Pauli problem for a spin of arbitrary length: A simple method to determine its wave function

Stefan Weigert
Institut für Physik der Universität Basel, Klingelbergstrasse 82, 4056 Basel, Switzerland
(Received 2 October 1991)

The problem of determining a pure state vector from measurements is investigated for a quantum spin of arbitrary length. Generically, only a finite number of wave functions is compatible with the intensities of the spin components in two different spatial directions, measured by a Stern-Gerlach apparatus. The remaining ambiguity can be resolved by one additional well-defined measurement. This method combines efficiency with simplicity: only a small number of quantities have to be measured and the experimental setup is elementary. Other approaches to determine state vectors from measurements, also known as the "Pauli problem," are reviewed for both spin and particle systems.

PACS number(s): 03.65.Bz

I. INTRODUCTION

It is straightforward to determine the expectation values of operators with respect to a given (pure or mixed) state of a quantum system, at least in principle. The inverse problem, to determine the quantum state of an ensemble of identically prepared individual systems by performing measurements, is a nontrivial task, even in principle.

Apparently, this question was raised for the first time in 1933: Pauli [1] pointed out that it was not known to him whether knowledge of the probability distribution of position and momentum, $|\psi(x)|^2$ and $|\psi(p)|^2$, would be sufficient to determine the wave function $|\psi\rangle$ of a particle. In the meantime interest in this problem continued, and various contributions [2–24] have been made to this field. Since there are numerous formulations of the original "Pauli problem" and since a variety of approaches to its solution exist (together with some erroneous statements), a review of the literature may be useful.

Instead of following the historical order of the contributions, a systematic approach seems to be more appropriate. Considering time-independent systems with bound states only, the Pauli problem decomposes naturally into eight different types, at least. First of all, one may restrict the performance of measurements to *one* instant of time only. Then the dimension of the Hilbert space $\mathcal{H}$ associated with the system under investigation may be finite or infinite. The first case occurs for spin systems, whereas the (countably) infinite-dimensional Hilbert space is typical for particle systems. In both cases one may either assume the system to be prepared in a *pure* state $|\psi\rangle$ or, what is more general, in a *mixed* state, described by a density matrix $\hat{\rho}$. The notation for the various cases is exhibited in Table I. Correspondingly, the four classes which arise if measurements are performed at *different* times or, equivalently, if the knowledge of time derivatives is required, are denoted by $\mathrm{I}_P(t)$, $\mathrm{II}_P(t)$, etc.

The aim of the present paper is to demonstrate the existence of a simple solution for the pure spin system (class II_P). It will be shown that measurements performed with a simple Stern-Gerlach apparatus are sufficient for the determination of a pure spin state. The number of measured quantities exceeds the number of free parameters of a pure state only by one, combining thus simplicity with efficiency.

The paper is organized as follows. The next section reviews a number of contributions to the Pauli problem. Section III describes the problem studied in this paper and discusses briefly the method to its solution. Subsequently, in Sec. IV, the derivation of the central statement is given. Then, in Sec. V, it is shown how a specific symmetry, due to the present approach, is reflected in the solution. Section VI contains a brief summary.

II. APPROACHES TO THE PAULI PROBLEM

Much work has been devoted to the problem of determining the pure state $|\psi\rangle$ of a particle in a known potential $V(x)$, that is, to class I_P. The number of positive statements, however, is limited. Instead, many authors, from Bargmann in Reichenbach's book [4] to Stulpe and Singer [24] provide counterexamples to the seemingly plausible guess that given values for the position and momentum distributions $|\psi(x)|^2$ and $|\psi(p)|^2$ might suffice to single out one and only one wave function $|\psi\rangle$. Reichenbach [4], Prugovečki [13], Vogt [15], Moroz [17], Wiesbrock [21], Friedman [19], and Stulpe and Singer [24] present examples of *pairs* of states, also called Pauli partners, which do entail equivalent position and momentum distributions. Typically, these constructions involve wave functions with specific behavior under reflection at the origin or under spatial translation by a certain amount, be it in one or three dimensions. Furthermore, the complex conjugate of a given wave function plays an important role in this context. However, exploiting properties of the eigenstates of the harmonic oscillator in one dimension, Corbett and Hurst [14] show the existence of

TABLE I. Notation for the types of the Pauli problem.

State	$\dim\mathcal{H}=\infty$	$\dim\mathcal{H}<\infty$
Pure	I_P	II_P
Mixed	I_M	II_M

 ©1992 The American Physical Society

a dense set of Pauli nonunique states in Hilbert space: The associated Pauli partners are not complex conjugate to each other and do not necessarily have a definite parity. To my knowledge the basic problem underlying the construction of counterexamples, namely, to enunciate actually the full set of states compatible with $|\psi(x)|^2$ and $|\psi(p)|^2$, has not yet been solved.

A number of constructive results can be found in the paper by Corbett and Hurst. For example, some general conditions for the existence of nonunique states are given. The question whether all real states (i.e., states with either a real position or momentum wave function) might be Pauli unique, raised by the same authors, has been answered in the negative by Pavičić [20]. In addition, one of the main theorems on particular real states seems to be incorrect, as is argued by Friedman [19], who presents a counterexample to the theorem. This author, in turn, proves the restricted set of "nonnegative states" to be Pauli unique.

