag{A6}$$

One might be tempted to use the smallness of h to approximate $R(\mathbf{p})$ as

$$\int_0^1 dx\left[\frac{-1}{\sqrt{hx}}\sqrt{h/x}+\frac{(1-x)}{x}\right]=-1\ . \tag{A7}$$

Indeed, for $|\mathbf{p}|=\gamma$ the integrand of $R(\mathbf{p})$ is within 1% of -1 for $0.05\le x\le 1$. However, for very small x the integrand differs sharply from -1. A correct evaluation of the first term in $R(\mathbf{p})$ is

$$\begin{aligned}R_1(\mathbf{p})&=\int_0^1 dx\left[\frac{-W}{\sqrt{hx}}\arctan\left[\frac{1}{W}\sqrt{h/x}\right]\right]\\&=-2\int_{\gamma/W}^{\infty}ds\frac{\arctan(s)}{|\mathbf{p}|^2/W^2+s^2}\\&=-2\int_0^{\infty}ds\frac{\arctan(s)}{|\mathbf{p}|^2/W^2+s^2}+2\int_0^{\gamma/W}ds\frac{\arctan(s)}{|\mathbf{p}|^2/W^2+s^2}\\&=-2+\ln\left[\frac{|\mathbf{p}|^2+\gamma^2}{W^2}\right]+\cdots\ ,\end{aligned} \tag{A8}$$

where

$$s=\frac{1}{W}\sqrt{h/x}\ . \tag{A9}$$

We have used the formula [24]

$$\int_0^{\infty}ds\frac{\arctan(s)}{1+a^2s^2}=\frac{1}{2a}\left[\mathrm{Li}_2\left[\frac{1}{a}\right]-\mathrm{Li}_2\left[\frac{-1}{a}\right]+\ln a\ln\left[\frac{1+1/a}{1-1/a}\right]\right]\approx\frac{1}{a^2}(1+\ln a)+O\left[\frac{1}{a^4}\right] \tag{A10}$$

with $a=W/|\mathbf{p}|$ to evaluate the first integral in Eq. (A8). The second term in $R(\mathbf{p})$ is unexceptional:

$$R_2(\mathbf{p})=\int_0^1 dx\frac{(1-x)W^2}{h+xW^2}=-1-\ln\left[\frac{|\mathbf{p}|^2+\gamma^2}{W^2}\right]+\cdots\ . \tag{A11}$$

In all, one has

$$R(\mathbf{p})=R_1(\mathbf{p})+R_2(\mathbf{p})\approx -3\ . \tag{A12}$$

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