Gale, Guth, and Trammell [5] claim that knowledge of the position distribution $|\psi(\mathbf{x})|^2$ in combination with the probability current $\mathbf{j}(\mathbf{x})$ allows one to determine the wave function $|\psi\rangle$. Writing $\psi(\mathbf{x})=f(\mathbf{x})\exp[iS(\mathbf{x})/\hbar]$ with real functions f and S, they argue that the relations $\rho(\mathbf{x})=|\psi(\mathbf{x})|^2=f(\mathbf{x})$ and

$$\begin{aligned}\mathbf{j}(\mathbf{x})&=\frac{\hbar}{2mi}[\psi^*(\mathbf{x})\nabla\psi(\mathbf{x})-\psi(\mathbf{x})\nabla\psi^*(\mathbf{x})]\\&=\rho(\mathbf{x})\nabla S(\mathbf{x})/m \qquad (1)\end{aligned}$$

admit only one solution, namely, $\psi(\mathbf{x})$, apart from an irrelevant total phase. But one easily works out that for a real wave function $[S(\mathbf{x})=0]$ the solution of the problem is not necessarily unique. Consider, for simplicity, only one spatial direction. Two wave functions $|\psi_\pm\rangle$ with the properties

$$\psi_\pm(-x)=\pm\psi_\pm(x) \quad \text{and} \quad \psi_\pm(0)=0\ , \qquad (2)$$

the moduli of which are equal almost everywhere, $|\psi_+(x)|^2=|\psi_-(x)|^2$, *cannot* be distinguished by the measurement of the probability $\rho(x)$ and the associated current $j(x)\equiv 0$. This construction of Pauli partners is easily generalized to more than one dimension.

Band and Park [16] treat the more general situation when the particle system under investigation is in a state to be described by a density matrix $\hat{\rho}$. Clearly, this type of Pauli problem, I_M, includes as a special case the problem for a particle in a pure state, I_P. These authors show that it is possible to expand the particle density matrix $\hat{\rho}$ in a series of expectation values containing only (appropriately symmetrized) products of powers of the basic variables position $\hat{x}$ and momentum $\hat{p}$. But the corresponding operators, termed a "quorum" for the determination of the state associated with $\hat{\rho}$, are not considered as "physically meaningful" by the authors, since the actual measurement of these quantities is not straightforward. Adopting the hypothesis that expectation values of the operators $d^n\hat{x}/dt^n$ and its powers, instead of $\hat{p}$ and its powers, are accessible in experiments, Band and Park arrive at a physically meaningful quorum, allowing one to determine the density matrix in its position representation. As a result, they have shifted to case $\mathrm{I}_M(t)$, since this method requires measurements of expectation values at different times. Apart from this investigation, the only attempt to determine a mixed particle state by measurements at different times, to my knowledge, is given by Gale, Guth, and Trammell [5], on the level of a thought experiment.

Feenberg's idea [2] to use $|\psi(x,t)|^2$ and its time derivative $\partial t|\psi(x,t)|^2/\partial t$ in order to work out the underlying particle wave function [class $\mathrm{I}_P(t)$] is reported in Kemble's book [3]; another discussion of this work is given by Reichenbach [4]. According to Gale, Guth, and Trammell, Kemble's generalization of Feenberg's argument from one to three spatial dimensions is not correct, and a detailed analysis of the error can be found in Royer [22].

It follows from this list that quite different sets of measurable quantities can be used to deal with the Pauli problem. For example, one can try to express the state as functions of expectation values of projection operators $|x\rangle\langle x|$ and $|p\rangle\langle p|$ or the probability density $j(x)$, of powers and products of the basic variables $\hat{x}$ and $\hat{p}$ or $\dot{\hat{x}}$, respectively. Such a variety of approaches is also present in the study of the Pauli problem for a spin to be considered now.

The case of pure spin states II_P is less intricate because of the finite-dimensional Hilbert space involved. Gale, Guth, and Trammell [5] present an approach which makes use of an advanced version of a Stern-Gerlach apparatus, called a Feynman filter (cf. Feynman, Leighton, and Sands [25]). It allows one to stop all components of a beam of spins in the state $|\psi\rangle$ except two, without disturbing their phase relation. Subsequently, the relative phase of the two remaining components can be determined. Combined with a measurement of the intensities of the components constituting the beam, one is able to derive from $6s$ numbers ($2s$ intensities plus 2 for each of the $2s$ relative phases) the underlying spin state. Since a pure state $|\psi\rangle$ is defined unambiguously by $4s$ real parameters, one might suspect that other methods exist which would require a smaller number of quantities to be measured. The particularly simple cases with $s=\frac{1}{2}$ or 1 have been analyzed by various authors in detail, including Band and Park [7], Busch and Lahti [23], and Stulpe and Singer [24]. For $s=\frac{1}{2}$ the study of a density matrix $\hat{\rho}$, corresponding to class II_M, is possible analytically and is also given by these authors. Systematic studies of this problem for arbitrary $s=\frac{1}{2},1,\frac{3}{2}\ldots$ are presented in two papers. The aforementioned method of using Feynman filters by Gale, Guth, and Tramell [5] can be adapted easily to the analysis of spin mixtures. It allows one, in principle, to determine directly the modulus and phase of each element of the density matrix $\hat{\rho}$ (cf. in this context d'Espagnat's general remark [11] on the measurability of the density matrix). Band and Park [8,9] choose a different approach. They show that there are $(2s+1)^2-1=4s(s+1)$ linearly independent "spin multipoles," the measurement of which fixes all $4s(s+1)$ free parameters of the density matrix. Nevertheless, according to the conclusion of the authors, one is left with the problem of actually performing the measurement of these multipoles, being a nontrivial task in general.

Two more paths have to be mentioned. On the one hand, Prugovečki [13] discusses the notion of "informational completeness" without referring explicitly to the Pauli problem. What set of operators are informationally complete; i.e., the expectation values of what sets of operators contain the same amount of information as the wave function does? Related work was done by Wiesbrock [21] and Busch and Lahti [23] (cf. Moroz [17] also). In the second of these papers, a method of state determination is described, requiring the simultaneous unsharp measurement of noncommuting observables. Royer [18,22] investigates the Pauli problem in combination with Wigner functions and their experimental determination.

On the other hand, Lamb [6], when formulating an operational interpretation of nonrelativistic quantum mechanics, proposed a method to work out a quantum state by placing it in different potentials; other contributions to this formulation of the Pauli problem are due to Kreinovitch [12] and Wiesbrock [21].

In summary, the statements about the particle version of the Pauli problem are not coherent. Clearly, the probability distributions $|\psi(x)|^2$ and $|\psi(p)|^2$ are not sufficient to determine the quantum state, but it is not known how to characterize in a sensible way the set of pure states compatible with given Pauli data. No theoretically *and* experimentally convincing approach to the state determination via measurements exists, even neglecting the more cumbersome case of mixtures. For spin systems the situation is more satisfactory, since solutions of the Pauli data problem are known, in principle. Nevertheless, it appears that the finite Hilbert space calls for a treatment simpler than the methods described above.

III. PAULI PROBLEM FOR A PURE SPIN STATE

This section consists of two parts. First, a particular formulation of the problem of Pauli data for a spin system is given, along with some general remarks. Second, the result, to be derived in the following section, is presented in a nontechnical way.

For a given quantum system, it is by no means obvious which observables have to be chosen in order that their measured expectation values determine the quantum state $|\psi\rangle$. For example, the intuitively appealing guess that for a particle in a potential $V(\mathbf{x})$ (without spin) knowledge of the probability distributions of position and momentum might suffice turned out to be wrong. Furthermore, as was emphasized above, it should be kept in mind that there will be no unique answer: Completely different sets of observable may have the required property.

In the following a spin of length $s=n/2$, $n\in\mathbb{N}$, is considered. The states $|\psi\rangle$ of the system are elements of the $(2s+1)$-dimensional Hilbert space $\mathcal{H}$ and, therefore, are specified by $(2s+1)$ complex coefficients with respect to any basis of $\mathcal{H}$. Because of the fact that only *rays*

$$|\psi\rangle=\{|\psi\rangle\,|\,|\psi\rangle=\alpha|\psi'\rangle, |\alpha|=1\} \tag{3}$$

are physically distinct, the absolute phase of a state $|\psi\rangle$ is undetermined, and since the states $|\psi\rangle$ are normalized,

$$\langle\psi|\psi\rangle=1, \quad |\psi\rangle\in\mathcal{H} , \tag{4}$$

the set of all states $|\psi\rangle\in\mathcal{H}$ can be labeled by $4s$ real parameters.

It is assumed that there is a source Q emitting particles into the positive x direction, say, in a well-defined pure spin state $|\psi\rangle$. The particles enter a standard Stern-Gerlach apparatus orientated along the z axis, for example. Passing through the inhomogeneous magnetic field $\mathbf{B}$, the beam is split into $2s+1$ components, which correspond to the different spin quantum numbers. A counter determines the intensities $\{|\psi_m(z)|^2\}$, that is, the squared moduli of the coefficients of expansion into the basis associated with the z direction [26]. As usual, the ensemble of equally prepared states $|\psi\rangle$ is supposed to be infinite, and the Stern-Gerlach apparatus is assumed to work perfectly well.

A natural way to pose the Pauli data problem is presented now, which, after some refinement, will be shown to allow the state determination unambiguously. Suppose that in a first series of measurements the intensities $\{|\psi_m(z)|^2\}$ of the state $|\psi\rangle$ have been determined with respect to the z axis, and that in a second series the intensities $\{|\psi_m(z')|^2\}$ have been measured with respect to a *different* direction z'. Without loss of generality, the direction z' may be chosen to lie in the yz plane. In other words, the set of measured quantities consists of $2(2s+1)$ operators projecting onto the eigenstates of the spin operator $\hat{\mathbf{S}}$ in the z and z' directions, respectively [27]. Does the set of these $4s+2$ numbers $\{|\psi_m(z)|^2, |\psi_m(z')|^2\}$ determine the state $|\psi\rangle$ uniquely? Because of the normalization conditions

$$\sum_{m=-s}^{s}|\psi_m(z)|^2=\sum_{m=-s}^{s}|\psi_m(z')|^2=1 , \tag{5}$$

only $4s$ out of all $4s+2$ intensities are independent. Consequently, this choice of Pauli data for a spin system is sensible: The number of independent real parameters derived from experiment agrees with the dimension of the manifold of physically distinct states $|\psi\rangle$.

Indeed, the considerations of the subsequent section lead to the result that only a finite number of states out of the $4s$-dimensional set of states is compatible with the observed intensities, measured along two noncollinear directions z and z'. In the generic case the remaining ambiguity turns out to be 2^{2s}-fold and can be resolved by measuring the expectation value of the x component of the spin $\hat{\mathbf{S}}$. Its origin can be understood as follows. As functions of the intensities, the unknown relative phases fulfill a set of $2s$ *quadratic* equations. Intuitively speaking, each of these relations contributes two roots, giving rise to a set of totally 2^{2s} solutions. The invariance of the measured intensities with respect to a specific transformation of the apparatus effects that solutions occur in pairs. This phenomenon will be discussed in more detail in Sec. V.

IV. DETERMINATION OF THE SPIN STATE

The eigenfunctions of the third component $\hat{S}^z$ of the spin operator $\hat{\mathbf{S}}$ are denoted by $|z;m\rangle$, $-s\le m\le s$, and they fulfill

$$\hat{S}^z|z;m\rangle=\hbar m|z;m\rangle\ ,\quad -s\le m\le s\ . \tag{6}$$

Constituting a complete orthonormal set of basis vectors in Hilbert space $\mathcal{H}$, any pure spin state $|\psi\rangle$ may be expressed as a linear superposition of these $2s+1$ states. The system under study in the following is supposed to be in a particular state:

$$|\psi\rangle=\sum_{m=-s}^{s}\psi_m(z)|z;m\rangle\equiv\sum_{m=-s}^{s}e^{i\phi_m}|\psi_m(z)|\,|z;m\rangle\ , \tag{7}$$

with complex coefficients $\psi_m(z)$, which from now on are assumed to be different from zero. If one or more coefficients $\psi_m(z)$ happen to be zero, an infinitesimal change in the definition of the z direction generically is sufficient to deal with a state having nonzero coefficients only.

Suppose that in a first series of measurements with the Stern-Gerlach apparatus the moduli $\{|\psi_m(z)|^2, -s\le m\le s\}$ have been determined. Clearly, in Hilbert space $\mathcal{H}$, there is a $2s$-dimensional submanifold $\mathcal{M}\subset\mathcal{H}$ of physically distinct states which is compatible with these numbers. The set $\mathcal{M}$ is given by

$$\mathcal{M}:\ |\psi(\bar{\chi})\rangle=\sum_{m=-s}^{s}e^{i\bar{\chi}_m}|\psi_m(z)|\,|z;m\rangle\ , \tag{8}$$

with the angles $\bar{\chi}_m\in[-\pi,\pi)$, $-s\le m\le s$. The set of states $\mathcal{M}$ is more conveniently parametrized by the angles $\chi_m\in[-\pi,\pi)$ instead of $\bar{\chi}_m$, defined by the relation $\bar{\chi}_m=\phi_m+\chi_m$, $-s\le m\le s$, corresponding to a shift of the origin. Then the value $\chi=0$ denotes the state $|\psi\rangle$,

$$|\psi(\chi=0)\rangle\equiv|\psi\rangle\ . \tag{9}$$

Now consider a rotation R_x of the Stern-Gerlach apparatus transforming the z direction into z' not parallel to z, and z' lies in the yz plane. Associated with the direction z' there is another basis $\{|z';m\rangle, -s\le m\le s\}$, representing the complete orthonormal set of eigenvectors of the third component of the transformed spin operator $\hat{S}^{z'}=\hat{U}(R_x)\hat{S}^z\hat{U}^{-1}(R_x)$. Here $\hat{U}(R_x)$ is the unitary operator which represents the spatial rotation R_x about the x axis in Hilbert space $\mathcal{H}$. The state $|\psi\rangle$ reads, with respect to the primed basis,

$$|\psi\rangle=\sum_{m=-s}^{s}\psi_m(z')|z';m\rangle\ , \tag{10}$$

and the coefficients ψ_m transform according to

$$\psi_m(z')=\sum_{m'=-s}^{s}U_{mm'}(z',z)\psi_{m'}(z)\ . \tag{11}$$

The matrix elements $U_{mm'}(R_x^{-1})\equiv\langle z';m|z;m'\rangle$ of $\hat{U}(R_x^{-1})\equiv\hat{U}(z',z)$ are known, in principle.

Consider the manifold $\mathcal{M}$ of states $|\psi(\chi)\rangle$ being compatible with the first series of measurements, expressed with respect to the primed basis $|z';m\rangle$. The coefficients of expansion are given by

$$\psi_m(z',\chi)=\sum_{m'=-s}^{s}U_{mm'}(z',z)e^{i\chi_{m'}}\psi_{m'}(z)\ , \tag{12}$$

leading to

$$|\psi_m(z',\chi)|^2=\sum_{m',\bar{m}=-s}^{s}U_{mm'}(z',z)U^*_{m\bar{m}}(z',z)\times e^{i(\chi_{m'}-\chi_{\bar{m}})}\psi_{m'}(z)\psi^*_{\bar{m}}(z)\ . \tag{13}$$

The question of whether or not a second series of measurements which fixes the numerical values of the quantities $|\psi_m(z')|^2$ is sufficient to determine the state $|\psi\rangle$ amounts to studying the following problem. Do the quantities $\{|\psi_m(z',\chi)|^2, -s\le m\le s\}$ represent a unique parametrization of the manifold $\mathcal{M}$? In other words, is it possible to invert the transformations Eq. (13) unambigously,

$$\chi=\chi(\{|\psi_m(z')|^2\})\ , \tag{14}$$

so that the phases χ (more precisely $2s$ *relative* phases) are determined by the moduli $\{|\psi_m(z')|^2\}$? The relations in Eq. (13) are *not* uniquely invertible: A finite number of ambiguities arise which can be resolved by one additional well-defined measurement. The proof will proceed in two steps. First, it is shown that Eq. (13) generically has 2^{2s} solutions, and in the second step it is demonstrated that, generically, these solutions lead to different values of the expectation value of the first component $\hat{S}^x$ of the spin operator.

It turns out that a simple way to obtain all possible solutions of Eq. (13) consists in assuming the directions z and z' to differ only infinitesimally. Experimentally, this requires the intensities to be measured up to first order. The matrix of rotation about the x axis, $\hat{U}(R_x)$, has nonzero elements in the diagonal and on the adjacent lines only, to first order in the infinitesimal angle of rotation ϵ. This immediately follows from the properties of the creation and annihilation operators [28] $\hat{S}^{\pm}$,

$$\hat{S}^{\pm}|z;m\rangle=\hbar\sqrt{s(s+1)-m(m\pm1)}|z;m\pm1\rangle \equiv C_m^{\pm}|z;m\pm1\rangle\ , \tag{15}$$

since $\hat{S}^x=\frac{1}{2}(\hat{S}^++\hat{S}^-)$. The operator for an infinitesimal rotation reads

$$\hat{U}(z',z)=\exp(i2\pi\epsilon\hat{S}^x/\hbar)\simeq1+2i\bar{\epsilon}\hat{S}^x+O(\bar{\epsilon}^2)\ , \tag{16}$$

with $\bar{\epsilon}\equiv\pi\epsilon/\hbar$ or, explicitly in the $(2s+1)$ dimensional matrix representation,

$$\begin{aligned}U_{mm'}(z',z)&=\delta_{mm'}+2i\bar{\epsilon}\langle z;m|\hat{S}^x|z;m'\rangle+O(\bar{\epsilon}^2)\\&=\delta_{mm'}+i\bar{\epsilon}(C^-_{m'}\delta_{mm'-1}+C^+_{m'}\delta_{mm'+1})\\&\quad+O(\bar{\epsilon}^2)\ .\end{aligned} \tag{17}$$

Using this expression in Eq. (13), one obtains, to first order in $\bar{\epsilon}$,

$$|\psi_m(z',\chi)|^2=|\psi_m(z)|^2+2i\bar{\epsilon}|\psi_m(z)|[|\psi_{m+1}|C^-_{m+1}\sin(\Delta\phi_{m+1}+\Delta\chi_{m+1})-|\psi_{m-1}|C^+_{m-1}\sin(\Delta\phi_m+\Delta\chi_m)] , \quad (18)$$

for $-s\le m\le s$, where the notation $\Delta\phi_m=\phi_m-\phi_{m-1}$, etc., has been introduced. Requiring these expressions for the intensities to be equal to those associated with the coefficients $\psi_m(z)\equiv\psi_m(z,\chi=0)$ of the original state $|\psi\rangle$, one obtains, to first order,

$$|\psi_m(z)|\,|\psi_{m+1}(z)|C^-_{m+1}[\sin(\Delta\phi_{m+1}+\Delta\chi_{m+1})-\sin(\Delta\phi_{m+1})]$$
$$=|\psi_m(z)|\,|\psi_{m-1}(z)|C^+_{m-1}[\sin(\Delta\phi_m)-\sin(\Delta\phi_m+\Delta\chi_m)] . \quad (19)$$

Choosing $m=-s$, the right-hand side (RHS) is equal to zero (because of $C^+_{-s-1}\equiv 0$), so that

$$\sin(\Delta\phi_{-s+1}+\Delta\chi_{-s+1})=\sin(\Delta\phi_{-s+1}) \quad (20)$$

is required for the existence of additional solutions. Remember that all moduli $|\psi_m(z)|$ are assumed to be nonzero and that all $C^\pm_m$, $-s\le m\le s$, are fixed nonzero numbers. If Eq. (20) is fulfilled, the RHS of Eq. (19) vanishes for $m=-s+1$; therefore one is left with

$$\sin(\Delta\phi_{-s+2}+\Delta\chi_{-s+2})=\sin(\Delta\phi_{-s+2}) , \quad (21)$$

and repeating this argument, one finally obtains $2s$ equations

$$\sin(\Delta\phi_m+\Delta\chi_m)=\sin(\Delta\phi_m) , \quad -s+1\le m\le s , \quad (22)$$

which simultaneously have to be fulfilled in order that $|\psi(\chi)\rangle$ give the same Pauli data $\{|\psi_m(z,\chi)|,|\psi_m(z',\chi)|\}$ as the state $|\psi\rangle$ does. Any of the Eqs. (22) has two solutions because from the requirement

$$\sin(\alpha+\beta)=\sin\alpha , \quad (23)$$

for any given number $\alpha\in[0,2\pi)$, two values of $\beta\in(-\pi,\pi]$ follow, namely,

$$\beta_-=0 \text{ and } \beta_+=\begin{cases}\pi-2\alpha, & \alpha\in[0,\pi)\\ 3\pi-2\alpha, & \alpha\in[\pi,2\pi) .\end{cases} \quad (24)$$

The values $\alpha=\pi/2,3\pi/2$ are exceptions since only β_- is a solution.

As a consequence, the set of all wave functions compatible with the set of Pauli data, $\{|\psi_m(z)|,|\psi_m(z')|\}$, generically consists of 2^{2s} elements. It will be demonstrated in a moment that indeed all 2^{2s} states are different. To exhibit these states explicitly, it is convenient to write $\beta_\pm\equiv\beta(\tau)$, $\tau=\pm1$,

$$\beta(\tau)=(1+\tau)[(1+\sigma)\pi-\alpha] , \quad (25)$$

with

$$\sigma(\alpha)=\begin{cases}-\frac{1}{2}, & \alpha\in[0,\pi)\\ +\frac{1}{2} , & \alpha\in[\pi,2\pi) .\end{cases}$$

For the set of allowed states $|\psi(\tau)\rangle$ labeled by the vector $\tau=(\tau_{-s+1},\tau_{-s+2},\dots,\tau_s)$, $\tau_n=\pm1$, one finds

$$|\psi(\tau)\rangle=\sum_{m=-s}^{s}|\psi_m(z)|\,|z;m\rangle\, e^{i(\phi_m+i\chi_m)}$$
$$=e^{i(\phi_{-s}+\chi_{-s})}\sum_{m=-s}^{s}|\psi_m(z)|z;m\rangle e^{i(\Delta\phi_m+\Delta\chi_m+\Delta\phi_{m-1}+\Delta\chi_{m-1}+\cdots+\Delta\phi_{-s+1}+\Delta\chi_{-s+1})} . \quad (26)$$

Suppressing the phase factor $\exp[i(\phi_{-s}+\chi_{-s})]$ does not change the ray $|\psi\rangle$ under consideration; hence, using Eq. (25) $2s$ times in the form

$$\Delta\chi_m=(1+\tau_m)[(1+\sigma_m)\pi-\Delta\phi_m] , \quad -s+1\le m\le s \quad (27)$$

($\sigma_m=\pm\frac{1}{2}$), one obtains the 2^{2s} states

$$|\psi(\tau)\rangle=\sum_{m=-s}^{s}|\psi_m(z)|\,|z;m\rangle\exp\left[i\sum_{m'=-s+1}^{m}(1+\tau_{m'})(1+\sigma_{m'})\pi-\tau_{m'}\Delta\phi_{m'}\right]$$
$$\equiv\sum_{m=-s}^{s}|\psi_m(z)|\,|z;m\rangle\exp[i\mu_m(\tau)] . \quad (28)$$

Choosing all elements of τ equal to -1, one recovers the state $|\psi\rangle$. It is a straight-forward calculation to verify that the relative phases of neighboring coefficients $\Delta\mu_m(\tau)$ indeed fulfill Eq. (22). Since

$$\Delta\mu_m(\tau)=\pi(1+\tau_m)(1+\sigma_m)-\tau_m\Delta\phi_m$$
$$=\begin{cases}3\pi-\Delta\phi_m, & \tau_m=+1, \quad \sigma_m=+\frac{1}{2}\\ \pi-\Delta\phi_m, & \tau_m=+1, \quad \sigma_m=-\frac{1}{2}\\ \Delta\phi_m, & \tau_m=-1, \quad \sigma_m=\pm\frac{1}{2}\ ,\end{cases} \quad (29)$$

one has $\sin\Delta\mu_m(\tau)=\sin\Delta\phi_m$, $-s+1\le m\le s$, what was to be shown.

The 2^{2s} wave functions $|\psi(\tau)\rangle$ compatible with a set of intensities with respect to the z and z' directions can be distinguished by their expectation values of the x component $\hat{S}^x$ of the spin operator $\hat{\mathbf{S}}$. Including this measurement, one has obtained information related to all three spatial directions, which intuitively seems to be necessary to get full knowledge about the state of the system. As a result, in total, $4s+1$ numbers are required for the specification of a pure wave function according to the method described here.

The value of $\langle\psi(\tau)|\hat{S}^x|\psi(\tau)\rangle$ can be calculated explicitly for all states $|\psi(\tau)\rangle$. Using Eq. (28) and the $(2s+1)$-dimensional matrix representation of $\hat{S}^x_{mm'}=\frac{1}{2}(C_m^+\delta_{mm'+1}+C_m^-\delta_{mm'-1})$, one finds

$$\langle\psi(\tau)|\hat{S}^x|\psi(\tau)\rangle = \sum_{m=-s+1}^{s}|\psi_m(z)|\,|\psi_{m-1}(z')|C_m^+\cos[\Delta\mu_m(\tau)]\ , \quad (30)$$

where $C_{m+1}^+=C_m^-$ has been used. From Eq. (29) it follows that

$$\cos[\Delta\mu_m(\tau)]=\cos\left[(1+\tau_m)\frac{\pi}{2}-\tau_m\Delta\phi_m\right] =-\tau_m\cos\Delta\phi_m \quad (31)$$

and $\tau_m=\pm1$. This leads to

$$\langle\psi(\tau)|\hat{S}^x|\psi(\tau)\rangle = \sum_{m=-s+1}^{s}\tau_m|\psi_m(z)|\,|\psi_{m-1}(z')|C_m^+\cos\Delta\phi_m\ . \quad (32)$$

Generically, every τ gives rise to a different expectation value $\langle\psi(\tau)|\hat{S}^x|\psi(\tau)\rangle$. By the way this result is one way to prove the fact that all $|\psi(\tau)\rangle$ represent distinct states. Another method, not requiring the evaluation of $\langle\hat{S}^x\rangle$, consists in showing that $|\psi(\tau)\rangle=\exp(i\alpha)|\psi(\tau')\rangle$ entails $\tau\equiv\tau'$ for almost every $|\psi(\tau)\rangle$.

Consequently, by measuring $\langle\hat{S}^x\rangle$ for the system under study, one can single out the correct set of numbers τ_0, implying the exact determination of the state $|\psi(\tau_0)\rangle$ which was the ultimate goal. It must be noted that in order to measure the expectation value $\langle\hat{S}^x\rangle$ the direction of the beam has to be changed by appropriate fields; this is assumed to be possible without changing the state of the spin.

There is an obvious pairing of states with expectation values of $\hat{S}^x$ of equal magnitudes but opposite signs since according to Eq. (32) one has

$$\langle\psi(-\tau)|\hat{S}^x|\psi(-\tau)\rangle=-\langle\psi(\tau)|\hat{S}^x|\psi(\tau)\rangle\ . \quad (33)$$

The origin of this structure in the set 2^{2s} solutions is indicated in the next section.

V. PAIRING OF STATES

As long as measurements are performed with respect to the yz plane only, one might expect the set of solutions compatible with the intensities along the z and z' directions to possess a particular structure. Indeed, it is shown in the following that every solution of Eq. (13) has a nontrivial partner presenting another solution. To give an example, a system with spin $\frac{1}{2}$ is appropriate. An explicit calculation shows that there are two rays compatible with prescribed intensities along y and z. The expectation values of $\hat{S}^x$ associated with them have equal moduli but opposite signs. In fact, this is the situation for all values s, and it is due to the invariance of the measured quantities with respect to a specific transformation of the experimental setup.

Consider the first series of measurements which lead to the determination of the intensities $\{|\psi_m(x)|^2\}$. If this experiment is performed with a magnetic field $\mathbf{B}$ of opposite direction, the sequence of intensities $\{|\psi_m(z)|^2\}$ occurs in reverse order with respect to the z axis. A subsequent rotation of the Stern-Gerlach apparatus by an angle π about any axis through the origin perpendicular to the z axis (or, equivalently, a rotation of the coordinate system about the same axis by the angle $-\pi$) restores the original order of the intensities. A corresponding set of transformations ($\mathbf{B}$ into $-\mathbf{B}$ and a subsequent rotation about an axis perpendicular to the direction z') leaves invariant the intensities measured with respect to z'. Both sets of quantities are unchanged only if the axis of rotation coincides with the x axis. Consequently, there are two physically distinct configurations of the apparatus leading to the same set of observed quantities. This situation, however, is equivalently described by stating that there are two spin states not to be distinguished by measurements in the xy plane alone.

The transformation described above is induced by the antiunitary operator

$$\hat{V}=\hat{K}\,\hat{U}(R_x^\pi)\ , \quad (34)$$

where $\hat{K}$ is the time-inversion operator, and $\hat{U}(R_x^\pi)=\exp(-i\pi\hat{S}^y/\hbar)$ is a rotation by π about the x direction. The operator $\hat{K}$ can be written as

$$\hat{K}=\hat{U}(R_y^\pi)\hat{K}_0\ , \quad (35)$$

$\hat{K}_0$ denoting the operator of complex conjugation. It is straightforward to show that for the spin state $|\psi\rangle$ and its partner

$$|\bar{\psi}\rangle=\hat{V}|\psi\rangle\ , \quad (36)$$

the following relations hold ($p=0,1,2,\ldots,2s$):

$$\langle\bar\psi|(\hat S^z)^p|\bar\psi\rangle=\langle\psi|(\hat S^z)^p|\psi\rangle\ ,$$
$$\langle\bar\psi|(\hat S^{z'})^p|\bar\psi\rangle=\langle\psi|(\hat S^{z'})^p|\psi\rangle\ . \tag{37}$$

According to the Appendix, these identities guarantee the invariance of the intensities $\{|\psi_m(z)|^2,|\psi_m(z')|^2\}$. Since

$$\hat K\hat S^x\hat K^+=-\hat S^x\ , \tag{38}$$

one obtains

$$\langle\bar\psi|\hat S^x|\bar\psi\rangle=-\langle\psi|\hat S^x|\psi\rangle\ , \tag{39}$$

indicating that the states $|\psi\rangle$ and $|\bar\psi\rangle$ can be distinguished by the expectation value of the x component of the spin.

The explicit form of $|\bar\psi\rangle$ follows from writing

$$\hat V=\hat U(R_y^\pi)\hat K_0\hat U(R_x^\pi)=\hat U(R_y^\pi)(\hat K_0\hat U(R_x^\pi)\hat K_0^{-1})\hat K_0$$
$$=\hat U(R_y^\pi)\hat U(R_x^{-\pi})\hat K_0\equiv\hat{\mathcal R}\hat K_0\ , \tag{40}$$

where $\hat K_0\hat S^x\hat K_0^{-1}=\hat S^x$ has been used. From the relation

$$\hat{\mathbf S}\cdot\mathbf n=\exp(i\pi\hat{\mathbf S}\cdot\mathbf k/\hbar)\hat{\mathbf S}\cdot\mathbf m\exp(-i\pi\hat{\mathbf S}\cdot\mathbf k/\hbar)\ , \tag{41}$$

where the vector $\mathbf n$ is obtained by rotating the unit vector $\mathbf m$ by an amount $|\mathbf k|$ about the vector $\mathbf k$, one finds that

$$\hat{\mathcal R}\hat S^x\hat{\mathcal R}^+=-\hat S^x,\quad \hat{\mathcal R}\hat S^y\hat{\mathcal R}^+=-\hat S^y,\quad \hat{\mathcal R}\hat S^z\hat{\mathcal R}^+=\hat S^z\ . \tag{42}$$

Consequently, apart from an irrelevant phase factor, the operator $\hat{\mathcal R}$ describes a rotation about the z axis by an amount of $+\pi$ (or $-\pi$). Therefore

$$|\bar\psi\rangle=\hat U(R_z^\pi)\hat K_0\sum_{m=-s}^{s}\psi_m(z)|z;m\rangle$$
$$=\hat U(R_z^{-\pi})\sum_{m=-s}^{s}\psi_m^*(z)|z;m\rangle$$
$$=\sum_{m=-s}^{s}\exp(-im\pi)\psi_m^*(z)|z;m\rangle$$
$$=\sum_{m=-s}^{s}(-1)^{-m}\psi_m^*(z)|z;m\rangle\ , \tag{43}$$

and the same ray is obtained by using $\hat U(R_z^{-\pi})$ instead. A direct calculation shows that the coefficients $\bar\psi_m(z)=(-1)^{-m}\psi_m^*(z)$ indeed fulfill Eq. (13) to first order in ϵ.

One may ask how the existence of paired solutions is reflected within the explicit form of all solutions $|\psi(\tau)\rangle$. To see this it is sufficient to indicate that for any of the 2^{2s} solutions

$$|\psi(\tau)\rangle=\sum_{m=-s}^{s}\psi_m(\psi_m(\tau)|z;m\rangle\ , \tag{44}$$

a partner exists with

$$\psi_m(\tau')=(-1)^{-m}\psi_m^*(\tau)\ . \tag{45}$$

From the formula for $|\psi(\tau)\rangle$ [Eq. (28)], it follows that

$$\psi_m^*(\tau)=\exp\left[-i2\pi\sum_{m'=-s+1}^{m}(1+\sigma_{m'})\right]|\psi_{m'}(z)|\exp\left[-i2\pi\sum_{m'=-s+1}^{m}[1+(-\tau_{m'})](1+\sigma_{m'})-\tfrac12(-\tau_{m'})\Delta\phi_{m'}\right]$$
$$=(-1)^{s+m}\psi_m(-\tau)\ . \tag{46}$$

The last step is a consequence of

$$\exp\left[-i2\pi\sum_{m'=-s+1}^{m}(1+\sigma_{m'})\right]$$
$$=\exp\left[-i2\pi(s+m)-i\pi\sum_{m'=-s+1}^{m}2\sigma_{m'}\right]$$
$$=\exp\left[-i\pi\sum_{m'=1}^{m+s}1\right]=(-1)^{s+m}\ , \tag{47}$$

since $2\sigma_m$ is equal to ±1. The result of Eq. (46) coincides with Eq. (45), choosing $\tau'=-\tau$, because suppressing the factor $(-1)^s$ does not change the ray in Hilbert space. Furthermore, Eqs. (33) and (39) are compatible, as it is necessary.

VI. SUMMARY

It is possible to determine unambiguously the wave function of a spin-s system, making use of an elementary Stern-Gerlach apparatus only. The intensities of a (generic) pure wave function along the z axis and an infinitesimally twisted one z' are sufficient for this purpose in combination with the expectation value of the spin component perpendicular to the zz' plane. Therefore this method requires knowledge of $4s+1$ real numbers. The state vector $|\psi\rangle$ in the $(2s+1)$-dimensional Hilbert space $\mathcal H$ of the problem is defined by $4s$ parameters. The additional measurement, being necessary in this approach, is due to the fact that the $4s$ independent intensities measured fulfill *nonlinear* equations, which turn out to have 2^{2s} roots. Note that the actual calculation of the phases ϕ has to be performed numerically because no analytic formula has been derived for them.

The approach presented by Band and Park [7–9] effectively requires only the minimal number of quantities ($=4s$) to be measured since in that case the defining relations are *linear*. However, the measurement of the appropriate "spin multipoles" is much more intricate, if not unaccessible at all. On the other hand, Gale, Guth, and Trammell [5] use the more refined "technology" of Feynman filters, and according to their prescription, one has to determine $6s$ independent numbers. Hence the present approach to find out a spin wave function from measurements combines the advantage of a particularly simple experimental setup with the necessity to perform a small number of measurements only. It would be still more satisfactory to generalize this result to axes z,z' separated by a *finite* angle of rotation. For a spin-$\frac{1}{2}$ system, two states are compatible with intensities measured along the y and z axes. This supports the idea that no additional solutions bifurcate from those which have been found for infinitesimally close directions z,z'.

From a mathematical point of view, the following problem has been considered. A normalized ray in Hilbert space is defined most conveniently in terms of its (complex) coefficients, i.e., moduli and phases, with respect to any set of orthonormal basis vectors. The transformation of the coefficients under a definite change of basis then is a straightforward procedure. The question investigated in this paper reads as follows. Is a ray also defined unambiguously by the moduli of its coefficients with respect to *two* different orthonormal bases? The bases involved are assumed to be obtained from each other by a unitary transformation [corresponding to an element of the group SU(2)] having the particular property that none of the basis vectors is left unchanged. As a result, the ray is defined in this way only up to a finite ambiguity, which, in the generic case, can be resolved easily by one additional "orthogonal" information.

Investigating the Pauli problem for a particular system (class I_P), the situation is formally identical, except that the dimension of the Hilbert space $\mathcal{H}$ is infinite. The question whether or not this method can be adapted successfully to the more general case is under study presently. One may hope to determine constructively along these lines the set of all states which are compatible with Pauli data $\{|\psi(x)|^2, |\psi(p)|^2\}$ for the particle system.

ACKNOWLEDGMENTS

H. Thomas's comments on the manuscript and, in particular, clarifying discussions about the symmetry aspect presented in Sec.V are gratefully acknowledged.

APPENDIX

Measuring all moments $S_p = \langle \psi|(\hat{S}^z)^p|\psi\rangle$, $p=0,1,2,\ldots,2s$, of the spin operator $\hat{S}^z$ is shown to be equivalent to the knowledge of all intensities $\{|\psi_m(z)|^2\}$. The expectation values of moments and intensities are related by the matrix equation

$$\underline{S} = \mathbf{M}\,\underline{C}\ , \tag{A1}$$

where $\underline{S} \equiv (1,S_1,S_2,\ldots,S_{2s})$ and

$$\underline{C} = (|\psi_{-s}(z)|^2, |\psi_{-s+1}(z)|^2, \ldots, |\psi_s(z)|^2)$$

are $(2s+1)$-dimensional vectors and $\mathbf{M}$ is a $(2s+1)\times(2s+1)$ matrix given by

$$M_{kl} = (m_l)^k\ ,\quad k,l=0,1,2,\ldots,2s\ , \tag{A2}$$

with $m_l \equiv -s+l$. The inversion of Eq. (A1) is only possible if

$$\det M \neq 0\ . \tag{A3}$$

Being of Vandermonde type, the determinant of $\mathbf{M}$ can be given explicitly as

$$\det\mathbf{M} = \prod_{0\le\lambda\le\mu\le 2s} (m_\mu - m_\lambda) \tag{A4}$$

and therefore is nonzero whenever all m_μ are different. In the case under consideration, all m_μ *are* different. Consequently, Eq. (A1) is invertible globally,

$$\underline{C} = \mathbf{M}^{-1}\underline{S}\ . \tag{A5}$$

By the way, the actual value of the determinant of $\mathbf{M}$ is easily found to be equal to $\prod_{p=0}^{2s} p!$.

[1] W. Pauli, in *Quantentheorie*, edited by H. Geiger and K. Scheel, Handbüch der Physik Vol. 24 (Springer, Berlin, 1933), Pt. 1, p. 98.
[2] E. Feenberg, Ph.D. thesis, Harvard University, 1933.
[3] E. C. Kemble, *Fundamental Principles of Quantum Mechanics* (McGraw-Hill, New York, 1937), p. 71.
[4] H. Reichenbach, *Philosophical Foundations of Quantum Mechanics* (University of California Press, Berkeley, 1946), Sec. 20.
[5] W. Gale, E. Guth, and G. T. Trammell, Phys. Rev. **165**, 1414 (1968).
[6] W. E. Lamb, Phys. Today **22**(4), 23 (1969); G. T. Trammell, *ibid.* **22**(10), 9 (1969).
[7] W. Band and J. L. Park, Found. Phys. **1**, 133 (1970).
[8] J. L Park and W. Band, Found. Phys. **1**, 211 (1971).
[9] W. Band and J. L. Park, Found. Phys. **1**, 339 (1971).
[10] B. Z. Moroz, Zap. Nauchn. Semin. LOMI **39**, 189 (1974).
[11] B. d'Espagnat, *Conceptual Foundations of Quantum Mechanics* (Benjamin, Reading, MA, 1976), Sec. 7.1.
[12] V. Ja. Kreinovitch, Theor. Math. Phys. **28**, 623 (1977).
[13] E. Prugovečki, Int. J. Theor. Phys. **16**, 321 (1977).
[14] J. V. Corbett and C. A. Hurst, J. Aust. Math. Soc. B **20**, 182 (1978).
[15] A. Vogt, in *Mathematical Foundations of Quantum Theory*, edited by A. R. Marlow (Academic, New York, 1978).
[16] W. Band and J. L. Park, Am. J. Phys. **47**, 188 (1979).
[17] B. Z. Moroz, Int. J. Theor. Phys. **22**, 329 (1983); **23**, 497(E) (1984).
[18] A. Royer, Phys. Rev. Lett. **55**, 2745 (1985).
[19] C. N. Friedman, J. Aust. Math. Soc. B **30**, 298 (1987).
[20] M. Pavičić, Phys. Lett. A **122**, 280 (1987).
[21] H.-W. Wiesbrock, Int. J. Theor. Phys. **26**, 1175 (1987). This author refers to a paper by Moroz [10] presenting another example for this ambiguity. However, this work and that one by Feenberg [2] were not available to me.
[22] A. Royer, Found. Phys. **19**, 3 (1989).
[23] P. Busch and P. J. Lahti, Found. Phys. **19**, 633 (1989).

[24] W. Stulpe and M. Singer, Found. Phys. Lett. **3**, 153 (1990).
[25] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1965), Vol. III, Chap. 5.
[26] In spite of the notation $|\psi_m(z)\rangle$, the quantity z should not be understood as a variable.
[27] It is demonstrated in the Appendix that, equivalently, the measurement of all moments $(\hat{S}^z)^p$ and $(\hat{S}^{z'})^p$, $p=0,1,2,\ldots,2s$, could be considered.
[28] The phases of the matrix elements are chosen according to the convention due to E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, England, 1935).